Super-sech soliton dynamics in optical metamaterials with generally parabolic law of nonlinearity using Lagrangian Variational Method

Original Research Article

Abstract

Aims/ objectives: This paper studies the impact of the generally parabolic law of nonlinearity on the evolution of the energy of super-sech soliton dynamics.
Study design: generally parabolic law of nonlinearity terms study.
Place and Duration of Study: Department of Physics, Faculty of Sciences and Technology(FAST), University of Abomey Calavi, Bénin. between Febuary 2018 and January 2019.
Methodology: Variational approach, namely, the Lagrangian Variational Method (LVM) is presented. The different results are obained using standard fourth order Runge-Kutta method for integration of the system of ordinary differential equation systems.
Results: Dynamics of the different parameters (amplitude, center position, pulse width, chirp, frequency and phase) has been presented with respect to propagating distance.
Conclusion: This study reveals that the generally parabolic law of nonlinearity terms don't affect the energy of the system but influence the pulse phase.

Keywords: Lagrangian approach, generaly parabolic law, super-sech soliton, metamaterial.
2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Introduction

The metamaterial is a new type of microstructured material which has been extensively used and studied during the recent years. Metamaterials are artificial composite structures with both negative permittivity and negative permeability. They also have fascinating physical properties and spectacular uses [Veselago (1968); Pendry (2000); Shalaev (2007); Zharova (2005); Veljkovic (2017, 2015);

Biswas (2017c, 2014a,b); Green (2008); Solymar (2009)]. Metamaterials are an emerging technology with applications in a range of diverse areas. Metamaterials are artificially engineered materials with properties not available in natural systems such as negative permeability and permittivity, display anomalous behaviour, such as negative refraction, superlensing, backward wave propagation and reverse Doppler shifting. Consequently the are many applications including energy harvesting, objet cloaking, high data rate communications, sensors and detectors, imaging, anti-vibration, noise reduction, seismic protection and antennae [Steve (2015)]. Metamaterials can either be used to improve the performance of existing applications. Nowadays, it is possible to use this material as waveguide in order to optimize the data transmission. This is precisely the framework of the present research. This research aims to study the dynamics of a soliton pulse, super-sech soliton which is propagated in a metamaterial, in order to assess the impact of the generally parabolic law of nonlinearity on the pulse profile along its path in the metamaterial. The dynamics of solitons in optical metamaterials is governed by the model [Agrawal (1989); Biswas (2014b); Douvagai (2017); Faroutan (2018); Zhou (2017a, 2014b); Veljkovic (2017, 2015)]:

$$\begin{aligned} iq_z + aq_{tt} + b|q|^2 q = i\alpha q_t + i\lambda(|q|^2 q)_t + i\nu(|q|^2)_t q \\ +\theta_1(|q|^2 q)_{tt} + \theta_2 |q|^2 q_{tt} + \theta_3 q^2 q^\star_{tt} \end{aligned} \tag{1.1}$$

This equation was recently used by Douvagai et al. where they showed an additional nonlinear term. These last ones used the complex envelope ansatz method and the F-expansion method to solve the generally nonlinear Schrödinger equation (GNLSE) with an additional parabolic law nonlinearity. Bright and dark soliton soliton solutions are obtained [Douvagai (2017)].

$$\begin{aligned} iq_z + aq_{tt} + b(|q|^2 + \sigma|q|^4) q = i\alpha q_t + i\lambda(|q|^2 q)_t + i\nu(|q|^2)_t q \\ +\theta_1(|q|^2 q)_{tt} + \theta_2 |q|^2 q_{tt} + \theta_3 q^2 q^\star_{tt} \end{aligned} \tag{1.2}$$

Recent work by Biswas et al. has taken this additional term into account [Biswas (2018)]. Similarly, Foroutan et al. studied disturbances of the optical soliton in a metamaterial using two approachs: the extented trial equation method and the improved G'/Gexpansion method. The bright, dark and singular soliton are retrieved in this research [Faroutan (2018)]. The study equation is:

$$\begin{aligned} iq_z + aq_{tt} + (b_1|q|^{-4} + b_2|q|^2 + b_3|q|^4) q = i\alpha q_t + i\lambda(|q|^2 q)_t + i\nu(|q|^2)_t q \\ +\theta_1(|q|^2 q)_{tt} + \theta_2 |q|^2 q_{tt} + \theta_3 q^2 q^\star_{tt} \end{aligned} \tag{1.3}$$

This equation is the nonlinear Schrödinger equation with an additional anti-cubic nonlinear term. In this work, we propose to solve by the Lagrangian method the equation (1.2) with generally parabolic law of nonlinearity [Biswas (2017c); Cai (2010); Fujioka (2011); Saha (2013); Zhou (2017a,b, 2014a)]. The objective of such a study would be to exhibit the contribution of these terms on the dynamics of the optical soliton. These terms appear in the metamaterial context when considered as centrosymmetric materials and high order polarization vectors are taken into account in the Maxwell equation. The new equation is therefore given by (1.4) and is named the general parabolic law nonlinearity equation.

$$\begin{aligned} iq_z + aq_{tt} + \sum_{k=1}^{n} b_k |q|^{2k} q = i\alpha q_t + i\lambda(|q|^2 q)_t + i\nu(|q|^2)_t q \\ +\theta_1(|q|^2 q)_{tt} + \theta_2 |q|^2 q_{tt} + \theta_3 q^2 q^\star_{tt} \end{aligned} \tag{1.4}$$

In eq.(1.4), the unknown or dependent variable $q(z,t)$ represents the wave profile, while z and t are the spatial and temporal variables respectively. The first and second terms are the linear spatial evolution terms and the group velocity dispersion, while third term introduces the generaly parabolic law of nonlinearity, fourth, fifth and sixth terms represent inter-modal dispersion, self steepening and the nonlinear dispersion respectively. Finally, the last three terms with θ_k for $k = 1, 2, 3$ appear in the context of metamaterials [Veljkovic (2017)].

2 Lagrangian Variational Method

The main idea of LVM is based on extending Euler-Lagrange least-action principles to dissipative systems. LVM is used to express the generalized NLSE in terms of fondamental parameters (collective variables). This consists in finding the Lagrangian of NLSE, then choosing any convenient trial function f (ansatz) assumed to best approximate the behaviour of the pulse in order to derive the set of variational equations [Adrian (2008); Biswas (2018, 2017a); Cheng (2009); Edah (2014); Moubissi (2001); Nakkeeran (2005)]. Let's write the NLSE (1.4) in the form

$$iq_z + aq_{tt} + \sum_{k=1}^{n} b_k|q|^{2k}q = \zeta \tag{2.1}$$

where

$$\zeta = i\alpha q_t + i\lambda(|q|^2 q)_t + i\nu(|q|^2)_t q + \theta_1(|q|^2 q)_{tt} + \theta_2|q|^2 q_{tt} + \theta_3 q^2 q^{\star}_{tt} \tag{2.2}$$

is considered as a perturbation term. Consider the equation (2.1) without perturbation term ($\zeta = 0$) and look for the solution q on the form

$$q(z,t) = u(z,t) + iv(z,t) \tag{2.3}$$

where u and v are real functions. Substituting (2.3) in (2.1), one obtains

$$u_z + av_{tt} + \sum_{k=1}^{n} b_k(u^2+v^2)^k v = 0 \tag{2.4}$$

$$-v_z + au_{tt} + \sum_{k=1}^{n} b_k(u^2+v^2)^k u = 0 \tag{2.5}$$

The equations (2.4) and (2.5) can be deduced respectively from Euler-Lagrange equations given by

$$\frac{\partial L_0}{\partial v} - \frac{\partial}{\partial z}\left(\frac{\partial L_0}{\partial v_z}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_0}{\partial v_t}\right) = 0 \tag{2.6}$$

$$\frac{\partial L_0}{\partial u} - \frac{\partial}{\partial z}\left(\frac{\partial L_0}{\partial u_z}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_0}{\partial u_t}\right) = 0 \tag{2.7}$$

where the Lagrangian L_0 is given by:

$$L_0 = \frac{1}{2}\left(u_z v - v_z u\right) + \sum_{k=2}^{n} \frac{b_{k-1}}{2k}\left(u^2+v^2\right)^k - \frac{a}{2}\left(u_t^2 + v_t^2\right) \tag{2.8}$$

When we express respectively u and v as follows: $u = \frac{1}{2}(q+q^*)$; $v = \frac{i}{2}(q^*-q)$, the Lagrangian L_0 can be rewritten as follows:

$$L_0 = \frac{i}{4}\left(q_z q^* - q_z^* q\right) + \sum_{k=2}^{n} \frac{b_{k-1}}{2k}|q|^{2k} - \frac{a}{2} \mid q_t \mid^2 \tag{2.9}$$

The averaged Lagrangian of equation without right hand side is defined as:

$$L = \int_{-\infty}^{+\infty} L_0 dt \tag{2.10}$$

. Then

$$L = \int_{-\infty}^{+\infty} \left[\frac{i}{4}\left(q_z q^* - q_z^* q\right) + \sum_{k=2}^{n} \frac{b_{k-1}}{2k}|q|^{2k} - \frac{a}{2} \mid q_t \mid^2\right] dt \tag{2.11}$$

3 Super-sech Parameter Dynamics

The ansatz function f that we assume in this paper is the super sech soliton [Veljkovic (2017)]:

$$f = X_1 sech^m \left[\frac{t-X_2}{X_3}\right] \exp\left[i\left(\frac{X_4}{2}(t-X_2)^2 + X_5(t-X_2) + X_6\right)\right]; \tag{3.1}$$

where X_1 represents the amplitude of the pulse, X_2 the temporal position, X_3 the width, X_4 the chirp, X_5 the frequency and X_6 the phase. m is the parameter of the super-sech. In this paper, m is set equal to 2. Substituting $q = f$ in (2.11), one obtains:

$$L = L_1 + \sum_{k=2}^{n} \frac{b_{k-1}}{2k} \int_{-\infty}^{+\infty} sech^{2k}\left[\frac{t-X_2}{X_3}\right] dt \tag{3.2}$$

where

$$\begin{aligned} L_1 = {} & \frac{2}{3}X_1^2 X_3 X_5 \dot{X}_2 + \frac{6-\pi^2}{36} X_1^2 X_3^3 \dot{X}_4 - \frac{2}{3} X_1^2 X_3 \dot{X}_6 \\ & -\frac{a}{90}\frac{X_1^2}{X_3}(48 + 60X_3^2X_5^2 - (30-5\pi^2)X_3^4X_4^2) \end{aligned} \tag{3.3}$$

so for $n = 6$, the average Lagrangien is:

$$\begin{aligned} L = {} & \frac{2}{3}X_1^2 X_3 X_5 \dot{X}_2 + \frac{6-\pi^2}{36} X_1^2 X_3^3 \dot{X}_4 - \frac{2}{3} X_1^2 X_3 \dot{X}_6 + \frac{8}{35} b_1 X_1^4 X_3 \\ & -\frac{a}{90}\frac{X_1^2}{X_3}(48 + 60X_3^2X_5^2 - (30-5\pi^2)X_3^4X_4^2) + \frac{256}{2079} b_2 X_1^6 X_3 \\ & + \frac{512}{6435} b_3 X_1^8 X_3 + \frac{71}{1251} b_4 X_1^{10} X_3 + \frac{127}{2948} b_5 X_1^{12} X_3 \end{aligned} \tag{3.4}$$

$\dot{X}_j$, $(j = 1,2,3,4,5,6)$ stands for derivative of X_j with respect to z. Now, let's come back to the full equation (2.1) where the term of right-hand side ζ is non zero. When one applies the Euler-Lagrange equations to (1.4), the variational equations are written as:

$$\frac{\partial L}{\partial X_j(z)} - \frac{d}{dz}\frac{\partial L}{\partial \dot{X}_j(z)} = \int_{-\infty}^{+\infty} \zeta f^*_{X_j} dt + c.c \tag{3.5}$$

Substituting the expression of the average Lagrangian given in equation (2.11) and the ansatz function f in ζ, then performing the integration of the right-hand side of (3.5), we obtain the following set of variational equations:

$$
\begin{aligned}
\dot{X}_1 &= -aX_1X_4 + \frac{2X_1^3X_4}{35(\pi^2-6)}\left((24\pi^2-235)\theta_1 + (24\pi^2-157)(\theta_2-\theta_3)\right) \qquad (3.6)\\
\dot{X}_2 &= 2aX_5 - 2\alpha - \frac{24}{35}\left((3\lambda+2\nu)X_1^2 + (6\theta_1+2\theta_2-2\theta_3)X_1^2X_5\right)\\
\dot{X}_3 &= 2aX_3X_4 - \frac{4}{35(-6+\pi^2)}\left(\left(-307+36\pi^2\right)\theta_1 + \left(-85+12\pi^2\right)(\theta_2-\theta_3)\right)X_1^2X_3X_4\\
\dot{X}_4 &= -2aX_4^2 + \frac{672a}{35(\pi^2-6)X_3^4} - \frac{1}{(\pi^2-6)X_3^2}\Big(\frac{144b_1}{35}X_1^2 + \frac{1024b_2}{231}X_1^4 + \frac{3072b_3}{715}X_1^6\\
&+\frac{568b_4}{139}X_1^8 + \frac{1648b_5}{425}X_1^8 + \frac{288}{35}\lambda X_1^2X_5\Big) + \frac{4}{175}\frac{X_1^2}{X_3^4}\left((30\pi^2-245)X_3^4X_4^2 - 360X_3^2X_5^2 - 3168\right)\theta_1\\
&+\frac{4}{175}\frac{X_1^2}{X_3^4}\left((30\pi^2-245)X_3^4X_4^2 - 360X_3^2X_5^2 - 864\right)(\theta_2+\theta_3)\\
\dot{X}_5 &= -\frac{2(X_1X_4+2X_5)aX_5}{X_1} + \frac{4\alpha X_5}{X_1} - \frac{4}{35}\frac{X_1X_5\left(24\pi^2X_5 - 13X_1X_4 - 144X_5\right)}{\pi^2-6}\\
&+\frac{4}{35}\frac{X_1X_5\left(72\pi^2X_5 - 91X_1X_4 - 432X_5\right)\theta_1}{\pi^2-6} - \frac{48}{35}(X_1X_4-3X_5)\lambda X_1\\
&+\frac{4}{35}\frac{X_1X_5\left(24\pi^2X_1X_4 + 24\pi^2X_5 - 157X_1X_4 - 144X_5\right)\theta_2}{\pi^2-6} - \frac{48}{35}(X_1X_4-2X_5)\nu X_1\\
\dot{X}_6 &= \frac{1}{35}\frac{35aX_3^2X_5^2 - 56a}{X_3^2} + \frac{30}{35}b_1X_1^2 + \frac{512}{693}b_2X_1^4 + \frac{896}{1365}b_3X_1^6 + \frac{497}{834}b_4X_1^8\\
&+\frac{309}{500}b_5X_1^{10} + \frac{1}{350}\frac{X_1^2}{X_3^2}\left((30\pi^2-245)X_3^4X_4^2 - 840X_3^2X_5^2 + 928\right)\theta_1\\
&-\frac{2}{35}(6\lambda+24\nu)X_1^2X_5 + \frac{1}{350}\frac{X_1^2}{X_3^2}\left((30\pi^2-245)X_3^4X_4^2 - 120X_3^2X_5^2 + 928\right)(\theta_2+\theta_3)
\end{aligned}
$$

4 Results and Discussion

The numerical study of the evolution of the different parameters of the super-sech soliton momentum has been made in order to appreciate the impact of the generally parabolic law nonlinearity terms on the dynamics of such an pulse in a metamaterials. The different results are obtained using standard fourth order Runge-Kutta method for integration of the system of ordinary differential equation systems [Balac (2013)]. The dynamics of the system have been presented in Figure2 for the following parameter values: $a = 0.1$, $b_1 = -20$, $\alpha = -0.25$, $\lambda = 0.1$, $\nu = 0.1$, $\theta_1 = -0.01$, $\theta_2 = -0.02$, $\theta_3 = -0.3$, $b_2 = 0.001$, $b_3 = 0.1$, $b_4 = 0.1$, $b_5 = 2$.
The analysis of this curve shows that the amplitude, the pulse width, the chirp and the frequency slip vary periodically as a function of z. Indeed, it should be noted that the choice of the initial condition is of paramount importance for such a study. These parameters have been chosen so that the super-sech soliton propagates itself without attenuation. The variationals equations $\dot{X}_1$, $\dot{X}_2$, $\dot{X}_3$ obtained are identical to those of Veljkovic et al. [Veljkovic (2017)]. This explains the resemblance of the representative curves of the amplitude, the center position and the pulse width. The terms of high order added to the equation don't influence the evolution of these parameters ($\dot{X}_1$, $\dot{X}_2$, $\dot{X}_3$). On the other hand the variationals equations: $\dot{X}_4$, $\dot{X}_5$, $\dot{X}_6$ are functions of the terms of high order introduced and show dissimilarities. The different terms b_k, $k = 1, ...6$, rather influence the parameters of the pulse phase. This confirms the absence of these terms in the expression that describes the variation of the energy (4.3). A particular attention has been carried on the energy of the system. The energy

is defined as:

$$L = \int_{-\infty}^{+\infty} |q|^2 \, dt. \tag{4.1}$$

In the case of the super-sech soliton, one has:

$$E = \frac{4X_1^2 X_3}{3} \tag{4.2}$$

The evolution of the energy is given by:

$$\frac{dE}{dz} = \left[\theta_1 \left(\frac{192 - 32\pi^2}{35(-6+\pi^2)}\right) + (\theta_2 - \theta_3)\left(\frac{-976 + 128\pi^2}{35(-6+\pi^2)}\right)\right] X_1^4 X_3 X_4. \tag{4.3}$$

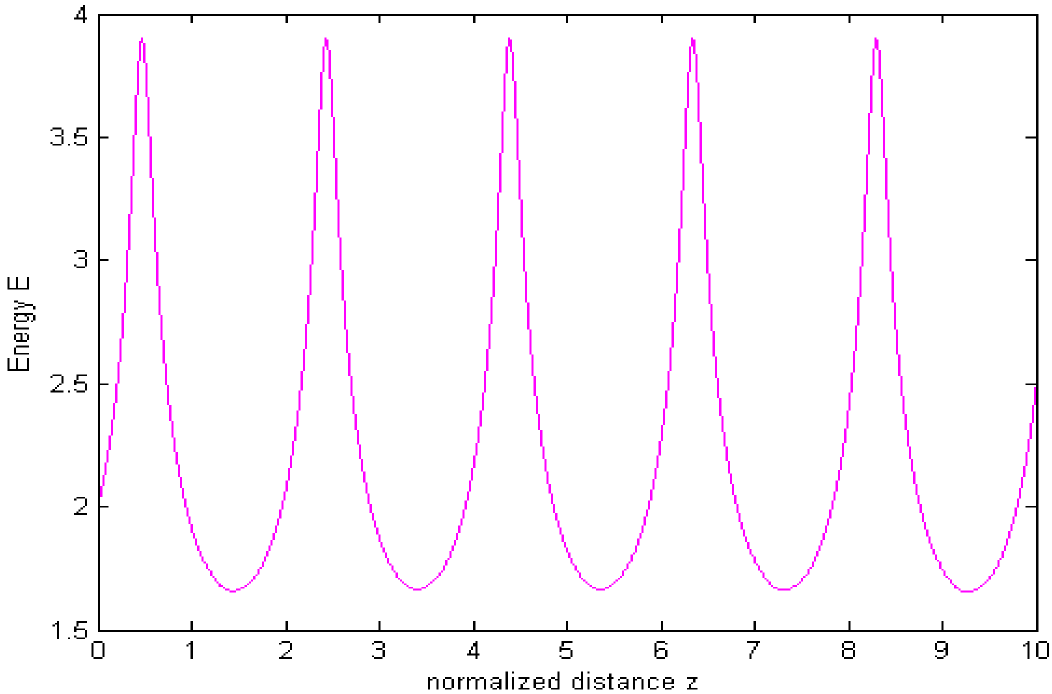

Figure 1: Variation of energy

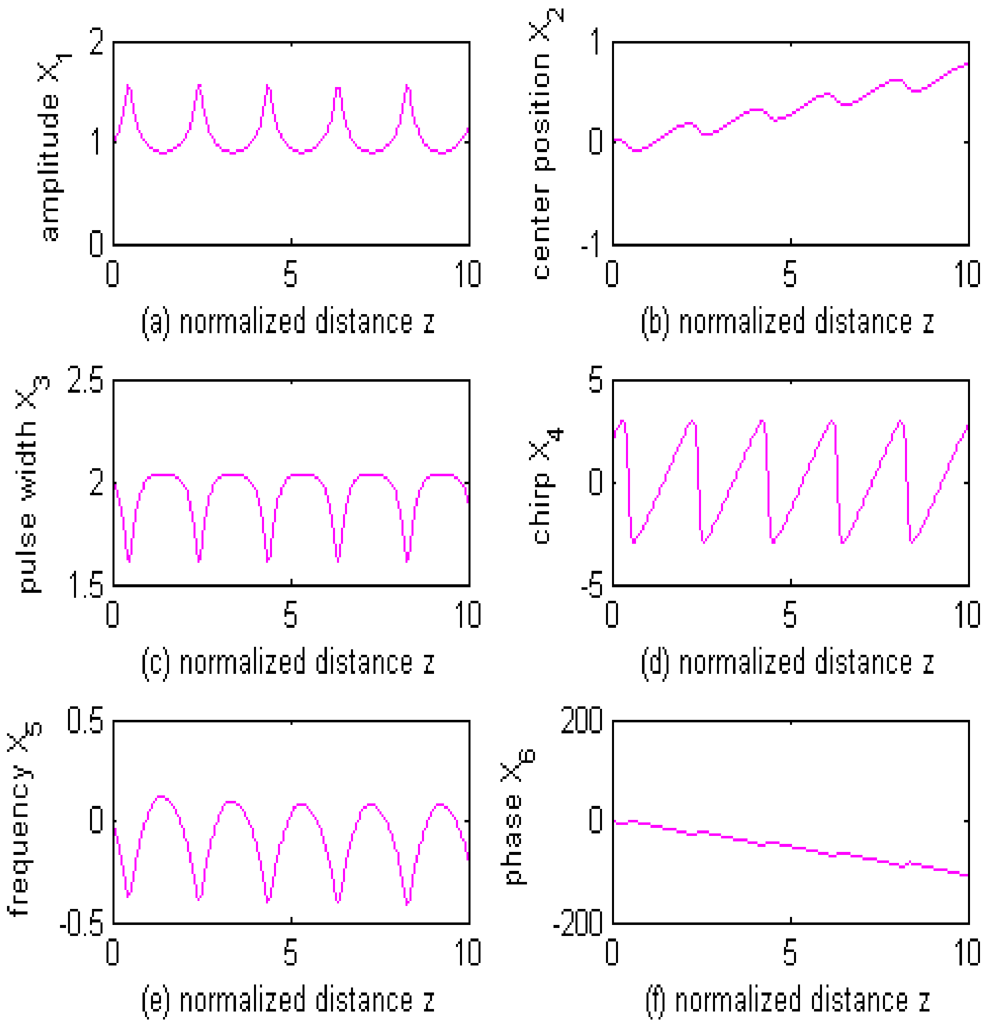

Figure 2: Variation of normalized pulse parameters(X_1-soliton amplitude, X_2-center position of the soliton, X_3-pulse width, X_4-soliton chirp, X_5-soliton frequency, X_6-soliton phase) with propagation distance

5 CONCLUSION

This paper presents lagrangian variational approach for super sech soliton dynamics in optical metamaterials. The optical soliton dynamics is governed by the generalized nonlinear Schrödinger equation including generally parabolic law of nonlinearity. This equation is solved by lagrangian approach where a six paramater (amplitude, center position, pulse width, chirp, frequency and phase) super-sech soliton test function has been used to approximate the exact solution. Numerical simulations have made it to represent these parameters graphically as a function of the propagation distance. This study reveals that the generally parabolic law of nonlinearity terms don't affect the energy of the system, but affect the pulse phase. Finally, the analysis of these results revealed that the choice of the initial condition is crucial for such a study. A comparison with other results gave excellent agreement. This work could be proposed in telecommunication to optimize the transmission of information. The results with those additional laws of nonlinearity will be reported in future.

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