A SEQUENTIAL THIRD ORDER ROTATABLE DESIGN IN FOUR DIMENSIONS

ABSTRACT.

In research, experiments must be performed at pre-determined levels of the controllable factors, meaning that an experimental design must be selected prior to experimentation. Once an experimenter has chosen a polynomial model of suitable order, the problem arises on how best to choose the settings for the independent variables over which he has control. A particular selection of settings or factor levels at which observations are to be taken is called a design. A design may become inappropriate under special circumstances requiring an increase in factors or levels to make it more desirable. In agriculture for instance, continuous cultivation of crops may exhaust previously available mineral elements necessitating a sequential appendage of the mineral elements which become deficient in the soil over time. In this study, a fifty six points third order rotatable design is constructed by adding a set of factors to a second order rotatable design in four dimensions and a practical hypothetical example is given by converting coded level to natural levels. This design permits a response surface to be fitted easily and provide spherical information contours besides the economic use of scarce resources in relevant production processes.

Key words*:* Response surface; Rotatable designs; Third order.

1.0 INTRODUCTION

Response surface methodology is a collection of mathematical and statistical techniques useful for analysing problems where several independent variables influence a dependent variable and its objective is to optimize the dependent variable. In the recent years, response surface methodology has been widely recognised as a very important tool for use in various fields such as in Medicine, Agriculture and Industry. The Kenyan economy for instance is mainly dependent on agriculture to produce food for both domestic consumption and export. The Kenyan population is growing at an alarming rate but the natural resources especially land which the population depends on have remained constant and minimal. This has necessitated proper utilization of the scarce commodity of land for maximum returns. In the past, unproductive land could be left fallow to naturally regain the exhausted nutrients, but today, the exhausted nutrients are sequentially appended to the soils through application of deficient elements(fertilizers) courtesy of response surface methodology. The fitting of the response surface can be complex and costly if done haphazardly thus the process requires expert knowledge on design and analysis of experiments. To cut on costs, an experimenter has to make a choice of the experimental design prior to experimentation. Rotatability is a natural and desirable property, which requires that the variance of a predicted response at a point remains constant at all such points that are equi-distant from the design centre. In this context, rotatable designs were introduced by Box and Hunter(2) in order to explore the response surface. They developed second order rotatable designs through geometrical configurations. Bose and Draper (1) point out that the technique of fitting a response surface is one widely used to aid in the statistical analysis of experimental work in which the response of a product depends in some unknown fashion, on one or more controllable variables. Draper and Beggs (6) state that once an experimenter has a polynomial model of suitable order, the problem arises as how best to choose the settings for the independent variables over which he has control. A Particular selection of settings or factor levels at which observations are to be taken is called a design.Designs are usually selected to satisfy some desirable criteria chosen by the experimenter. These criteria include the rotatability criterion and the criterion of minimizing the mean squire error of estimation over a given region in the factor space. The moment and non-singularity conditions for third order rotatability were derived and developed by Gardiner *et al* (9). They considered a problem arising in the design of experiments for empirically investigating the relationship between a dependent and several independent variables assuming that the form of the functional relationship is unknown but that within the region of interest, the function may be represented by a Taylor series expansion of moderately low order.Draper (8) constructed a third order rotatable design in four dimensions. Mutiso (20), constructed specific and sequential second and third order rotatable designs in three dimensions but did not give the optimality criteria for the designs. Kosgei (13) gave the alphabetic optimality criteria for the designs constructed by mutiso (20).Kosgei et al (14) gave criteria of selecting the optimality of a design based known classical optimality criteria.Koske*et al* (15, 17) and Keny*et al* (11) contructed optimal second order rotatable designs and gave practical hypothetical examples.Koske and Mutai*et al* (16,19,18 and12) used the methods laid down by Huda(10) to construct third order rotatable designs of different factors through balanced incomplete block designs.Cheruiyot (5) evaluated the efficiencies of the six specific second order rotatable designs constructed by Mutiso,(20). There is need to give hypothetical examples to all the existing designs to make them ready for the experimenters to apply in the production processes. The current study solves, in part, this problem. In this study, we construct a third order rotatable design in four dimensions with fifty six points and give a practical hypothetical example to this design.

2.0 MOMENTS AND NON-SINGULARITY CONDITIONS FOR THIRD ORDER ROTATABILITY

A set of points is said to form a third order rotatable design in k dimensions if it satisfies the following moment conditions according to Draper (8).

$\sum_{u=1}^{N} x_{iu}^2$=A (i=1, 2 . . . k), (2.1)

$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2$ =3C, (2.2)

$\sum_{u=1}^{N} x_{iu}^6$= 5 $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^4 = 15\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 x_{lu}^2$=15D, (2.3)

I$i \neq j \neq l = 1,2,\ldots,k, \quad u = 0,1,. \ . \ .,N,$

And all other sums of powers and products up to order six are zero, where

A=Nλ_2, C=Nλ_4, and D= Nλ_6 (2.4)

The arrangement of points is said to form a rotatable design of third order only if it forms a non-singular third order design (if the points give rise to a non- singular matrix). Gardiner et al. (9) derived the non –singularity conditions as;

$$\frac{NC}{A^2} > \frac{K}{K+2},$$
$$\frac{AD}{C^2} > \frac{(K+2)}{(K+4)}. \qquad (2.5)$$

These are the non-singularity conditions required for a third order rotatable arrangement of points to form third order rotatable designs.

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3.0 CONSTRUCTION OF FIFTY SIX POINTS THIRD ORDER ROTATABLE DESIGN IN FOUR DIMENSIONS.

Four dimensional third order rotatable design is constructed by combining a set of twenty four points to a second order rotatable design of thirty two points in four dimensions.

Table 1: A summary of the excess functions for fifty six points third order rotatable design in four dimensions.

Set composition of class	**S(f, f, o, o)**	**S(a, a, a, a)**	**S(c_1,o,o,o)**	**S(c_2,o,o,o)**
Number of points	**24**	**16**	**8**	**8**
A_X	$12f^2$	$16a^2$	$2c_1^2$	$2c_2^2$
E_X	**0**	$-32a^4$	$2c_1^4$	$2c_2^4$
H_X	$12f^6$	$-224a^6$	$2c_1^6$	$2c_2^6$
I_X	$4f^6$	$-32a^6$	**0**	**0**

We shall consider the combination of a set of twenty four points with another set of thirty two points second order rotatable design in four dimensions to obtain fifty six points third order rotatable design in four dimensions.

s_1=[s (a, a, a, a) + s (c_1, o, o, o) + s (c_2, o, o, o)] (3.1) with thirty two points

s_2 = [s (f, f, o, o)] (3.2) with twenty four points

For the point set in (3.1) and (3.2) above, the following conditions hold:

$\sum_{u=1}^{56} x_{iu}^4$ - $3\sum_{u=1}^{56} x_{iu}^2 x_{ju}^2$ = $2c_1^4 + 2c_2^4 - 32a^4$=0

$\sum_{u=1}^{56} x_{iu}^6$ - $15\sum_{u=1}^{56} x_{iu}^2 x_{ju}^2 x_{lu}^2$= $2c_1^6 + 2c_2^6 + 8f^6 - 112a^6$= 0

$\sum_{u=1}^{56} x_{iu}^4\, x_{ju}^2$ - $3\sum_{u=1}^{56} x_{iu}^2 x_{ju}^2 x_{lu}^2$=$4f^6 - 32a^6$= 0 (3.3)

And all the other sums of powers and products up to order six are zero. Conditions in (3.3) give,

f=1.414213562a,c_1=1.999124809a and c_2=0.741713873a (3.4)

The point sets in (3.1) and (3.2) form a rotatable arrangement of order three for the values of constants given in (3.4). For the set to form a rotatable design of order three, the non-singularity conditions given in (2.5) must be satisfied.

From the solution set, we have,

$\lambda_2=\frac{\sum_{u=1}^{56} N_i\rho_1^2}{NK}$= 0.87671871$a^2$

$\lambda_4=\frac{\sum_{u=1}^{56} N_i\rho_1^4}{NK(k+2)}$ =0.574698401a^4

$\lambda_6=\frac{\sum_{u=1}^{56} N_i\rho_1^6}{NK(k+2)(k+4)}$ =0.285711068a^6

For k=4 and N=56.

These satisfy condition (2.5) which implies that the point sets constitute a rotatable design of order three in four dimensions with fifty six points.

4.0 A PRACTICAL HYPOTHETICAL EXAMPLE

A design was set up to investigate the effects of four fertilizer ingredients on the yield of hybrid maize in Trans-Nzoia to illustrate the use of the sequential third order rotatable design of fifty six points in four dimensions under field conditions.

The fertilizer ingredients and actual amount applied were phosphoric acid(p_2o_5)x_1, $\psi_{1.}$=30 miligram/hole; Nitrogen(N)$x_2\psi_{2.}$=25 miligram/hole ;potash (k_2o)$x_3\psi_{3.}$=40 miligram/hole and sodium (Na) $x_4\psi_{4.}$=15mligram/hole. The response of interest is the average yield in mg per hole of hybrid maize.

As a result ofsoil mapping investigations which indicate deficiencies of these mineral elements in the Trans-Nzoia loam soils. We shall point out that the original letterparametrs represent the variation in quantity application of a factor due to soil fertility gradient culminating in several radii manifestations of rotatabilitycriterion. According to Box(3) and Box and Wilson (4) it can be reverted that the natural levels of these mineral elements denoted ψ_{iu} where Bose and Draper(1) scaling down condition fixes a particular design when $\lambda_{2=1}$ whence,

$$x_{iu}=\frac{\psi_{iu}-\psi_{i.}}{s_i} \quad (4.1)$$

$$\psi_{i.}=\frac{\sum_{u=1}^{N}\psi_{iu}}{N} \quad (4.2)$$

$$s_i=\left[\frac{\sum_{u=1}^{N}(\psi_{iu}-\psi_{iu.})^2}{N}\right]^{0.5} \quad (4.3)$$

$$\psi_{iu}=x_{iu}s_i+\psi_{iu.} \quad (4.4)$$

For $\sum_{u=1}^{N} x_{iu}^2$= N and $\sum_{u=1}^{N} x_{iu}$=0

An example illustrating the conversion of coded levels to natural levels

Let the natural level $x_{1u} = 0.5$

And the amount of potash applied per hole($\psi_{3.}$)= 40milligram/hole

Further let S=0.3,

Then using, $\psi_{iu}=x_{iu}s_i+\psi_{iu.}$,

ψ_{iu}=0.3*0.5+40,

=40.15milligram/hole

The design matrix can be constituted but the evaluation of the inverse will be a major computational project to estimate the coefficients of the third order rotatable design model which give the optimum response yield. This requires a separate discourse but the actual responses or yields can be obtained if a field experiment is conducted as explained. Let the scale parameter,

s_i, assume s_1=0.5, $s_2 = 0.3$, s_3=1 and $s_4 = 0.6$ to estimate the co-efficients;

β_0,β_1,β_2, β_3, β_4,β_{11},β_{22},β_{33},$\beta_{44,}$$\beta_{111}$,$\beta_{222}$,$\beta_{333}$,, $\beta_{444}$$\beta_{12}$,$\beta_{13}$,$\beta_{14,}$$\beta_{23}$, β_{24}, β_{122},β_{133}, $\beta_{144}$$\beta_{211}$$\beta_{233}$,$\beta_{244}$$\beta_{311}$,$\beta_{322}$,$\beta_{344}$, β_{123}, β_{1122}, β_{1133}, β_{1144}, β_{2233}, β_{2244}, $\beta_{3344}$and β_{1234} in the expected third order rotatable design model in four dimensions, we require field observations of the yield y_u(u=1,2,…,56)

The complete third order model to be fitted to yield values is,

$$y_u=\beta_o+\sum_{i=1}^{56}\beta_i x_i+\sum_{i=1}^{56}\beta_{ii}x_i^2+\sum_{i=1}^{56}\beta_{iii}x_i^3+\sum_{i=1}^{56}\sum_{j=1}^{56}\beta_{ij}x_ix_j+\sum_{i}^{56}\sum_{j}^{56}\sum_{l}^{56}\beta_{ijl}x_ix_jx_l$$
$$+\sum_{i=1}^{56}\sum_{j=1}^{56}\beta_{iijj}x_{ii}^2x_{jj}^2+e \quad (4.5)$$

For the fifty six points third order rotatable design in four dimensions, we have the following coded and natural levels respectively as treatments in the table below:

Coded levels				Natural levels			
x_{1u}	x_{2u}	x_{3u}	x_{4u}	ψ_{1u}	ψ_{2u}	ψ_{3u}	ψ_{4u}
1	1	1	1	30.1	25.3	41.0	15.6
1	-1	1	1	30.1	24.7	41.0	15.6
1	1	1	-1	30.1	25.3	41.0	14.4
-1	1	-1	1	29.9	25.3	39	15.6
1	-1	-1	1	30.1	24.7	39	15.6
1	1	-1	-1	30.1	25.3	39	14.4
-1	-1	1	-1	29.9	24.7	41.0	14.4
-1	1	1	1	29.9	25.3	41.0	15.6
1	1	-1	1	30.1	25.3	39.0	15.6
-1	-1	1	1	29.9	24.7	41.0	15.6
-1	1	1	-1	29.9	25.3	41.0	14.4
1	-1	1	-1	30.1	24.7	41.0	14.4
-1	-1	-1	1	29.9	24.7	39.0	15.6
-1	1	-1	-1	29.9	25.3	39.0	14.4
1	-1	-1	-1	30.1	24.7	39.0	14.42
-1	-1	-1	-1	29.9	24.7	39.0	14.42
1.99913	0	0	0	30.1999	25.0	40.0	15.0
-1.99913	0	0	0	29.8001	25.0	40.0	15.0
0	1.99913	0	0	30.0	25.5997	40.0	15.0
0	-1.99913	0	0	30.0	24.4003	40.0	15.0
0	0	1.99913	0	30.0	25.0	41.9991	15.0
0	0	-1.99913	0	30.0	25.0	38.00087	15.0
0	0	0	1.99913	30.0	25.0	40.0	16.1995
0	0	0	-1.99913	30.0	25.0	40.0	13.8005

0.74171	0	0	0	30.07417	25.0	40.0	15.0
-0.74171	0	0	0	29.9258	25.0	40.0	15.0
0	0.74171	0	0	30.0	25.0025	40.0	15.0
0	-0.74171	0	0	30.0	24.7775	40.0	15.0
0	0	0.74171	0	30.0	25.0	40.74171	15.0
0	0	-0.74171	0	30.0	25.0	39.2583	15.0
0	0	0	0.74171	30.0	25.0	40.0	15.445
0	0	0	-0.74171	30.0	25.0	40.0	14.555
1.14142	1.14142	0	0	30.01141	25.3424	40.0	15.0
1.14142	-1.14142	0	0	30.01141	24.6576	40.0	15.0
1.14142	0	1.14142	0	30.01141	25.0	41.1424	15.0
1.14142	0	-1.14142	0	30.01141	25.0	38.8586	15.0
1.14142	0	0	1.14142	30.01141	25.0	40.0	15.6849
1.14142	0	0	-1.14142	30.01141	25.0	40.0	14.3151
0	1.14142	1.14142	0	30.0	25.34242	41.1414	15.0
0	1.14142	-1.14142	0	30.0	25.34242	38.8586	15.0
-1.14142	1.14142	0	0	29.8859	25.34242	40.0	15.0
-1.14142	-1.14142	0	0	29.8859	24.6576	40.0	15.0
-1.14142	0	1.14142	0	29.8859	25.0	41.1414	15.0
-1.14142	0	-1.14142	0	29.8859	25.0	38.8586	15.0
-1.14142	0	0	1.14142	29.8859	25.0	40.0	15.6849
-1.14142	0	0	-1.14142	29.8859	25.0	40.0	14.3151
0	-1.14142	1.14142	0	30.0	24.6576	41.1414	15.0
0	-1.14142	-1.14142	0	30.0	24.6576	38.8586	15.0
0	1.14142	0	1.14142	30.0	25.34242	40.0	15.6849
0	1.14142	0	-1.4142	30.0	25.34242	40.0	14.3151

0	0	1.14142	1.14142	30.0	25.0	41.1414	15.6849
0	0	1.14142	-1.14142	30.0	25.0	41.1414	14.3151
0	-1.14142	0	1.14142	30.0	24.6575	40.0	15.6849
0	-1.14142	0	-1.14142	30.0	24.6575	40.0	14.3151
0	0	-1.14142	1.14142	30.0	25.0	38.8586	15.6849
0	0	-1.14142	-1.14142	30.0	25.0	38.8586	14.3151

APPLICATIONS

Experiments of this kind are widely applied in the field of agriculture,human medicine, veterinary science and industry to provide useful information. The design under consideration permits a response surface to be fitted easily and provides spherical information contours besides optimum combinations of treatments in agriculture, medicine and industry which results in economic use of scarce resources in relevant production processes.However we note that; practical applications of this methods is not automatic, judgement is required, if an experimenter applies insufficient intellect to his results, he is likely to suffer as in any other method of experiment and it is always possible especially in the new field of experiment to make an unfortunate selection of units.The expected third order rotatable design model in four factors will be available when an experimenter would carry out an experiment where the responses would facilitate the estimation of the linear, quadratic and cubic coefficients

CONCLUSIONS

The study presented an illustration on how to obtain the mathematical parameters of the coded values and its corresponding natural levels of a third order rotatable design in four dimensions by utilizing response surface methodology to approximate the functional relationship between the performance characteristics and design variables. After experimentation, the resulting response is used to construct response surface approximation model using least squires' regression analysis.

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