On gso-Closed Sets in Topological Spaces

M.I. Irshad and P. Elango*

Department of Mathematics, Faculty of Science, Eastern University, Sri Lanka.

Type of Article

Received: XX December 20XX
Accepted: XX December 20XX
Online Ready: XX December 20XX

Abstract

In this paper, a new kind of closed sets called generalized semi open-closed sets (briefly gso-closed sets) are introduced in topological spaces. A subset A of a topological space X is called a gso-closed set if A is both a g-closed set and a semi-open set in X. The properties of the gso-closed sets are investigated and they are compared with the existing relevant generalized closed sets. The generalized semi-open continuous function between topological spaces are also defined and their properties are investigated.

Keywords: Generalized closed sets; continuous functions; generalized continuous functions

2010 Mathematics Subject Classification: 54A05;54C10

1 Introduction

The concept of closed sets in topological spaces have many important properties such as closed subset of a compact space is compact; closed subset of a normal space is normal; closed subset of a complete uniform space is complete, etc. The generalized closed sets, simply g-closed sets, which are weaker form of the closed sets were introduced and studied by N.Levine [1] in a topological space in 1970. He defined a set A to be a g-closed set in a topological space if its closure is contained in every open super set of A. Since then many mathematicians introduce and investigate different kind of generalized closed sets in topological spaces. Some recent works on generalized closed sets can be found in [2], [3], [4], [5] and [6]. Continuous functions are one of the core concepts of topological spaces defined in terms of open sets as well as closed sets. With the introduction of different kind of generalized closed sets, there are many different kind of continuous functions defined in the literature. In this paper, we introduce a new kind of closed set called the generalized semi open-closed set (briefly gso-closed set) in topological spaces. A subset A of a topological space (X, τ) is called a gso-closed set if A is both a g-closed set and a semi-open set in the (X, τ). We study its properties and compare this with the existing relevant generalized closed sets. In the second part of this paper, we define the generalized semi-open continuous functions between topological spaces as an application of these gso-closed sets and investigate their properties.

*Corresponding author: E-mail: elangop@esn.ac.lk

1.1 Generalized Closed Sets

Throughout this paper, we represent X, Y and Z as the topological spaces (X, τ), (Y, σ) and (Z, ν) respectively, on which no separation axioms are assumed unless otherwise stated. For a subset of A of X, $cl(A)$ denotes the closure of A and $int(A)$ denotes the interior of A, respectively.

In a topological space, we recall the definitions for some of the relevant open and closed sets.

Definition 1.1. A subset A of a topological space X is called a

(i) semi-open [7] if $A \subseteq cl(int(A))$.

(ii) regular open [8] if $A = int(cl(A))$.

(iii) regular closed [8] if $A = cl(int(A))$.

(iv) pre-closed [9] if $cl(int(A)) \subseteq A$.

A semi-closed set is the complement of a semi-open set.

We now recall the definitions of some of the relevant generalized closed sets in a topological space.

Definition 1.2. A subset A of a topological space X is called a

(i) generalized closed set (briefly g-closed set) [1] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X. The complement of a g-closed set is called a g-open set in X.

(ii) generalized semi closed (briefly gs-closed) [10] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X, where $scl(A)$ is the intersection of all semi-closed sets of X containing A.

(iii) weakly generalized closed (briefly wg-closed) [11] if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

(iv) regular weakly generalized closed (briefly rwg-closed) [11] if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.

1.2 Generalized Continuous Functions

Continuity of functions between topological spaces is one of the core concepts of topological spaces which is defined in terms of open sets as well as closed sets. Here we recall some kind of generalized continuous function defined by the relevant generalized closed sets.

Definition 1.3. Let X and Y be two topological spaces. A function $f : X \to Y$ is called a

(i) g-continuous [12] if $f^{-1}(V)$ is a g-closed set of X for every closed set V of Y.

(ii) gs-continuous [13] if $f^{-1}(V)$ is a gs-closed set of X for every closed set V of Y.

(iii) wg-continuous [11] if $f^{-1}(V)$ is a wg-closed set of X for every closed set V of Y.

(iv) rwg-continuous [12] if $f^{-1}(V)$ is a rwg-closed set of X for every closed set V of Y.

2 Results and Discussion

Definition 2.1. A subset A of a topological space X is called a *generalized semi open-closed set* (briefly gso-closed set) if A is both a g-closed set and a semi-open set in X.

It is clear that every gso-closed set is a g-closed set in a topological space.

Example 2.1. *Consider a linearly ordered topological space, defined in Faber, [14]. A basic example is the real numbers with the standard order. Or more generally, a preference-ordered topological space, defined in Alcantud, [15]. Then the sets with the form*

$$\{x \in X : a \leq x \leq b\}$$

are gso-closed sets when $a < b$.

Theorem 2.2. *The union of two gso-closed sets in a topological space X is also a gso-closed set in X.*

Proof. Let A and B be two gso-closed sets in a topological space X. Then A and B are both g-closed sets and semi-open sets in X. Therefore, $A \cup B$ is both g-closed set and semi-open set and hence it is a gso-closed set in X. □

The intersection of two gso-closed sets in a topological space X is generally not a gso-closed set in X. We see this by the following example. Let $X = \{a, b, c\}$ be with topology $\tau = \{X, \emptyset, \{a\}\}$. Then, $A = \{a, b\}$ and $B = \{a, c\}$ are gso-closed sets. But their intersection, $A \cap B = \{a\}$ is not a gso-closed set in X.

Theorem 2.3. *If A and B are both closed and open sets in a topological space X, then $A \cap B$ is a gso-closed set in X.*

Proof. Suppose that A and B are both open and closed sets in a topological space X. Then, $A \cap B$ is a closed set and so is a g-closed set. Also, $A \cap B$ is an open set and so is a semi-open set in X. Hence $A \cap B$ is a gso-closed set in X. □

Theorem 2.4. *If A is a gso-closed set in a topological space X and B is an open set in X such that $B \subseteq A$, then B is a gso-closed set in X.*

Proof. Suppose that A is a gso-closed set in a topological space X. Then, $cl(A) \subseteq U$ whenever $A \subseteq U$, U is open in X and $A \subseteq cl(int(A))$. We now suppose that $B \subseteq A$, B is open in X and $A \subseteq U$, U is open in X. Then, we get $cl(B) \subseteq U$ whenever $B \subseteq U$, U is open in X. Thus, B is a g-closed set in X. As $cl(B) \subseteq cl(int(B))$, we also get $B \subseteq cl(int(B))$. Thus, B is a semi-open set in X. Hence B is a gso-closed set in X. □

Theorem 2.5. *If A is both closed and open set in a topological space X, then A is a gso-closed set in X.*

Proof. Suppose that A is both open and closed set in a topological space X. Then, clearly A is a g-closed set and a semi-open set. Hence A is a gso-closed set in X. □

Theorem 2.6. *If a gso-closed set A is a pre-closed set in a topological space X, then A is a regular closed set in X.*

Proof. Suppose that the gso-closed set A is a pre-closed set in a topological space X. Then, we get $A \subseteq cl(int(A))$ and $cl(int(A)) \subseteq A$. Thus $A = cl(int(A))$. Hence A is a regular closed set in X. □

Theorem 2.7. *Every gso-closed set in a topological space X is a wg-closed set in X.*

Proof. Let A be a gso-closed set in a topological space X. Then, $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X and $A \subseteq cl(int(A))$. Let $A \subseteq U$, U is open in X. Then, $cl(int(A)) \subseteq cl(A) \subseteq U$. Hence A is a wg-closed set in X. □

The converse of the above theorem need not be true as seen in the following example. Let $X = \{a, b, c, d\}$ be with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then, the set $A = \{d\}$ is a wg-closed set in X but not a gso-closed set in X.

Theorem 2.8. *Every gso-closed set in a topological space X is a gs-closed set in X.*

Proof. Let A be a gso-closed set in a topological space X. Then, $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X and $A \subseteq cl(int(A))$. Let $A \subseteq U$, U is open in X. Since $scl(A) \subseteq cl(A)$, we get $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open set in X. Hence A is a gs-closed set in X. □

The converse of the above theorem need not be true as seen in the following example. Let $X = \{a, b, c, d\}$ be with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then, the set $A = \{a\}$ is a gs-closed set in X but not a gso-closed set in X.

Theorem 2.9. *Every gso-closed set in a topological space X is a rwg-closed set in X.*

Proof. Let A be a gso-closed set in a topological space X. Then, $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X and $A \subseteq cl(int(A))$. Let $A \subseteq U$, U is open in X. Then, $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X. Hence A is a rwg-closed set in X. □

Definition 2.2. Let X and Y be two topological spaces. A function $f : X \to Y$ is called a *generalized semi open-continuous function* (briefly gso-continuous function) if $f^{-1}(V)$ is a gso-closed set of X for every closed set V of Y.

Example 2.10. *Let $X = \{a, b\}$ be with topology $\tau = \{X, \emptyset, \{a\}\}$ and let $Y = \{a, b, c\}$ be with topology $\sigma = \{Y, \emptyset, \{a, b\}\}$. Define $f : X \to Y$ by $f(a) = a$ and $f(b) = c$. Then, f is a gso-continuous function.*

In general, the composition of two gso-continuous functions need not be a gso-continuous function. This can be seen in the following example. Let $X = \{a, b\}$ be with topology $\tau = \{X, \emptyset, \{a\}\}$ and $Y = \{a, b, c\}$ be with topology $\sigma = \{Y, \emptyset, \{a, b\}\}$ and $Z = \{a, b, c\}$ be with topology $\eta = \{Z, \emptyset, \{a, c\}\}$. Define $f : X \to Y$ by $f(a) = a$ and $f(b) = c$, and $g : Y \to Z$ by $g(a) = a$ and $g(c) = b$. Then, f and g are gso-continuous functions but the composition $g \circ f$ is not a gso-continuous function.

Theorem 2.11. *Let X and Y be two topological spaces. Let $f : X \to Y$ be a gso-continuous function and $g : Y \to Z$ be a continuous function. Then, $g \circ f$ is a gso-continuous function.*

Proof. Suppose that A be a closed set in Z. Then, $g^{-1}(A)$ is a closed set in Y and so $f^{-1}(g^{-1}(A))$ is a gso-closed set in X. That is, $(g \circ f)^{-1}(A)$ is a gso-closed set in X. Hence $g \circ f$ is a gso-continuous function. □

Theorem 2.12. *Let X and Y be two topological spaces. Let $f : X \to X$ be an identity function and $g : X \to Y$ be a gso-continuous function. Then, $g \circ f$ is a gso-continuous function.*

Proof. Suppose that A be a closed set in Y. Then, $g^{-1}(A)$ is a gso-closed set in X and then, $f^{-1}(g^{-1}(A))$ = $g^{-1}(A)$ is a gso-closed set in X. That is, $(g \circ f)^{-1}(A)$ is a gso-closed set in X. Hence $g \circ f$ is a gso-continuous function. □

Theorem 2.13. *Every gso-continuous function is a g-continuous function.*

Proof. Suppose that $f : X \to Y$ is a gso-continuous function and let A be a closed set in Y. Then, $f^{-1}(A)$ is a gso-closed set in X. Then by the definition of gso-closed set, $f^{-1}(A)$ is a g-closed set in X. Hence f is a g-continuous function. □

Theorem 2.14. *Every gso-continuous function is a wg-continuous function.*

Proof. Suppose that $f : X \to Y$ is a gso-continuous function and let A be a closed set in Y. Then, $f^{-1}(A)$ is a gso-closed set in X. Then by Theorem 2.7, $f^{-1}(A)$ is a wg-closed set in X. Hence f is a wg-continuous function. □

Theorem 2.15. *Every gso-continuous function is a gs-continuous function.*

Proof. Suppose that $f : X \to Y$ is a gso-continuous function and let A be a closed set in Y. Then, $f^{-1}(A)$ is a gso-closed set in X. Then by Theorem 2.8, $f^{-1}(A)$ is a gs-closed set in X. Hence f is a gs-continuous function. □

Theorem 2.16. *Every gso-continuous function is a rwg-continuous function.*

Proof. Suppose that $f : X \to Y$ is a gso-continuous function and let A be a closed set in Y. Then, $f^{-1}(A)$ is a gso-closed set in X. Then by Theorem 2.9, $f^{-1}(A)$ is a rwg-closed set in X. Hence f is a rwg-continuous function. □

3 CONCLUSIONS

In this paper, we defined a new kind of generalized closed set, called generalized semi open-closed sets (briefly gso-closed sets), in a topological space X. A subset A of X is called a gso-closed set if A is both a g-closed set and a semi-open set in X. This gso-closed set became some of the relevant generalized closed sets but not conversely. The generalized semi-open continuous function (briefly gso-continuous function) between two topological spaces are also defined as an application of the gso-closed set. From these gso-continuous functions some of the relevant generalized continuous functions were derived.

Acknowledgment

The authors acknowledged the reviewers for their valuable comments.

Competing Interests

Authors have declared that no competing interests exist.

Authors' Contributions

Author PE designed the study, wrote the protocol. Author MII managed the literature searches, did the theoretical work and wrote the first draft of the manuscript. All authors read and approved the final manuscript.

References

[1] Levine N. Generalized closed sets in topology. Rend. Circ. Mat. Palermo. 1970;19:89-96.

[2] Bishwambhar Roy, Rit Sen. On a class of sets between μ-closed sets and μg-closed sets. Journal of Taibah University for Science. 2017;11:268-273.

[3] Ramadhan A, Mohammed, Tahir H. Ismail, Allam AA. A comments on generalized $\alpha\beta$-closed sets. Journal of the Egyptian Mathematical Society. 2017;25:57-58.

[4] Ahmed I El-Maghrabi. More on γ-generalized closed sets in topology. Journal of Taibah University for Science. 2013;7:114-119.

[5] Tyagi BK, Harsh VS Chauhan. On generalized closed sets in generalized topological spaces. CUBO A Mathematical Journal. 2016;18(1):27-45.

[6] Vithyasangaran K, Elango P. On $\tau_1\tau_2$-$\bar{g}$-Closed Sets in Bitopological Spaces. Asian Research Journal of Mathematics. 2018;11(2):1-8.

[7] Levine N. Semi-open sets and semi-continuity in topological spaces. Amer. Math. Monthly. 1963;70:36-41.

[8] Stone AH. Absolutely FG Spaces. Proc. Amer. Math. Soc. 1980;80:515-520.

[9] Mashhour AS, Abd. El-Monsef ME, El-Deeb SN. On pre continuous mappings and weak pre-continuous mappings. Proc Math, Phys. Soc. Egypt. 1982;53:47-53.

[10] Arya SP, Nour TM. Characterizations of s-normal spaces. Indian J. Pure Appl. Math. 1990;21:717-719.

[11] Nagaveni N. Studies on Generalizations of Homeomorphisms in Topological Spaces. Ph.D. Thesis, Bharathiar University, Coimbatore, 1999.

[12] Balachandran K, Sundram P, Maki H. On generalized continuous maps in topological spaces. Mem. Fac. Kochi. Univ. Ser.A.(Math). 1991; 12:5-13.

[13] Devi R, Balachandran K, Maki H. Semi generalized homeomorphisms and generalized semi homeomorphisms in topological spaces. Indian J.Pure.Appl.Math. 1995;26(3):271-284.

[14] Faber MJ. Metrizability in generalized ordered spaces. Mathematisch Centrum, Amsterdam; 1974.

[15] Alcantud JCR. Topological properties of spaces ordered by preferences. International Journal of Mathematics and Mathematical Sciences. 1999;22(1):17-27.
