MODIFIED LAPLACE DISTRIBUTION, ITS STATISTICAL PROPERTIES AND APPLICATIONS

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ABSTRACT

Increasing the parameter of a distribution helps to capture the skewness and peakedness characteristic in the data sets. This allows a more realistic modeling of data arising from different real life situations. In this paper, we modified Laplace distribution using the exponentiation method. The study proved that the modified Laplace distribution (MLD) is a probability density function. Some of the basic statistical properties of the modified Laplace distribution are obtained. We applied the proposed modified Laplace distribution on two life datasets and simulated data. Parameters of the distributions were estimated using method of maximum likelihood estimation. The study compared the modified Laplace distribution with Laplace distribution and Generalized error distribution using Schwartz Criteria (SC) measure of fitness. The results obtained revealed that the modified Laplace distribution has a better fit than the Laplace and Generalized error distributions and can be used for more realistic modeling of data arising from different real life situations. The simulation results obtained shows that as the sample size increases, the Biasedness and Root Mean Square Error (RMSE) of the proposed modified Laplace distribution reduces.

Keywords: Laplace distribution, probability density function, cumulative density function, exponentiated distributions.

1. Introduction

One important family of distributions is the exponentiated family. Many exponentiated distributions have been introduced in several literatures with the goal to provide a better fitting distributions on data sets than the conventional parent distributions.

Srivastava and Mudholkar (1993) proposed the exponentiated Weibull distribution as an extension of the Weibull family. The statistical properties of the distribution were derived. The parameters of the distribution were obtained using maximum likelihood estimation, the distribution was applied on bathtub failure datasets and the result of the study revealed that the proposed distribution has better fitness than the Weibull distribution when compared using Alkaike Information Criteria.

Nadarajah and Kotz (2006) proposed the exponentiated gamma and exponentiated Gumbel distributions. The statistical properties of the proposed distributions were studied and were observed to have similar statistical properties to their respective parent distributions. They applied the proposed distributions on real life data set and observed that they were better than the baseline distributions.

Kozubowski and Podgorski (2001) study the Asymmetric Laplace distribution (AL) which arises as the limits of sums of independent and identically distributed random variables with finite second moment and the number of terms are geometrically distributed and independent of each other.

The AL distribution studied by Kozubowski and Podgorski has fat tail and high peaks just similar to the proposed Modified Laplace distribution. However, AL distribution has just two parameters while the proposed modified Laplace distribution has three parameters. It is expected that on applications to real life situations, the modified three parameters Laplace distribution will perform better because of the additional shape parameter which helps to capture the skewness and peakness inherent in the data groups and allow a more realistic modeling of data arising from different real life situations.

El-Bassiouny, Abdo and Shahen (2015) proposed Exponentiated Lomax (ELOMAX) distribution. The moment, moment generating function, hazard rate function, characteristic function and order statistics of the proposed ELOMAX distribution were derived.

Mahmoudvand *et al* (2015) introduced a modified symmetric version of the classical Laplace distribution with comprehensive theoretical description. Some of the basic statistical properties of the modified classical Laplace distribution including the maximum likelihood estimators of the parameters and the properties were investigated via simulation and the real life applications was demonstrated on three real-world datasets. The results suggest that further improvement to classical Laplace distribution fitting is possible and the new model provides an attractive alternative to the classical Laplace distribution.

Ownuk (2015) proposed the Beta Exponentiated Gumbel (BEG) distribution. The distribution was applied on real life data. The parameters of the proposed distribution were estimated using method of moment estimation. The proposed distribution was compared with Beta and Gumbel distributions using the Alkaike Information Criteria and the result revealed that BEG gave a significantly better fitness than the Beta and Gumbel distributions and recommend that the proposed BEG distribution be used to model data arising from different life situations instead of Beta and Gumbel distributions.

Dina, Abdel and Rania (2016) proposed transmuted Laplace distribution and applied it on real life data sets. The result of the study revealed that the proposed distribution provides a better fit than the symmetric Laplace distribution when compared.

Shittu, Chukwu and Adepoju (2016) proposed an exponentiated F-test for one way analysis of variance (ANOVA) in the presence of outliers. The statistical properties of the proposed distribution were derived.

Okorie, Akpanta and Ohakwe (2016) proposed an exponentiated Gumbel type-2 distribution and the statistical properties were derived. Natalie, Edwin and Gauss (2017) proposed Exponentiated-Log-logistic Geometric Distribution (ELLGD) and the statistical properties were derived.

Elgarhy and Shawki (2017) proposed a three parameter exponentiated Sushila (ES) distribution, applied it on life datasets and all these distributions were found to be more flexible than their respective parent distributions.

Natalie, Edwin and Gauss (2017) proposed Exponentiated-Log-logistic Geometric Distribution (ELLGD) and they observed that the proposed distribution has a better fitness when compared with the exponentiated Log-logistic distribution (ELLD) and Log-logistic distribution (LLD).

2.0 Exponentiation method and the Laplace distribution

The cumulative density function (cdf) of an exponentiated random variable X proposed by Gupter *et al* (1998) is given by:

$$G(x) = [F(x)]^{\alpha},\ \alpha > 0, \qquad (1)$$

where $F(x)$ is the cumulative density function of the baseline distribution. Differentiating (1) we obtain the probability density function (pdf) of the exponentiated random variable X proposed by Gupter *et al* (1998) as

$$z(x) = \alpha[F(x)]^{\alpha-1} f(x), \qquad (2)$$

where $f(x)$ is the probability density function of the baseline distribution. The baseline distribution here is the two parameter Laplace distribution.

The probability density function $f(x)$ and the cumulative density function $F(x)$ of Laplace distribution proposed by Pierre-Simon Laplace (1819) are given by:

$$f(x) = \begin{cases} \frac{1}{2b} e^{\left(\frac{x-\mu}{b}\right)}, & if\ x \leq \mu, \\ \frac{1}{2b} e^{-\left(\frac{x-\mu}{b}\right)}, & if\ x > \mu, \end{cases} \qquad (3)$$

$$F(x) = \begin{cases} \frac{1}{2} e^{\left(\frac{x-\mu}{b}\right)} & if\ x \leq \mu \\ 1 - \frac{1}{2} e^{-\left(\frac{x-\mu}{b}\right)} & if\ x > \mu \end{cases} \qquad (4)$$

where, $b > 0, -\infty \leq x \leq \infty$.

b is the scale parameter, μ the location parameter.

2.1 The new distribution

Using the exponentiated link function proposed by Gupter *et al* (1998) given in equation (1), we obtain the cumulative density function (cdf) of the new modified Laplace distribution as:

$$Z(x) = \begin{cases} \left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha}, \; if \; x \leq \mu, \\ \left(1 - \frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha}, if \; x > \mu, \end{cases} \tag{5}$$

where, $\alpha, b > 0, -\infty \leq x \leq \infty$.

Similarly, using equation (2), we obtain the probability density function (pdf) of the new modified Laplace distribution as:

$$z(x) = \begin{cases} \alpha\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b}e^{\frac{(x-\mu)}{b}}, if \; x \leq \mu, \\ \alpha\left(1 - \frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b}e^{-\frac{(x-\mu)}{b}} \;, if \; x > \mu, \end{cases} \tag{6}$$

where,$\alpha, b > 0, -\infty \leq x \leq \infty$.

α is the shape parameter, b is the scale parameter and μ is the location parameter.

2.2 Statistical properties of the modified Laplace distribution

A random variable X is said to have a modified Laplace distribution $X \sim \text{MLD}\ (\alpha, \mu, b)$ if the probability density function is given by

$$z(x) = \begin{cases} \alpha\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b}e^{\frac{(x-\mu)}{b}}, if \; x \leq \mu, \\ \alpha\left(1 - \frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b}e^{-\frac{(x-\mu)}{b}} \;, if \; x > \mu, \end{cases}$$

where, $\alpha, b > 0, -\infty \leq x \leq \infty$.

We wish to proof that the probability density function of the random variable X given above is a density function.

Let us prove that

$$\int_{-\infty}^{\infty} z(x)dx = 1. \qquad \text{(required)**}$$

Let

$$\int_{-\infty}^{\infty} z(x)dx = \int_{-\infty}^{\mu} g_1(x)dx + \int_{\mu}^{\infty} g_2(x)dx, \tag{7}$$

where

$$g_1(x) = \alpha \left(\frac{1}{2} e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b} e^{\frac{(x-\mu)}{b}}, if\ x \leq \mu,$$

$$g_2(x) = \alpha \left(1 - \frac{1}{2} e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b} e^{-\frac{(x-\mu)}{b}}\ , if\ x > \mu.$$

We have

$$\int_{-\infty}^{\infty} z(x)dx = \int_{-\infty}^{\mu} \alpha \left(\frac{1}{2} e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b} e^{\frac{(x-\mu)}{b}} dx + \int_{\mu}^{\infty} \alpha \left(1 - \frac{1}{2} e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b} e^{-\frac{(x-\mu)}{b}} dx.$$

Integrate $g_1(x)$ first, we have

$$\int_{-\infty}^{\mu} g_1(x)dx = \frac{1}{2^{\alpha}}. \tag{8}$$

Now, integrate $g_2(x)$, we have

$$\int_{\mu}^{\infty} g_2(x)dx = 1 - \frac{1}{2^{\alpha}}. \tag{9}$$

Sum equation (8) and equation (9). The result shows that the modified Laplace distribution $z(x)$ is a density function since the integral of the pdf is equal to 1.

If X~MLD (α, μ, b), $lim Z(x) = 0\ as\ x \to -\infty$ and $lim Z(x) = 1\ as\ x \to \infty$.

Proof

Recall that $Z(x) = \begin{cases} \left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha}, \; if \; x \leq \mu, \\ \left(1 - \frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha}, if \; x > \mu, \end{cases}$

For $limZ(x) = 0 \; as \; x \to -\infty$.

$$limZ(x) = \left(\frac{1}{2}e^{\frac{(-\infty-\mu)}{b}}\right)^{\alpha} = 0. \tag{10}$$

For $limZ(x) = 1 \; as \; x \to \infty$.

$$limZ(x) = \left(1 - \frac{1}{2}e^{-\frac{(\infty-\mu)}{b}}\right)^{\alpha} = 1 - 0 = 1.$$

If X~MLD (α, μ, b), $limz(x) = 0 \; as \; x \to -\infty$ and $limz(x) = 0 \; as \; x \to \infty$.

Proof

Recall that $z(x) = \begin{cases} \alpha\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b}e^{\frac{(x-\mu)}{b}}, if \; x \leq \mu, \\ \alpha\left(1 - \frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b}e^{-\frac{(x-\mu)}{b}} \;, if \; x > \mu, \end{cases}$

For $limz(x) = 0 \; as \; x \to -\infty$.

$$limz(x) = \alpha\left(\frac{1}{2}e^{\frac{(-\infty-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b}e^{\frac{(-\infty-\mu)}{b}} = 0. \tag{11}$$

For $limz(x) = 0 \; as \; x \to \infty$.

$$limz(x) = \alpha\left(1 - \frac{1}{2}e^{-\frac{(\infty-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b}e^{-\frac{(\infty-\mu)}{b}} = 0. \tag{12}$$

Hence $limz(x) = 0 \; as \; x \to -\infty$ and $limz(x) = 0 \; as \; x \to \infty$.

This implies that the modified Laplace distribution (MLD) has a unique mode.

If $X \sim$MLD (α, μ, b), the moment generating function (mgf) of the modified Laplace distribution can be derive as:

By definition of moment generating function,

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} z(x) e^{tx} dx. \tag{13}$$

where $z(x)$ is the pdf of the MLD defined as

$$z(x) = \begin{cases} \alpha \left(\frac{1}{2} e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b} e^{\frac{(x-\mu)}{b}}, if\ x \leq \mu, \\ \alpha \left(1 - \frac{1}{2} e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b} e^{-\frac{(x-\mu)}{b}} \ , if\ x > \mu, \end{cases}$$

Let

$$g_1(x) = \alpha \left(\frac{1}{2} e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b} e^{\frac{(x-\mu)}{b}} e^{tx}, if\ x \leq \mu,$$

$$g_2(x) = \alpha \left(1 - \frac{1}{2} e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b} e^{-\frac{(x-\mu)}{b}} e^{tx} \ , if\ x > \mu,$$

We have

$$\int_{-\infty}^{\mu} g_1(x) e^{tx} dx = \int_{-\infty}^{\mu} \alpha \left(\frac{1}{2} e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b} e^{\frac{(x-\mu)}{b}} e^{tx} dx = \frac{\alpha e^{\mu t}}{2^{\alpha}(\alpha+bt)}, \tag{14}$$

$$\int_{\mu}^{\infty} g_2(x) e^{tx} dx = \int_{\mu}^{\infty} \alpha \left(1 - \frac{1}{2} e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1} \frac{1}{2b} e^{-\frac{(x-\mu)}{b}} e^{tx} dx.$$

Using series expansion

$$\left(1 - \frac{1}{2} e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1} = 1 + \sum_{r=1}^{\infty} \prod_{i=1}^{r} \frac{(\alpha-i)(-1)^r e^{-r\frac{(x-\mu)}{b}}}{r! 2^r}.$$

Let

$$\emptyset_r = \prod_{i=1}^{r} \frac{(\alpha-i)(-1)^r}{r! 2^r}.$$

Then, we obtain

$$\int_{\mu}^{\infty} g_2(x)e^{tx}dx = \frac{\alpha e^{\mu t}}{2(1-bt)} + \frac{\alpha e^{\mu t}}{2}\sum_{r=1}^{\infty} \emptyset_r \frac{1}{(r+1-bt)}. \quad (15)$$

sum equation (14) and equation (15) gives the mgf of the modified Laplace distribution as

$$M_x(t) = \frac{\alpha e^{\mu t}}{2^{\alpha}(\alpha+bt)} + \frac{\alpha e^{\mu t}}{2(1-bt)} + \frac{\alpha e^{\mu t}}{2}\sum_{r=1}^{\infty} \emptyset_r \frac{1}{(r+1-bt)}.$$

The hazard function of a random variable X is defined as

$$F_h = \frac{z(x)}{1-Z(x)}. \quad (16)$$

Therefore, using equation (16) the hazard function of the modified Laplace distribution is given as:

$$F_h = \begin{cases} \dfrac{\alpha\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{\frac{(x-\mu)}{b}}}{1-\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha}}, & if\ x \le \mu, \\ \dfrac{\alpha\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{-\frac{(x-\mu)}{b}}}{1-\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha}}, & if\ x > \mu, \end{cases} \quad (17)$$

where, $\alpha, b > 0$, $-\infty \le x \le \infty$.

The survival function of a random variable X is defined as

$$S_f = 1 - Z(x). \quad (18)$$

Therefore, using equation (18) we obtain the survival function of the modified Laplace distribution as:

$$S_r = \begin{cases} 1 - \left(\frac{1}{2} e^{\frac{(x-\mu)}{b}}\right),^{\alpha} & if\ x \le \mu, \\ 1 - \left(1 - \frac{1}{2} e^{-\frac{(x-\mu)}{b}}\right),^{\alpha} & if\ x > \mu, \end{cases} \tag{19}$$

where, $\alpha, b > 0,\ -\infty \le x \le \infty$.

The cumulant of a random variable X is simply the log of the moment generating function of the random variable. Therefore,

If $X \sim$MLD (α, μ, b), then the cumulant $[k_x(t)]$ is defined as:

$$k_x(t) = log\big(M_x(t)\big).$$

Where $M_x(t)$ is the moment generating function of the random variable. Hence, the cumulant of a random variable X having a modified Laplace distribution is given by

$$k_x(t) = log\left(\frac{\alpha e^{\mu t}}{2^{\alpha}(\alpha+bt)} + \frac{\alpha e^{\mu t}}{2(1-bt)} + \frac{\alpha e^{\mu t}}{2} \sum_{r=1}^{\infty} \emptyset_r \frac{1}{(r+1-bt)}\right). \tag{20}$$

If $X \sim$MLD(α, μ, b), then the first moment is

$$M_x'(t=0) = \frac{\mu}{2^{\alpha}} - \frac{b}{2^{\alpha}\alpha} + \frac{\alpha}{2}(\mu + b) + \ \alpha \sum_{r=1}^{\infty} \emptyset_r \left(\frac{\mu}{2(r+1)} + \frac{b}{2(r+1)^2}\right).$$

2.3 First and second moments of the modified Laplace distribution (MLD)

Recall that the moment generating function (mgf) of MLD is given as:

$$M_x(t) = \frac{\alpha e^{\mu t}}{2^{\alpha}(\alpha+bt)} + \frac{\alpha e^{\mu t}}{2(1-bt)} + \frac{\alpha e^{\mu t}}{2} \sum_{r=1}^{\infty} \emptyset_r \frac{1}{(r+1-bt)}.$$

Differentiating the above equation with respect to t, we obtain the first moment of the MLD as

$$M_x'(t=0) = \frac{\mu}{2^{\alpha}} - \frac{b}{2^{\alpha}\alpha} + \frac{\alpha}{2}(\mu + b) + \ \alpha \sum_{r=1}^{\infty} \emptyset_r \left(\frac{\mu}{2(r+1)} + \frac{b}{2(r+1)^2}\right). \tag{21}$$

The second moment is obtain by differentiating equation (21) and the result is

$$M_x''(t=0) = \frac{\mu^2 b}{2^\alpha \alpha} - \frac{3\mu b}{2^\alpha \alpha^2} + \frac{2b}{2^\alpha \alpha^2} - \frac{\alpha b(\mu^2 - 3\mu - 1)}{2} +$$

$$\varphi\left[\frac{[-\mu^2 b(r+1) + \mu b](r+1)^2 + 2b(r+1)[\mu(r+1) + b]}{(r+1)^4}\right]. \quad (22)$$

The mean of the modified Laplace distribution is the first moment derived above. Then the variance of the random variable X having a modified Laplace distribution is obtained as

$$var(X) = M_x''(t=0) - [M_x'(t=0)]^2.$$

$$var(X) = \frac{\mu^2 b}{2^\alpha \alpha} - \frac{3\mu b}{2^\alpha \alpha^2} + \frac{2b}{2^\alpha \alpha^2} - \frac{\alpha b(\mu^2 - 3\mu - 1)}{2} +$$

$$\varphi\left[\frac{[-\mu^2 b(r+1) + \mu b](r+1)^2 + 2b(r+1)[\mu(r+1) + b]}{(r+1)^4}\right] - \left[\frac{\mu}{2^\alpha} - \frac{b}{2^\alpha \alpha} + \frac{\alpha}{2}(\mu + b) +\right.$$

$$\left.\alpha \sum_{r=1}^{\infty} \emptyset_r \left(\frac{\mu}{2(r+1)} + \frac{b}{2(r+1)^2}\right)\right]^2. \quad (23)$$

2.4 Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_u$ be a random sample of size n from the modified Laplace distribution. Then, the log-likelihood function is given by:

$$lnL = \sum_{i=i}^{n} \ln z(x_i; \alpha, b, \mu)$$

Where

$$z(x_i; \alpha, b, \mu) = \begin{cases} \alpha\left(\frac{1}{2} e^{\frac{(x_i - \mu)}{b}}\right)^{\alpha - 1} \frac{1}{2b} e^{\frac{(x_i - \mu)}{b}}, & if\ x_i \le \mu, \\ \alpha\left(1 - \frac{1}{2} e^{-\frac{(x_i - \mu)}{b}}\right)^{\alpha - 1} \frac{1}{2b} e^{-\frac{(x_i - \mu)}{b}}, & if\ x_i > \mu, \end{cases}$$

Hence, the log-likelihood function of MLD is given by

$$lnL = \begin{cases} nln\left(\frac{\alpha}{b}\right) + \alpha nln\left(\frac{1}{2}\right) + (\alpha - 1)\sum_{i=1}^{n}\left(\frac{x_i - \mu}{b}\right) + \sum_{i=1}^{n}\left(\frac{x_i - \mu}{b}\right), & if\ x \le \mu, \\ nln\left(\frac{\alpha}{b}\right) + nln\left(\frac{1}{2}\right) - \sum_{i=1}^{n}\left(\frac{x_i - \mu}{b}\right) + (\alpha - 1)\sum_{i=1}^{n} ln\left(1 - \frac{1}{2} e^{-\left(\frac{x_i - \mu}{b}\right)}\right), & if\ x > \mu, \end{cases} \quad (24)$$

To obtain the estimate of the parameters $\alpha, b \ and \ \mu$ of equation (24), the solution may not be obtained in closed form, hence, R software is used to obtain the estimates numerically.

2.5 Quantile function and median

The quantile function $Q(p)$ is derived from the relation:

$$Q(p) = Z^{-1}(p). \tag{25}$$

Hence, the quantile function of the modified Laplace distribution is derived as

$$Q(p) = \begin{cases} \mu + \alpha^{-1} bln p + bln2, \ if \ x \le \mu, \\ \mu - blog2\left(1 - p^{\frac{1}{\alpha}}\right), \ if \ x > \mu, \end{cases} \tag{26}$$

for, $\alpha, b > 0, -\infty \le x \le \infty$.

where $p \backsim Uniform(0,1)$. This simply mean that random samples from the modified Laplace distribution can be generated using

$$x = \begin{cases} \mu + \alpha^{-1} bln p + bln2, \ if \ x \le \mu \\ \mu - blog2(1 - p^{\frac{1}{\alpha}}), \ if \ x > \mu \end{cases}$$

The median of the modified Laplace distribution can be obtained by making the substitution of $p = 0.5$ in equation (26) to have

$$median = \begin{cases} \mu + \alpha^{-1} bln(0.5) + bln2, \ if \ x \le \mu, \\ \mu - blog2\left(1 - (0.5)^{\frac{1}{\alpha}}\right), \ if \ x > \mu, \end{cases} \tag{27}$$

for, $\alpha, b > 0, \ -\infty \le x \le \infty$.

2.6 Order statistics

The probability density function of the jth order statistic for a random sample of size n from a distribution function $Z(x)$ and an associated $z(x)$ is given by:

$$h_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!} z(x)[Z(x)]^{j-1}[1-Z(x)]^{n-j}. \quad (28)$$

where $z(x)$ and $Z(x)$ are the probability density function (PDF) and the cumulative density function (CDF) of the modified Laplace distribution respectively.

The probability density function (PDF) of the jth order statistic for a random sample of size n from the Modified Laplace distribution is given by:

$$h_{j:n}(x) = \begin{cases} \frac{n!}{(j-1)!\,(n-j)!}\alpha\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{\frac{(x-\mu)}{b}}\left[\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha}\right]^{j-1}\left[1-\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha}\right]^{n-j}, & if\ x \le \mu, \\ \frac{n!}{(j-1)!\,(n-j)!}\alpha\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{-\frac{(x-\mu)}{b}}\left[\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha}\right]^{j-1}\left[1-\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha}\right]^{n-j}, & if\ x > \mu, \end{cases} \quad (29)$$

The probability density function of the minimum order statistics is obtained by setting $j = 1$ in equation (29) to have:

$$h_{1:n}(x) = \begin{cases} \frac{n!}{(n-1)!}\alpha\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{\frac{(x-\mu)}{b}}\left[1-\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha}\right]^{n-1}, & if\ x \le \mu, \\ \frac{n!}{(n-1)!}\alpha\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{-\frac{(x-\mu)}{b}}\left[1-\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha}\right]^{n-1}, & if\ x > \mu, \end{cases} \quad (30)$$

and the corresponding probability density function of the maximum order statistics is obtained by setting $j = n$ in equation (28) to have:

$$h_{n:n}(x) = \begin{cases} \frac{n!}{(n-1)!}\alpha\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{\frac{(x-\mu)}{b}}\left[\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha}\right]^{n-1}, & if\ x \le \mu, \\ \frac{n!}{(n-1)!}\alpha\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{-\frac{(x-\mu)}{b}}\left[\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha}\right]^{n-1}, & if\ x > \mu, \end{cases} \quad (31)$$

2.7 Entropies

The R'enyi entropy of a random variable X is given by:

$$I_R = \frac{1}{1-\varepsilon} log\left(\int_0^{\infty}[z(x)]^{\varepsilon}\, dx\right). \quad (32)$$

where $z(x)$ is the pdf of the modified Laplace distribution. Substituting equation (6) into (32) we obtain the R'enyi entropy expression for the random variable X having modified Laplace distribution as.

$$I_R = \begin{cases} \frac{1}{1-\varepsilon} log\left(\int_0^\infty \left[\alpha\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{\frac{(x-\mu)}{b}}\right]^\varepsilon dx\right), if\ x \le \mu, \\ \frac{1}{1-\varepsilon} log\left(\int_0^\infty \left[\alpha\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{-\frac{(x-\mu)}{b}}\right]^\varepsilon dx\right), if\ x > \mu. \end{cases} \quad (33)$$

The Shannon entropy of a random variable X is defined by:

$$n_\gamma = -E\{log[z(x)]\} \quad (34)$$

(Shannon, 1948)

Substituting equation (6) into equation (34) we obtain the expression for the Shannon entropy of the random variable X having modified Laplace distribution as:

$$n_\gamma = \begin{cases} -E\left\{log\left[\alpha\left(\frac{1}{2}e^{\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{\frac{(x-\mu)}{b}}\right]\right\}, if\ x \le \mu, \\ -E\left\{log\left[\alpha\left(1-\frac{1}{2}e^{-\frac{(x-\mu)}{b}}\right)^{\alpha-1}\frac{1}{2b}e^{-\frac{(x-\mu)}{b}}\right]\right\}, if\ x > \mu. \end{cases}$$

4.0 summary of the data collected

Data used in this study were the monthly exchange rate of Nigeria currency to US dollar and monthly Nigerian crude oil prices (USD/Barrel) from 2009-2017 which covered a period of 8 years. These data were collected through the official website of Central Bank of Nigeria (CBN https://www.cbn.gov.ng/rates/ExchRateByCurrency.asp). The choice of the data sets is to demonstrate the numerical application of the modified Laplace distribution.

4.1 Numerical applications

The three parameters modified Laplace distribution under study with the Laplace and Generalized error distributions were estimated by fitting the distributions to the data set of monthly exchange rate of Naira to US dollar and monthly crude oil prices per barrel collected through the official CBN website from 2009-2016.

Table 1: Parameter estimates of the fitted probability models on monthly exchange rate of Naira to US dollar

Models	Parameter estimates	Log likelihood (LL)	SC
MLD	$\alpha = 4.762 \times 10^{-1}$, $b = 1.207 \times 10^{-7}$, $\mu = -6.001 \times 10^{-3}$	**2405859**	**−6.816**
LD	$b = 3.401 \times 10^{-7}$ $\mu = -6.528 \times 10^{-2}$	2935.4317131	−2.971
GED	$\alpha = 1.0 \times 10^{-2}$, $b = 1.06 \times 10^{-7}$, $\mu = 2.0 \times 10^{-2}$	2002940.1	−6.656

Bolded values are the highest value of log likelihood (LL) function and the least value of Schwartz Criteria (SC) measure of fitness.

Table 2: Parameter estimates of the fitted probability models on Nigerian monthly crude oil price per barrel

Models	Parameter estimates	Log likelihood (LL)	SC
MLD	$\alpha = 2.455 \times 10^{-1}$, $b = 3.958 \times 10^{-7}$, $\mu = -5.767 \times 10^{-3}$	**346683.1**	**−5.133**
LD	$b = 3.633 \times 10^{-7}$ $\mu = -7.970 \times 10^{-1}$	343662.2603	−3.108
GED	$\alpha = 3.0 \times 10^{-2}$, $b = 4.122 \times 10^{-7}$, $\mu = 2.2 \times 10^{-2}$	45531.64	−3.360

Bolded values are the highest value of log likelihood (LL) function and the least value of Schwartz Criteria (SC) measure of fitness.

4.1 Simulation results

Table 3

$\mathbf{n = 50}$, $\alpha = 1, b = 1, \mu = 1$	α	b	μ
Means	1.05830	1.03650	0.98030
Bias	0.58310	0.03650	0.29700
RMSE	0.20060	0.21810	0.36905
$\mathbf{n = 100}$			
Means	1.02630	1.02840	0.99130
Bias	0.31630	0.02840	0.18472
RMSE	0.14210	0.16110	0.29150
$\mathbf{n = 200}$			
Means	1.00070	1.01560	1.02040
Bias	0.13480	0.01560	0.07400
RMSE	0.09800	0.11540	0.21280
$\mathbf{n = 300}$			
Means	0.99770	1.00640	1.02180

Bias	0.02830	0.00640	0.04218
RMSE	0.08350	0.09320	0.09140

Table 4

$\mathbf{n = 50}$, $\alpha = 0.5, b = 0.5, \mu = 0.5$			
Means	0.51460	0.53140	0.50070
Bias	0.71460	0.41401	0.31601
RMSE	0.69020	0.85900	0.41804
$\mathbf{n = 100}$			
Means	0.50690	0.51950	0.49880
Bias	0.51250	0.40300	0.30021
RMSE	0.48402	0.55240	0.31510
$\mathbf{n = 200}$			
Means	0.50060	0.50780	0.50800
Bias	0.50130	0.35140	0.15033
RMSE	0.33205	0.51930	0.28244
$\mathbf{n = 300}$			
Means	0.49920	0.50370	0.50820

Bias	0.34641	0.22603	0.12950
RMSE	0.27109	0.34739	0.23151

5. 0 Discussions

The study proposed a new distribution called the modified Laplace distribution using the exponentiation method to inject a shape parameter to the existing Laplace distribution. The study proved that the proposed modified Laplace distribution is a probability density function, some basic statistical properties of the modified Laplace distribution are derived. The numerical application of the proposed distribution is demonstrated using two datasets.

Table 1 shows the application of the modified Laplace distribution on monthly exchange rate of Naira to US dollar. The result obtained shows that the modified Laplace distribution gives the highest value of likelihood function and the least value of the Schwartz Criteria. The implication of this finding is that the modified Laplace distribution fits better than the Laplace distribution and the Generalized error distribution on the data.

Table 2 shows the application of the modified Laplace distribution on monthly price of crude oil per barrel. The result obtained shows that the modified Laplace distribution gives the highest value of likelihood function and the least value of the Schwartz Criteria indicating that the modified Laplace distribution fits better than the Laplace distribution and the Generalized error distribution on the data in all the cases. Hence, the modified Laplace distribution can be best used to model data arising from different real life situations than the Laplace and the Generalized error distributions.

The findings above are in line with the findings of (Adebayo, Badmus and Idowu, 2014, Famoye, Lee and Alzaatreh, 2013, Shittu, Chukwu and Adepoju, 2014, Okorie, Akpanta and Ohakwe, 2016, Nadarajah and Bakar, 2016) on the use of exponentiation method to generate more flexible distributions. All these researchers found that exponentiated forms of distributions were better than their respective parent distributions which is one of the major finding of this study.

Table 3-4 shows the simulation results. It is observed that as the sample size is increasing the modified Laplace distribution fits better with reducing biasedness and Root Mean Square error.

5.1 Conclusion

This study proposed a distribution called modified Laplace distribution which is an extension of the Laplace distribution by the injection of a shape parameter. The proposed distribution have been proven to be better in terms of fitness using both real life and simulated data. The finding showed that the proposed Laplace distribution can be used in place of Laplace and Generalized error distributions to model data sets arising from different life situations.

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