Two algorithms to determine the number pi (π)

Original research paper

ABSTRACT

Archimedes used the perimeter of inscribed and circumscribed regular polygons to obtain lower and upper bounds of π. Starting with two regular hexagons he doubled their side from 6 to 12, 24, 48, and 96. Using the perimeters of 96 side regular polygons, Archimedes showed that 3+10/71<π<3+1/7. His method can be realized as a recurrence formula called the Borchardt-Pfaff-Schwab algorithm. Heinrich Dörrie modified this algorithm to produce better approximations to π than these based on Archimedes' scheme. Lower bounds generated by his modified algorithm are the same as from the method discovered earlier by the cardinal Nicolaus Cusanus (XV century), and again re-discovered two hundred years later by Willebrord Snell (XVII century). Knowledge of Taylor series of the functions used in these methods allows to develop new algorithms. Realizing Richardson's extrapolation, it is possible to increase accuracy of the constructed methods by eliminating some terms in their series. Two new methods are presented. An approximation of squaring the circle with high accuracy is proposed.

Keywords: quadrature, circle, pi number, Archimedes, approximation, algorithm

1. INTRODUCTION

The first known rigours mathematical calculation of π was done by Archimedes. Archimedes' book "On the Measurements of a Circle", [1], written in the 3rd century B.C., contains three propositions. Proposition 3. represents the numerical computing of the number π. Archimedes used an algorithmic scheme based on doubling the number of sides in inscribed and circumscribed regular polygons. He started with the regular hexagons (N = 6) and doubled the number of their sides until N = 96. Archimedes obtained a series of two approximations, lower and upper, for length of the circumference of the circle with diameter equals to one (d = 1), thus consequently to the number π. Archimedes was able to determine the following bounds for the number π: $3+\frac{10}{71}<\pi<3+\frac{1}{7}$.

It's often suggested to combine these values to improve the approximation by taking their arithmetic average. Of course, it's correct but it's possible to realize a better combination (see Table 1) than an arithmetical mean of these two bounds. Archimedes' estimations can be improved using only information already generated by the constructed polygons. Here two such improvements are proposed and presented. The proposed algorithms use only values obtained from the traditional and well-known methods. New created algorithms produce faster convergence to π than original techniques. Such approach already was realized for some other numerical methods. Table 1 shows the results for the regular polygons (N = 6, 12, 24, 48, and 96) and their combinations proposed in XVII century by Snell and later proved by Huygens [2]. Archimedes' approach is true real algorithm to obtain the value of π. The method is capable to generate an arbitrarily precision of the number π. The process is relatively slow in its convergence. It is also difficult to use this algorithm in direct calculations for large number of sides. It is a similar situation as with Turing's machine and a modern computer. Theoretically all computable problems can be realized on both types of machines. It's only a difference and matter of time. There were many attempts to improve Archimedes' method. One such approach resulted in Pfaff-Borchardt-Schwab's (PBS) method developed in XIX century. It's realized without using trigonometric functions.

The method PBS is defined by the following formulas: $a'=\frac{2ab}{a+b}, b'=\sqrt{a'b}$, new values a′, b′ are determined by old values a, b - the values from the previous step. It's an iterative process and it's easy to realize on a computer. Starting with a = 2√3 and b = 3; the values for circumscribed and inscribed regular 6-gons, we can generate the sequence of the intervals [b, a], b < a. The intervals contain π. It's π for the circle of the diameter one (d = 1), or for a unit circle (r = 1), and in this case it's half of its perimeter, which, of course, it's also π.

Table 1. Approximations of the number pi obtained by Archimedes' method (values a and b, from the inscribed and circumscribed polygons, respectively), their arithmetic average ($c=\frac{(a+b)}{2}$), and by using Snell's approach ($d=\frac{(2a+b)}{3}$), Here N determines the number of sides in regular polygons and $x=180^o/N$ is the central angle in the circle.

N	a=N*sin(x)	b=N*tan(x)	c=a+(b-a)/2	d=a+(b-a)/3
6	3.00000	3.46410	3.23205	3.15470
12	3.10583	3.21539	3.16061	3.14235
24	3.12567	3.17389	3.14978	3.14174
48	3.13263	3.15966	3.14614	3.14164

96	3.13585	3.15313	3.14449	3.14161

Ludolph van Ceulen (1540-1610) was the last person who performed great Archimedean calculations. He used 2^{62}-gons and obtained 39 places with 35 correct digits. The number is still called Ludolph's number in some parts of Europe. For example, in Poland it is called in Polish "liczba ludolfina". Archimedes' method may be interpreted as a rectification problem. Its goal is to find the length of the arc of the considered circle. In this case, the method estimates the circumference of the circle (i.e. full arc for the angle 2π). Very simple and beautiful rectification method was developed by the Polish mathematician Adam Adamandy Kochański [3]. His construction results with π estimation equals to 3.141533. Kochański's geometrical construction can be done with only one opening of a compass. In this case the process is not iterative. It is one-time construction.

2. MATERIAL AND METHODS

For our purpose we here consider two basic methods, Snells' rectification method and Dörrie's method [2, 4]. Both methods were developed to accelerate Archimedes' process. Here, we are doing the next step further. Our two approaches use the values generated by Snell's and Dörrie's method to construct better approximations for the number π. We listed all used methods in this work in Table 2. In our notation we added X (after M) to indicate that the method (M) is the result of combinations. We assumed that combination occurred, when the composite method is defined by elements already calculated in its components, [5-7]. Consider three of the following methods: MX4: Snell-P based on perimeter (P) of the circle, MX5: Snell-A based on area (A) of the circle, (Huygens, 1654)) and MX6: Ch-H based on the methods M1, M2 and M3, [5]. Table 2 represents the applied methods, their short descriptions, and the results for using them with N=3 and 6. (π=3.14159265358979...). The method M8 was invented by Cusanus (XV), Snell-Huygens (XVII), and again by Dörrie (XX century). One of the results of this presentation is detection that one Dörrie's formula (for B) was already known in XV and XVII centuries.

Table 2. Methods, their descriptions and the results for pi using N=3 and 6. Method M8 was invented by Cusanus, Snell-Huygens, and Dörrie.

Method (X combined)	Description	N=3	N=6
M1, side, inscribed	sin(x)	2.598076	3.000000
M2, side, area circumscribed	tan(x)	5.196152	3.464101
M3, area, inscribed	sin(2x)/2	1.299038	2.598076
MX4=M1+(M2-M1)/3	Snell-P	3.464101	3.154700
MX5=M2+(M3-M2)/3	Snell-A	3.897114	3.175426
MX6=(32M1+4M2-6M3)/30	Ch-H	3.204293	3.142264
M7=(2 cos (x/3)+1) tan (x/3)	Snell-ArcU	3.144031	3.141740
M8=3 sin(x)/(2+cos (x))	Snell-ArcL	3.117691	3.140237
MX9=$(M2*M1*M1)^{1/3}$	A- Dörrie	3.273370	3.147345
MX10=M8+(MX9-M8)/5	Szysz-Dörrie	3.148827	3.141658
MX11=M7+(M8-M7)/10	Szyszkowicz	3.141397	3.141589

2.1 Snell's rectification

Cardinal Nicolaus Cusanus (1401-1464) has elaborated the following rectification of the arc in the circle for the corresponding angle x: $arc = 3sin(x)/(2 + cos(x))$. It corresponds to the first convergent of the continued fraction for $sin(x)/x$. This formula was once again proposed two hundred years later by the Dutch mathematician and physicist Snell (Willebrord Snellius, 1580-1626). We don't know it was an original invention or using the known result obtained by the cardinal. Snell developed two approximations for the length of the arc, lower (M8: Snell-ArcL) and upper (M7: Snell-ArcU), Huygens (1654). We combine these two methods to define better approximation (MX11; Szyszkowicz, 2015, [6]). To develop such approach, we used Taylor series for the corresponding methods (Table 3 and 4), in this case M7 and M8, and generated the new method as MX11=u*M7+v*M8. The coefficients u and v are determined by the following system of the equations (see Table 4) to improve its accuracy: $u + v = 1, u/1620 - v/180=0$. The solution allows us to define better method of the form MX11=M7+(M8-M7)/10. Table 4 shows that in its Taylor series the next term after x is x to power 7. We keep the element x (x to power one) but eliminate x to power 5. Here x = π/N and as N is growing N*MX11 goes to π.

.Table 3. Taylor series of the methods related to Archimedes' algorithm.

Method	Taylor series
M1: $\sin(x)$	$x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\frac{x^9}{362880}-\frac{x^{11}}{39916800}+O(x^{12})$
M2: $\tan(x)$	$x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\frac{1382x^{11}}{155925}+O(x^{12})$
M3: $\sin(2x)/2$	$x-\frac{2x^3}{3}+\frac{2x^5}{15}-\frac{4x^7}{315}+\frac{2x^9}{2835}-\frac{4x^{11}}{155925}+O(x^{12})$

The Ch-H (MX6, [5]) method can be developed differently than originally presented by its authors. The method can be determined (Richardson's approach) as the results of a linear combination $MX6 = a * M1 + b * M2 + c * M3$. Here we use all methods related to Archimedes' technique. We are able to improve the accuracy without additional calculations (increasing N). Using their Taylor representation, it's possible to keep the term with x (we need to satisfy the condition a+b+c=1) and to eliminate the terms with x in power 3 and 5. This request produces the following conditions on the coefficients $a+b+c=1, -\frac{a}{6}+\frac{b}{3}-\frac{2c}{3}=0, \frac{a}{120}+\frac{2}{15}b+\frac{2}{15}c=0$. The obtained linear system is easy to solve. The system results in the following formula $MX6 = (32M1 + 4M2 - 6M3)/30$. With new set of the parameters a and b, the method is also defined as $MX6 = a * MX2 + b * MX4$, with the following conditions: $a+b=1, \frac{a}{20}+\frac{2b}{15}=0$ (see Tables 3 and 4).

Table 4. Taylor series of the presented methods.

Method	Taylor series or analytic formula
M7	$x+\frac{x^5}{1620}+\frac{x^7}{40824}+\frac{7x^9}{6298560}+\frac{3931x^{11}}{78568237440}+O(x^{12})$
M8	$B=\frac{3ab}{2a+b}=\frac{3\tan(x)\sin(x)}{2\tan(x)+\sin(x)}=\frac{3\sin(x)}{2+\cos(x)}$
M8	$x-\frac{x^5}{180}-\frac{x^7}{1512}-\frac{x^9}{25920}+\frac{x^{11}}{391680}+O(x^{13})$
MX9	$A=\sqrt[3]{ab^2}=\sqrt[3]{\tan(x)\sin(x)^2}$
MX9	$x+\frac{x^5}{45}+\frac{4x^7}{567}+\frac{x^9}{405}+\frac{248x^{11}}{280665}+O(x^{13})$
MX10	$x+\frac{4x^7}{1134}+\frac{x^9}{2160}+\frac{227x^{11}}{1283040}+O(x^{13})$
MX11	$x-\frac{x^7}{22680}-\frac{x^9}{349920}+\frac{437x^{11}}{6235574400}+O(x^{13})$

2.2 Dörrie's sequence

In his book ("100 Great Problems of Elementary Mathematics") the German mathematician Heinrich Dörrie in the problem No. 38 presented another approach to improve Archimedes method, [4]. He constructed two new series B and A, ([B, A] interval) which give better approximation for the length of the circumference (C) of the circle. For a given values b, a (the [b, a] interval) are generated $B=\frac{3ab}{2a+b}$ *and* $A=\sqrt[3]{ab^2}$. He proved that the following inequalities hold b < B < C < A < a. The sequence of Bs increases to C, and the sequence of As decreases to C. Always the interval [b, a] contains the interval [B, A]. For example, starting with a regular hexagon d = 1, a = 2√3, b = 3 we have the following values from Dörrie's method B = 3.140237343, A = 3.14734519, a precision achieved by the Archimedes method first with a 96-gon. It's interesting that the method used to generate the sequence B is the same formula as proposed by the cardinal Cusanus and Snell (M8), also see Tables 2 and 4, and Figure 1. In a similar way as the method MX11 was obtained the method MX10 was determined. The method is constructed as follows MX10=M8+(MX9-M8)/5.

3. RESULTS AND DISCUSSION

Below is presented the program in R. It realizes some of the discussed methods. In its bottom the results are given for N=64. The listing of this program allows better understand the presented material and the realized formulae. The program can start with a square (N=4) or hexagon (N=6).

```
#Program realizes the following methods: M1, M2, M8, MX9, and MX10.
options(digits=15)
N=4; b=2*sqrt(2); a=4 #square
N=6; b=3; a=2*sqrt(3) #hexagon
```

```
for (k in 1:5){
cn=c(k-1,N); print(cn)
arch = c(b,a) #Archimedes' results
# Dörrie:
B=3*a*b/(2*a + b)
A=(a*b*b)^(1/3)
dor = c(B,A) # Dörrie's results
#Szyszkowicz
S=B+(A-B)/5 # Szyszkowicz's method
res=c(arch,dor,S)
print(res)
#Next Archimedes:
a=2*a*b/(a+b)
b=sqrt(a*b)
N=N+N}
method=c("M1","M2","M8","MX9","MX10")
print(method)
print(pi); #The end
#The results for 96-gon
M1: 3.14103195089051; M2: 3.14271459964537; M8: 3.14159263357057
MX9: 3.14159273368372; MX10: 3.14159265359320; pi: 3.14159265358979
```


The main results of this paper are two methods (MX10 and MX11), where we used Taylor series to justify their correctness and accuracy. The methods are very easy to program. Some calculations were executed (see the program). Table 5 shows the results for the Pfaff-Borchardt-Schwab algorithm (a, b values), Dörrie's method (A, B values) and the method MX10 proposed in this paper.

Table 5. The approximations generated by Archimedes, Dörrie's method, and Szyszkowicz's method(MX10).

Size	**Archimedes**		**Dörrie**		**Szyszkowicz**
N	**M1 (b)**	**M2 (a)**	**M8 (B)**	**MX9 (A)**	**M10 (B+(A-B)/5)**
6	3.0000000	3.4641016	3.1402373	3.1473452	3.1416589
12	3.1058285	3.2153903	3.1415100	3.1419279	3.1415936
24	3.1326286	3.1596599	3.1415875	3.1416133	3.1415927
48	3.1393502	3.1460862	3.1415923	3.1415939	3.1415927
96	3.1410320	3.1427146	3.1415926	3.1415927	3.1415927

Table 6 presents the obtained results for the method MX11 and a few other methods already known in literature.

Table 6. Estimated value of pi from various methods and N.

Size	**MX5**	**MX6**	**M7**	**M8**	**MX11**
N	**Snell-A**	**Ch-H**	**Snell-ArcU**	**Snell-ArcL**	**Szyszkowicz**
6	3.89711432	3.20429399	3.14403156	3.11769145	3.14139755
8	3.33333333	3.15032227	3.14234913	3.13444650	3.14155887
10	3.21435552	3.14368811	3.14189972	3.13874170	3.14158392
12	3.17542648	3.14226497	3.14174002	3.14023734	3.14158975
14	3.15948495	3.14185286	3.14167196	3.14086739	3.14159151
16	3.15194804	3.14170766	3.14163906	3.14116990	3.14159214
18	3.14800282	3.14164881	3.14162159	3.14132974	3.14159240
20	3.14577340	3.14162228	3.14161162	3.14142063	3.14159252
22	3.14443578	3.14160929	3.14160560	3.14147540	3.14159258
24	3.14359354	3.14160249	3.14160179	3.14150999	3.14159261

Method MX11 has interesting geometrical interpretation and one example is here presented. Figure 1 shows the rectification process for the arc corresponding to the angle x = 135 degree. It's relatively large angle and by a consequence the estimation is not very accurate. In this approach we have to realize two methods, M7 and M8, to obtain lower and upper estimations for the length of the arc. Their arithmetic average is less accurate than this generated by MX11 method. We have already as pi the value 3.11582354. The exact value for the length of this arc is $\frac{3}{4}\pi$. It allows us to determine our accuracy obtained for the angle x=135 degrees using MX11 method.

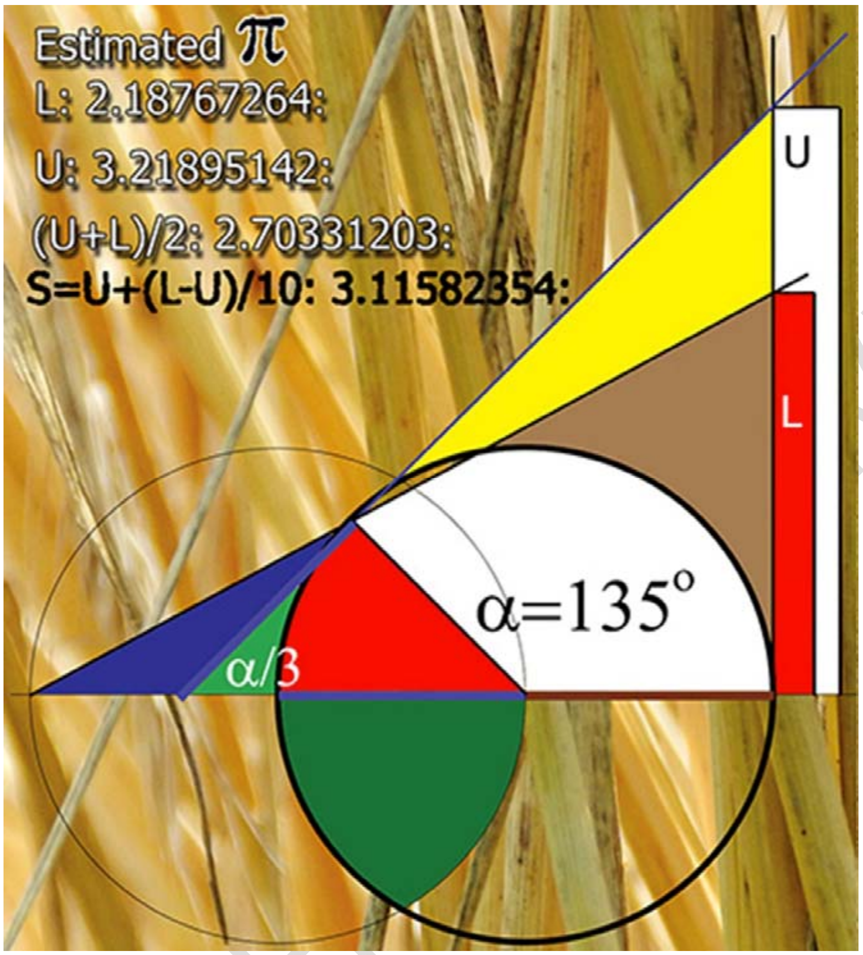

Figure 1: Rectification of the arc - Szyszkowicz's method (MX11).

As the method needs also the angle x/3, we should be able to do trisection of a given angle x. In this case it is possible to do this by a pure geometrical construction. It's easy to obtain the angle x/3. It's by using a half of the right angle (90/2 = 45 = 125/3). The lower (L) and upper (U) estimations are generated by the methods M8 and M7, respectively. They have geometrical interpretations: the angle's vertex has the distance r (radius) to the circle for L, and to the cutting point on the circumference for the angle x. We are using the method MX11 to obtained better approximation for the number π.

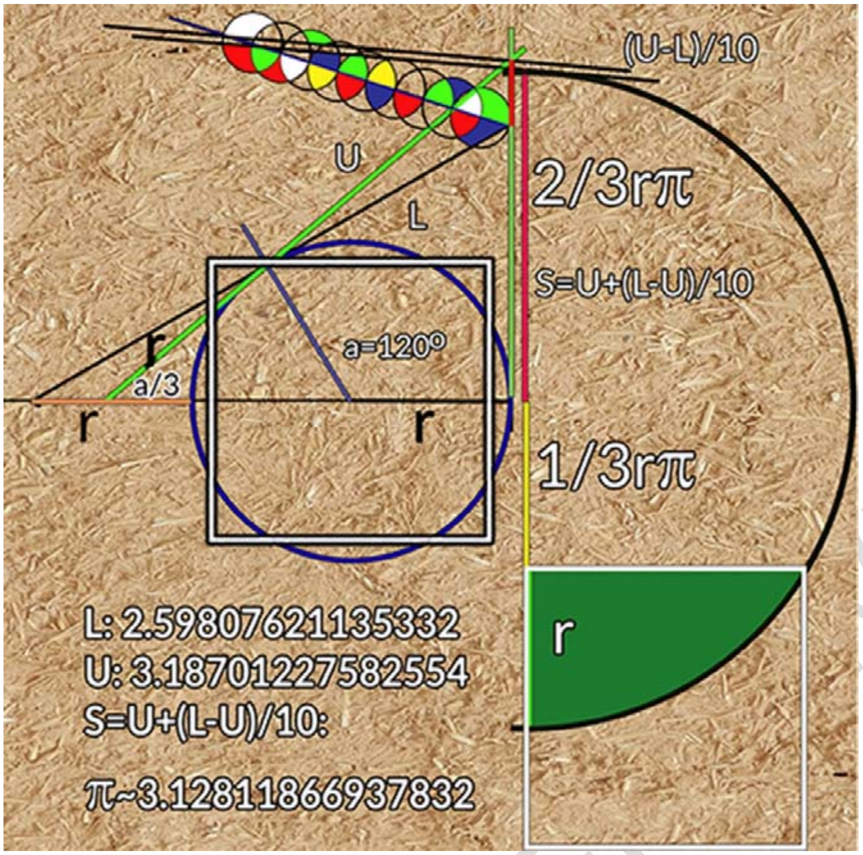

Figure 2. Quadrature based on the rectification of the arc - Szyszkowicz's method (MX11).

Figure 2 shows more difficult situation. The angle of 120 degrees can't be trisected. We need the angel of 40 degrees. We may use other sources of such angle, but not from pure geometrical construction process. In this case a used graphic software was asked to rotate horizontal segment by 40 degrees. The method MX11 is applied and determines the segment S = U + (L − U)/10. Here, the main problem (mainly constructional) is to determine the segment (U-L)/10. On Fig. 2 we a series of small circle used to realize the division in 10 equal parts. Thales' approach to divide a segment in a proportion is applied. The obtained segment (2/3πr) is extended by 1/3πr and r. It allows us to perform the squaring of the rectangle (interpreted as such) of the side πr and r. By the consequence we did approximated quadrature of our circle with estimated value of the number π. In the geometrical process Thales theorem on proportion is applied to divide the segment U-L into 10 equal parts.

4. CONCLUSION

The illustrative results summarize obtained approximations by various methods. As the values show the best approximation is produced by the MX10 method. The method is the result of the combination of two sequences generated by Dörrie's algorithm.

Well know methods to approximate the number pi are realized. Taylor series of these method (and Richardson's extrapolation) allow to produce new methods with better convergence properties. As the main results two methods are proposed: (i) combined Dörrie's sequence (MX10 method), (ii) combined Snell's sequence (MX11).

Two methods presented here improve Archimedes' technique. The method MX11 can be used geometrically for an angle x, if x/3 can be constructed to execute an approximate quadrature of the circle. In addition, the presented methodology has an educational aspect.

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