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SOME FIXED POINT RESULTS OF NON-NEWTONIAN EXPANSION MAPPINGS

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Abstract: We manifest some fixed point and common fixed point results for non-Newtonian expansive maps defined on non-Newtonian metric spaces. The results offered in this articlecomprisenon-Newtonian metric generalizations of some fixed point results in the literature.

Keywords: non-Newtonian metric space; non-Newtonian expansive mapping; fixed point.

1. INTRODUCTION

The idea of non-Newtonian calculus was firstly acquaintby Grossman and Katz [1]. Later, the non-Newtonian calculus is studied by Bashirov et al. [2], Ozyapici et al. [3], Cakmak and Basar [4] and others [5-17]. Cakmak and Basar [4] have studied the conceptof non-Newtonian metric. Several statements about them are proven in [7]. Binbasıoglu et al [18] defined the contractive mapping in non-Newtonian metric space. The non-Newtonian calculi are alternatives to the classical calculus of Newton and Leibnitz. They confer a wide variety of mathematical tools for usage in technologyand mathematics. The non-Newtonian calculus has great applications in various areas including fractal geometry, the economics of climate change, image analysis, physics, quantum physics, growth/decay analysis, finance, the theory of elasticity in economics, marketing and gauge theory, information technology, pathogen counts in treated water, actuarial science, tumor therapy and cancer-chemotherapy in medicine, materials science/engineering, demographics, finite-difference methods, differential equations, averages of functions, calculus of variations, means of two positive numbers, least-squares methods, multivariable calculus, weighted calculus, meta-calculus, approximation theory, probability theory, utility theory, Bayesian analysis, complex analysis, functional analysis, stochastics, chaos theory, dimensional spaces, decision making, dynamical systems etc.

The study of expansive maps is a very enthralling research area in fixed point theory. Wang et.al [19] deputized the concept of expanding maps and vouched some fixed point results in complete metric spaces. Daffer and Kaneko [20] vouched some common fixed point results in complete metric spaces for two expansive mappings. For more details, we refer the reader to [21-26].

In this article, we give someproperties of the relevant non-Newtonian metric space and non-Newtonian normed space. We alsointroduce the concept of non-Newtonian expansive mappings and presentsome fixed point results in non-Newtonian metric space. These results also generalize some results obtained previously.

2. PRELIMINARIES

Aninjective functionwhose domain is$\mathbb{R}$, the set of all real numbers, and whose range is a subset of $\mathbb{R}$ is called a generator. Eachgenerator generates exactly one type of arithmetic, andconversely each type of arithmetic is generated by exactlyone generator. As a generator, we choose the function exp from $\mathbb{R}$ to the set$\mathbb{R}^+$ of positive reals, that is to say,

$$\alpha: \mathbb{R} \longrightarrow \mathbb{R}^+,$$

$$r \longmapsto \alpha(r) = e^r = s$$

and $$\alpha^{-1}: \mathbb{R}^+ \longrightarrow \mathbb{R},$$

$$s \longmapsto \alpha^{-1}(s) = \ln s = r$$

If $I(r) = r$for all $r \in \mathbb{R}$, then$I$ is called identity function and we know that inverse of the identity function is itself. If $\alpha = I$, thenαgenerates the classical arithmetic and if $\alpha = exp$, then αgenerates geometrical arithmetic.All concepts of α-arithmetic have similar properties in classical arithmetic. α-zero, α-one and all α-integers are formed as

$$\ldots\ldots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2)\ldots\ldots..$$

The α-positive numbers are the numbers $j \in A$ such that $\dot{0} \dot{<} j$and the α-negative numbers are those for which $j \dot{<} \dot{0}$. The α-zero, $\dot{0}$, and the α-one, $\dot{1}$, turn out to be $\alpha(0)$ and $\alpha(1)$. The α-integers consist of $\dot{0}$ and all the numbers that result by successive α-addition of $\dot{1}$ and $\dot{0}$ and by successive α-subtraction of $\dot{1}$ and $\dot{0}$.

We denote by $\mathbb{R}(N)$ the range of generator$\alpha$and write $\mathbb{R}(N) = \{\alpha(r) : r \in \mathbb{R}\}$. $\mathbb{R}(N)$ is called Non-Newtonian real line. Non-Newtonian arithmetic operations on $\mathbb{R}(N)$are represented as follows:

α-addition $\quad i \dot{+} j = \alpha\big(\alpha^{-1}(i) + \alpha^{-1}(j)\big),$

α-subtraction $\quad i \dot{-} j = \alpha\big(\alpha^{-1}(i) - \alpha^{-1}(j)\big),$

α-multiplication $\quad i \dot{\times} j = \alpha\big(\alpha^{-1}(i) \times \alpha^{-1}(j)\big),$

α-division $\quad i \dot{/} j = \alpha(\alpha^{-1}(i) / \alpha^{-1}(j)),$

α-order $\quad i \dot{<} j (i \dot{\leq} j) \Leftrightarrow \alpha^{-1}(i) < \alpha^{-1}(j)\big(\alpha^{-1}(i) \leq \alpha^{-1}(j)\big),$

The α-square of a number $i \in A \subset \mathbb{R}(N)$ is denoted by $i \dot{\times} i = i^{2_N}$. For each α-nonnegative number v, the symbol $\sqrt{i}^N$ will be used to denote $v = \alpha\left(\sqrt{\alpha^{-1}(i)}\right)$ which is the unique α-square is equal to i, which means that $v^{2_N} = i$. Throughout this paper, i^{p_N}denotes the pth non-Newtonian exponent. Thus we have

$$i^{2_N} = i \dot{\times} i = \alpha\big(\alpha^{-1}(i) \times \alpha^{-1}(i)\big) = \alpha([\alpha^{-1}(i)]^2),$$

$$i^{3_N} = i^{2_N} \dot{\times} i = \alpha\big(\alpha^{-1}(i^{2_N}) \times \alpha^{-1}(i)\big)$$

$$= \alpha\left(\alpha^{-1}\left(\alpha\big(\alpha^{-1}(i) \times \alpha^{-1}(i)\big)\right) \times \alpha^{-1}(i)\right) = \alpha([\alpha^{-1}(i)]^3),$$

$$\vdots$$

$$i^{p_N} = i^{p-1_N} \dot{\times} i = \alpha([\alpha^{-1}(i)]^p)$$

$$\vdots$$

The α-absolute value of a number $i \in A \subset \mathbb{R}(N)$ is defined as $\alpha(|\alpha^{-1}(i)|)$ and is denoted by $|i|_N$. For each number $i \in A \subset \mathbb{R}(N), \sqrt{i^{2_N}}^N = |i|_N = \alpha(|\alpha^{-1}(i)|)$. In this case,

$$|i|_N = \begin{cases} i, & \text{if } i \dot{>} \dot{0} \\ \dot{0}, & \text{if } i = \dot{0} \\ \dot{0} \dot{-} i, & \text{if } i \dot{<} \dot{0} \end{cases}$$

Also $\mathbb{R}^+(N)$ denotes non-Newtonian positive real numbers and $\mathbb{R}^-(N)$ denotes non-Newtonian negative real numbers.α-intervals are represented by

Closedα-interval$\dot{[} i, j \dot{]} = [i, j]_N = \{s \in \mathbb{R}(N) : \; i \dot{\leq} s \dot{\leq} j\}$

$$= \{s \in \mathbb{R}(N) : \; \alpha^{-1}(i) \leq \alpha^{-1}(s) \leq \alpha^{-1}(j)\}$$

Open α-interval$\dot{(} i, j \dot{)} = (i, j)_N = \{s \in \mathbb{R}(N) : \; i \dot{<} s \dot{<} j\}$

$$= \{s \in \mathbb{R}(N) : \; \alpha^{-1}(i) < \alpha^{-1}(s) < \alpha^{-1}(j)\}$$

Likewise semi-closed and semi-open α-intervals can be represented. For the set $\mathbb{R}(N)$ of non-Newtonianreal numbers, the binary operations $(\dot{+})$addition and $(\dot{\times})$ multiplication are defined by

$$\dot{+} \; : \; \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(i, j) \longmapsto i \dot{+} j = \alpha\left(\alpha^{-1}(i) + \alpha^{-1}(j)\right)$$

$$\dot{\times} \; : \; \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

$(i, j) \longmapsto i \dot{\times} j = \alpha\left(\alpha^{-1}(i) \times \alpha^{-1}(j)\right)$.

The fundamental properties provided in the classical calculus is provided in non-Newtonian calculus, too.

Lemma 2.1 (see[4]). $(\mathbb{R}(N), \dot{+}, \dot{\times})$ is a topologicallycomplete field.

Lemma 2.2(see[4])$|i \dot{\times} j|_N = |i|_N \dot{\times} |j|_N \forall\, i, j \in \mathbb{R}(N)$.

Lemma 2.3(see [4])$|i \dot{+} j|_N \dot{\leq} |i|_N \dot{+} |j|_N, \forall\, i, j \in \mathbb{R}(N)$

The non-Newtonian metric spaces provide an alternative to the metricspaces introduced in [4].

Definition 2.4(see [4]). Let M be a non-empty set and $d_N: M \times M \longrightarrow \mathbb{R}^+(N)$be a function such that for all $i, j, k \in M$;

(NNM1). $d_N(i,j) = \dot{0} \Leftrightarrow i = j$

(NNM2). $d_N(i,j) = d_N(j,k)$

(NNM3). $d_N(i,j) \dot{\leq} d_N(i,k) \dot{+} d_N(k,j)$.

Then, the map d_N is called non-Newtonian metric and the pair $(M, d_N)$is called non-Newtonian metric space.

Definition 2.5(see [4]). Let Mbe a vector space on $\mathbb{R}(N)$. If a function $\| . \|_N : M \longrightarrow \mathbb{R}^+(N)$ satisfies the following axioms for all $i, j \in M$ and $\lambda \in \mathbb{R}(N)$:

(NNN1). $\|i\|_N = \dot{0} \Leftrightarrow i = \dot{0}$

(NNN2). $\|\lambda \dot{\times} i\|_N = |\lambda|_N \dot{\times} \|i\|_N$

(NNN3). $\|i \dot{+} j\|_N \dot{\leq} \|i\|_N \dot{+} \|j\|_N$.

then it is called a non-Newtonian norm on $M$and the pair $(M, \| . \|_N)$ is called a non-Newtonian normed space.

Remark 2.6(see [4]). Here it is easily seen that every non-Newtonian norm $\| . \|_N$on Mproduces a non-Newtonian metric $d_N$on Mgiven by

$$d_N(i,j) = \|i \dot{-} j\|_N, \forall\, i, j \in M$$

Definition 2.7(see [4]). (non-Newtonian convergentsequence)A sequence $\{j_n\}$ina non-Newtonian metric space$(M, d_N)$is said to be non-Newtonian convergent if for every given $\epsilon \dot{>} \dot{0}$,there exists an $n_0 = n_0(\epsilon) \in \mathbb{N}$ and $j \in M$ such that $d_N(j_n, j) \dot{<} \epsilon$ for all $n > n_0$and is denoted by ${}^{\mathrm{N}}\lim_{n \to +\infty} j_n = j$ or $j_n \xrightarrow{N} j$ as $n \to \infty$.

Definition 2.8 (see [4]). (non-Newtonian Cauchysequence) A sequence $\{j_n\}$ ina non-Newtonian metric space(M, d_N) is said to be non-Newtonian Cauchy if for every given $\epsilon \dot{>} \dot{0}$,there exists an $n_0 = n_0(\epsilon) \in \mathbb{N}$such that $d_N(j_n, j_m) \dot{<} \epsilon$ for all $m, n > n_0$.

Definition 2.9 (see [4]). (non-Newtonian complete metric space) The space M is said to be non-Newtonian complete if every non-Newtonian Cauchy sequencein Mconverges.

Definition 2.10(see [4]). (non-Newtonian bounded)Let (M, d_N) be a non-Newtonian metric space. The space M is said to benon-Newtonian bounded if there is a non-Newtonian constant $\kappa \dot{>} \dot{0}$ such that $d_N(i, j) \dot{\leq} \kappa$ for all $i, j \in M$. The space M is said to be non-Newtonian unbounded if it is not non-Newtonian bounded.

Proposition 2.11(see [4]). Suppose that the non-Newtonian metric d_N on$\mathbb{R}(N)$is such that $d_N(i, j) = |i \dot{-} j|_N$ for all $i, j \in \mathbb{R}(N)$, then $(\mathbb{R}(N), d_N)$is a non-Newtonian metric space.

Lemma 2.12 (see[18]). Let (M, d_N) be a non-Newtonian metric space. Then,

(1). A non-Newtonian convergent sequence in M is non-Newtonian bounded and its non-Newtonian limit is unique.

(2). A non-Newtonian convergent sequence in M is a non-Newtonian Cauchy sequence in M.

From the definition of non-Newtonian Cauchy sequence and Lemma 2.12, we can give the following corollary:

Corollary 2.13(see [18])Anon-Newtonian Cauchy sequence is non-Newtonian bounded.

Lemma 2.14(see [18])Suppose (M, d_N) is a non-Newtonian metric spaceand $i, j, k \in M$. Then

$$|d_N(i.j) \dot{-} d_N(j,k)|_N \dot{\leq} d_N(i,k)$$

Definition 2.15 Let Mbe a set and fa map from $M$to M. A fixed point of $f$is asolution of the functional equation $f(j) = j, j \in M$. A point $j \in M$ is called common fixed point of two self-mappings f and g on M if $f(j) = g(j) = j$.

Definition 2.16(see [18])Suppose (M, d_N) is a non-Newtonian complete metric space. A mapping $f: M \rightarrow M$is called non-Newtonian Lipschitzian if there exists a non-Newtonian number $\delta \in \mathbb{R}(N)$ such that

$$d_N\big(f(i), f(j)\big) \dot{\leq} \delta \dot{\times} d_N(i,j), \forall\, i,j \in M.$$

The mapping $f$is called non-Newtoniancontractive if $\delta \dot{<} \dot{1}$.

Binbasıoglu et al [18] establishedfollowing result in non-Newtonian metric space.

Theorem 2.17 Let f be a non-Newtonian contraction mapping on a non-Newtonian complete metric space M. Then f has a unique fixed point.

3. Main Results

Now, we give some properties related to non-Newtonian metricspaces and non-Newtonian normedspaces.

Proposition 3.1 The non-Newtonian distance is commutative.

Proof Let i and j be any two non-Newtonian numbers. Then

$$|i \dot{-} j|_N = \alpha(|\alpha^{-1}(i) - \alpha^{-1}(j)|)$$

$$= \alpha(|\alpha^{-1}(j) - \alpha^{-1}(i)|)$$

$$= |j \dot{-} i|_N \qquad (3.1)$$

This shows that non-Newtonian distance is commutative.

Proposition 3.2 Let(M, d_N) be a non-Newtonian metric space and let $i, j, k, l \in M$. Then

$|d_N(i,j) \dot{-} d_N(k,l)|_N \dot{\leq} d_N(i,k) \dot{+} d_N(j,l)$ (3.2)

Proof The triangle inequality with the NNM axioms yields first

$$d_N(i,j) \dot{\leq} d_N(j,k) \dot{+} d_N(k,j)$$

$$\dot{\leq} d_N(j,k) \dot{+} d_N(k,l) \dot{+} d_N(l,j)$$

Usingthe symmetry axiom, rearrangement of the above inequalitygives

$d_N(i,j) \dot{-} d_N(k,l) \dot{\leq} d_N(i,k) \dot{+} d_N(j,l)$ (3.3)

Similarly, we have

$$d_N(k,l) \dot{\leq} d_N(k,i) \dot{+} d_N(i,l)$$

$$\dot{\leq} d_N(k,i) \dot{+} d_N(i,j) \dot{+} d_N(j,l)$$

$$= d_N(i,k) \dot{+} d_N(i,j) \dot{+} d_N(j,l)$$

Therefore

$d_N(k,l) \dot{-} d_N(i,k) \dot{\leq} d_N(i,k) \dot{+} d_N(j,l)$ (3.4)

Thus from (3.3) and (3.4) it follows that (3.2).

Proposition 3.3 Let$(M, \|\,.\,\|_N)$ be a non-Newtonian normed space. Then

$|\|i\|_N \dot{-} \|j\|_N|_N \dot{\leq} \|i \dot{-} j\|_N, \forall i, j \in M$ (3.5)

ProofObserve that

$$\|i\|_N = \|i \dot{-} j \dot{+} j\|_N \dot{\leq} \|i \dot{-} j\|_N \dot{+} \|j\|_N$$

Therefore $\|i\|_N \dot{-} \|j\|_N \dot{\leq} \|i \dot{-} j\|_N$. Swapping the role of i and j, we also obtain $\|j\|_N \dot{-} \|i\|_N \dot{\leq} \|i \dot{-} j\|_N$. This implies (3.5).

Now, we introduce some definitions in non-Newtonian metric spaces.

Definition 3.4 Suppose (M, d_N) is a non-Newtonian complete metric space. A mapping $f: M \to M$ is called non-Newtonian expansive if there exists a non-Newtonian number $\delta \dot{>} \dot{1}$ such that

$d_N(fx, fy) \dot{\geq} \delta \dot{\times} d_N(x, y), \forall\, x, y \in M$. (3.6)

Definition 3.5 Let (M, d_N) be a non-Newtonian metric space and f be a self-mapping of M: (NN1) There exist non-Newtonian numbers a, b, c satisfying $b \dot{\geq} \dot{0}, c \dot{\geq} \dot{0}$ and $a \dot{>} \dot{1}$ such that

$$d_N\big(f(i), f(j)\big) \dot{\geq} a \dot{\times} d_N(i, j) \dot{+} b \dot{\times} d_N\big(i, f(i)\big) \dot{+} c \dot{\times} d_N\big(j, f(j)\big) \qquad (3.7)$$

for each $i, j \in M$. In this case f is called non-Newtonian expansive type mapping.

Now, we give a simple but a useful Lemma.

Lemma 3.6 Let $\{j_n\}$ be a sequence in a non-Newtonian metric space such that

$d_N(j_n, j_{n+1}) \leq \delta \dot{\times} d_N(j_{n-1}, j_n)$ (3.8)

where $\delta \dot{<} \dot{1}$ and $n \in \mathbb{N}$. Then $\{j_n\}$ is a non-Newtonian Cauchy sequence in M.

Proof By the simple induction with the condition (3.8), we have

$$d_N(j_n, j_{n+1}) \dot{\leq} \delta \dot{\times} d_N(j_{n-1}, j_n)$$

$$\dot{\leq} \delta^{2_N} \dot{\times} d_N(j_{n-2}, j_{n-1})$$

$\dot{\leq} \delta^{n-1_N} \dot{\times} d_N(j_0, j_1)$ (3.9)

Now, if $m < n$, we have

$$d_N(j_n, j_m) \dot{\leq} d_N(j_n, j_{n-1}) \dot{+} d_N(j_{n-1}, j_{n-2}) \dot{+} \ldots . \dot{+} d_N(j_{m+1}, j_m)$$

$$\dot{\leq} \delta^{n-1_N} \dot{\times} d_N(j_0, j_1) \dot{+} \delta^{n-2_N} \dot{\times} d_N(j_0, j_1) \dot{+} \ldots . \dot{+} \delta^{m_N} \dot{\times} d_N(j_0, j_1)$$

$$\dot{\leq} \delta^{m_N} \dot{\times} \left(\dot{1} \dot{+} \delta \dot{+} \delta^{2_N} \dot{+} \ldots \ldots . \dot{+} \delta^{n-m-1_N}\right) \dot{\times} d_N(j_0, j_1)$$

$\dot{\leq} \frac{\delta^{m_N} \dot{\times} d_N(j_0, j_1)}{\dot{1} \dot{-} \delta}$(3.10)

Since $\delta^{m_N} \dot{<} \dot{1}$ and $d_N(j_0, j_1) \in \mathbb{R}(N)$ is fixed, we can make$\frac{\delta^{m_N} \dot{\times} d_N(j_0, j_1)}{\dot{1} \dot{-} \delta}$as small as we want by taking $m$sufficiently large. This shows that $\{j_n\}$is anon-Newtonian Cauchy sequence.

Now, we give some fixed-point results for expansive mappings in a non-Newtonian complete metric space.Our first main result as follows.

Theorem 3.7 Let $f: M \to M$ be a surjection and non-Newtonian expansive mapping on a non-Newtonian complete metric space M. Then f has a unique fixed point.

Proof: Let $j_0 \in M$ be arbitrary. Since f is surjection, then there exists $j_1 \in M$ such that $j_0 = f(j_1)$. By continuing this process, we get

$j_n = f(j_{n+1}),\ n = 0, 1, 2, \ldots .$ (3.11)

In case $j_{n_0} = j_{n_0+1}$ for some n_0, then it is clear that $j_{n_0}$is a fixed point of f. Now assume that $j_n \neq j_{n-1}$ for all n. Since $f$non-Newtonian expansive mapping

$$d_N(j_{n-1}, j_n) = d_N\big(f(j_n), f(j_{n+1})\big) \dot{\geq} \delta \dot{\times} d_N(j_n, j_{n+1})$$

Consequently

$d_N(j_n, j_{n+1}) \dot{\leq} \left(\dot{1} \dot{/} \delta\right) \dot{\times} d_N(j_{n-1}, j_n) = \kappa \dot{\times} d_N(j_{n-1}, j_n)$(3.12)

where $\kappa = \dot{1} \dot{/} \delta \dot{<} \dot{1}$.

Then by Lemma 3.6,$\{j_n\}$is anNN-Cauchy sequence. Since (M, d_N) is non-Newtonian complete,there exists a point $j$in Msuch that $j_n \xrightarrow{N} j$.Since f is surjection on M, there exists $u \in M$ such that $j = f(u)$.We now show that $j$is a fixedpoint of the mapping f. It follows from (3.6) and (3.11) that

$$d_N(j_n, j) = d_N\big(f(j_{n+1}), f(u)\big) \dot{\geq} \delta \dot{\times} d_N(j_{n+1}, u)$$

Since $j_n \xrightarrow{N} j$, it follows that $d_N(j_{n+1}, u) \xrightarrow{N} \dot{0}$ and hence $j_{n+1} \xrightarrow{N} u$. By uniqueness of non-Newtonian limit, we have $j = u$. This shows that $j$is a fixed point of f. We conclude the proof by showing that $j$isthe only fixed point. Suppose that k is also a fixed point, that is, suppose $f(k) = k$, then

$$d_N(j, k) = d_N\big(f(j), f(k)\big) \dot{\geq} \delta \dot{\times} d_N(j, k)$$

Since $\delta \dot{>} \dot{1}$, this implies that $d_N(j, k) = \dot{0}$ and hence $j = k$.

Theorem 3.8 Let (M, d_N)be a non-Newtonian complete metric space and let $f$be a surjective self-mapping of M. If $f$satisfies condition $(NN1)$, then $f$has a unique fixed point in M.

Proof. Using the hypothesis, it can be easily seen that $f$isinjective. Indeed, if we take $f(i) = f(j)$, then, using (3.7), we get

$$\dot{0} = d_N\big(f(i), f(j)\big) \dot{\geq} a \dot{\times} d_N(i, j) \dot{+} b \dot{\times} d_N\big(i, f(i)\big) \dot{+} c \dot{\times} d_N\big(j, f(j)\big)$$

And so $d_N(i, j) = \dot{0}$; that is, we have $i = j$, since $a \dot{>} \dot{1}$.

Let us denote the inverse mapping of $f$by F. Let $j_0 \in M$and define the sequence $\{j_n\}$as follows:

$$j_1 = F(j_0), \qquad j_2 = F(j_1) = F^2(j_0),$$

$j_3 = F(j_2) = FF^2(j_0) = F^3(j_0), \ldots\ldots, j_{n+1} = F(j_n) = F^{n+1}(j_0)$,(3.13)

Suppose that $j_n \neq j_{n+1}$for all n. Using (3.7) and (3.13), we have

$$d_N(j_{n-1}, j_n) = d_N\left(ff^{-1}(j_{n-1}), ff^{-1}(j_n)\right)$$

$$\dot{\geq} a \dot{\times} d_N\left(f^{-1}(j_{n-1}), f^{-1}(j_n)\right) \dot{+} b \dot{\times} d_N\left(f^{-1}(j_{n-1}), ff^{-1}(j_{n-1})\right)$$

$$\dot{+} c \dot{\times} d_N\left(f^{-1}(j_n), ff^{-1}(j_n)\right)$$

$$\dot{\geq} a \dot{\times} d_N\left(F(j_{n-1}), F(j_n)\right) \dot{+} b \dot{\times} d_N(F(j_{n-1}), j_{n-1}) \dot{+} c \dot{\times} d_N(F(j_n), j_n)$$

$$\dot{\geq} a \dot{\times} d_N(j_n, j_{n+1}) \dot{+} b \dot{\times} d_N(j_n, j_{n-1}) \dot{+} c \dot{\times} d_N(j_{n+1}, j_n)$$

$$= (a \dot{+} c) \dot{\times} d_N(j_n, j_{n+1}) \dot{+} b \dot{\times} d_N(j_n, j_{n-1})$$

which implies that

$(\dot{1} \dot{-} b) \dot{\times} d_N(j_{n-1}, j_n) \dot{\geq} (a \dot{+} c) \dot{\times} d_N(j_n, j_{n+1})$(3.14)

Clearly, we have $a \dot{+} c \neq \dot{0}$. Hence, we obtain

$d_N(j_n, j_{n+1}) \dot{\leq} (\dot{1} \dot{-} b) \dot{/} (a \dot{+} c) \dot{\times} d_N(j_{n-1}, j_n) = \delta \dot{\times} d_N(j_{n-1}, j_n)$(3.15)

Where $\delta = (\dot{1} \dot{-} b) \dot{/} (a \dot{+} c)$, then we get $\delta \dot{<} \dot{1}$, since $a \dot{+} b \dot{+} c \dot{>} 1$. Repeating this process in condition (3.15), wefind

$$d_N(j_n, j_{n+1}) \dot{\leq} \delta^{n_N} \dot{\times} d_N(j_0, j_1)$$

and by Lemma 3.6,$\{j_n\}$is an NN-Cauchy sequence. Since(M, d_N) is non-Newtonian complete,there exists a point $j$in Msuch that $j_n \xrightarrow{N} j$and therefore

$d_N(j_n, j) \xrightarrow{N} \dot{0}, \;\; d_N(j_{n+1}, j_n) \xrightarrow{N} \dot{0}.$

Using the subjectivity of hypothesis, there exists $u \in M$ such that $j = f(u)$.From (3.7) and (3.13), we have

$$d_N(j_n, j) = d_N\left(f(j_{n+1}), f(u)\right)$$

$$\dot{\geq} a \dot{\times} d_N(j_{n+1}, p) \dot{+} b \dot{\times} d_N\left(j_{n+1}, f(j_{n+1})\right) \dot{+} c \dot{\times} d_N\left(u, f(u)\right)$$

$$= a \dot{\times} d_N(j_{n+1}, p) \dot{+} b \dot{\times} d_N(j_{n+1}, j_n) \dot{+} c \dot{\times} d_N(u, f(u))$$

If we take limit for $n \xrightarrow{N} \infty$, we obtain

$$\dot{0} \dot{\geq} (a \dot{+} c) \dot{\times} d_N(u, j)$$

which implies that $d_N(u, j) = \dot{0}$; that is, we have $j = u$, since $a \dot{+} c \dot{>} \dot{1}$. This shows that $j$is a fixed point of f.

Now we show the uniqueness of j. Let $k$be another fixed point of $f$with $j \neq k$. Using (3.7),we get

$$d_N(j, k) = d_N(f(j), f(k))$$

$$\dot{\geq} a \dot{\times} d_N(j, k) \dot{+} b \dot{\times} d_N(j, f(j)) \dot{+} c \dot{\times} d_N(k, f(k))$$

$$= a \dot{\times} d_N(j, k) \dot{+} b \dot{\times} d_N(j, j) \dot{+} c \dot{\times} d_N(k, k)$$

$= a \dot{\times} d_N(j, k)$(3.16)

which implies that $j = k$, since $a \dot{>} \dot{1}$. Consequently, $f$has aunique fixed point j.

If we take $b = c$ in condition $(NN1)$, then we obtain the following corollary.

Corollary 3.9 Let (M, d_N)be a non-Newtonian complete metric space and let $f$be a surjective self-mapping of M. If there exist real numbers a, b satisfying $b \dot{\geq} \dot{0}$and $a \dot{>} \dot{1}$ such that

$d_N(f(i), f(j)) \dot{\geq} a \dot{\times} d_N(i, j) \dot{+} b \dot{\times} max\{d_N(i, f(i)), d_N(j, f(j))\}$(3.17)

for each$i, j \in M$, then $f$has a unique fixed point in M.

Now, we prove following common fixed point result.

Theorem 3.10 Let $f, g: M \to M$ be two surjective mappings of a non-Newtonian complete metric space (M, d_N). Suppose that f and g satisfying inequalities

$$d_N\left(f\left(g(j)\right), g(j)\right) \dot{+} \kappa \dot{\times} d_N\left(f\left(g(j)\right), j\right) \dot{\geq} a \dot{\times} d_N(g(j), j) \quad (3.18)$$

$$d_N\left(g\left(f(j)\right), f(j)\right) \dot{+} \kappa \dot{\times} d_N\left(g\left(f(j)\right), j\right) \dot{\geq} b \dot{\times} d_N(f(j), j) \quad (3.19)$$

for $j \in M$ and some non-Newtonian real numbers a, b and κ with $a \dot{-} \kappa \dot{>} \dot{1} \dot{+} k$ and $b \dot{-} \kappa \dot{>} \dot{1} \dot{+} \kappa$. If f or g is non-Newtoniancontinuous, then $f$and g have a common fixed point in M.

Proof Let j_0 be an arbitrary point in M. Since f is surjective, there exists $j_1 \in M$ such that $j_0 = f(j_1)$. Also, since g is surjective, there exists $j_2 \in M$ such that$j_2 = g(j_1)$. Continuing this process, we construct a sequence $\{j_n\}$ in M such that $j_{2n} = f(j_{2n+1})$ and $j_{2n+1} = g(j_{2n+2})$ for all $n \in \mathbb{N}$. Now for $n \in \mathbb{N}$, by (3.18) we have

$$d_N\left(f\left(g(j_{2n+2})\right), g(j_{2n+2})\right) \dot{+} \kappa \dot{\times} d_N\left(f\left(g(j_{2n+2})\right), j_{2n+2}\right) \dot{\geq} a \dot{\times} d_N(g(j_{2n+2}), j_{2n+2})$$

Thus

$$d_N(j_{2n}, j_{2n+1}) \dot{+} \kappa \dot{\times} d_N(j_{2n}, j_{2n+2}) \dot{\geq} a \dot{\times} d_N(j_{2n+1}, j_{2n+2})$$

which implies that

$$d_N(j_{2n}, j_{2n+1}) \dot{+} \kappa \dot{\times} [d_N(j_{2n}, j_{2n+1}) \dot{+} d_N(j_{2n+1}, j_{2n+2})] \dot{\geq} a \dot{\times} d_N(j_{2n+1}, j_{2n+2})$$

Hence

$$d_N(j_{2n+1}, j_{2n+2}) \dot{\leq} \left[\left(\dot{1} \dot{+} \kappa\right) \dot{/} (a \dot{-} \kappa)\right] \dot{\times} d_N(j_{2n}, j_{2n+1}) \quad (3.20)$$

On other hand, from (3.19), we have

$$d_N\left(g\left(f(j_{2n+1})\right), f(j_{2n+1})\right) \dot{+} \kappa \dot{\times} d_N\left(g\left(f(j_{2n+1})\right), j_{2n+1}\right) \dot{\geq} b \dot{\times} d_N(f(j_{2n+1}), j_{2n+1})$$

Thus

$$d_N(j_{2n-1}, j_{2n}) \dot{+} \kappa \dot{\times} d_N(j_{2n-1}, j_{2n+1}) \dot{\geq} b \dot{\times} d_N(j_{2n}, j_{2n+1})$$

which implies that

$$d_N(j_{2n-1}, j_{2n}) \dot{+} \kappa \dot{\times} [d_N(j_{2n-1}, j_{2n}) \dot{+} d_N(j_{2n}, j_{2n+1})] \dot{\geq} b \dot{\times} d_N(j_{2n}, j_{2n+1})$$

Hence

$d_N(j_{2n}, j_{2n+1}) \dot{\leq} \left[(\dot{1} \dot{+} \kappa) \dot{/} (b \dot{-} \kappa) \right] \dot{\times} d_N(j_{2n-1}, j_{2n})$(3.21)

Let $\delta = maj\{[(\dot{1} \dot{+} \kappa) \dot{/} (a \dot{-} \kappa)], [(\dot{1} \dot{+} v) \dot{/} (b \dot{-} \kappa)]\} \dot{<} \dot{1}$

Then by combining (3.20) and (3.21), we have

$d_N(j_n, j_{n+1}) \leq \delta \dot{\times} d_N(j_{n-1}, j_n)$(3.22)

where $\delta \in \dot{[}0,1\dot{)}$, $\forall\, n \in \mathbb{N}$. Then by Lemma 3.6, the sequence $\{j_n\}$ is an NN-Cauchy sequence. Since (M, d_N) is non-Newtonian complete,there exists a point $j$in Msuch that $j_n \xrightarrow{N} j$. Therefore $j_{2n+1} \xrightarrow{N} j$ and $j_{2n+2} \xrightarrow{N} j$ as $n \to +\infty$. Without loss of generality, we may assume that f is continuous, then $f(j_{2n+1}) \xrightarrow{N} f(j)$as $n \to +\infty$. But $f(j_{2n+1}) = j_{2n} \xrightarrow{N} j$ as $n \to +\infty$. Thus, we have $f(j) = j$. Since g is surjection on M, there exists $u \in M$ such that $j = g(u)$.We now show that $j$is a common fixedpoint of the mapping f and g. It follows from (3.18) that

$$d_N\left(f(g(u)), g(u)\right) \dot{+} \kappa \dot{\times} d_N(f(g(u)), u) \dot{\geq} a \dot{\times} d_N(g(u), u)$$

$$\Longrightarrow \qquad \dot{0} \dot{+} \kappa \dot{\times} d_N(j, u) \dot{\geq} a \dot{\times} d_N(j, u)$$

$$\Longrightarrow \qquad \dot{0} \dot{\geq} (a \dot{-} \kappa) \dot{\times} d_N(j, u)$$

Since $a \dot{-} \kappa \dot{>} \dot{1} \dot{+} \kappa$, we conclude that $d_N(j, u) = \dot{0}$ and consequently $j = u$. Hence $f(j) = g(j) = j$. Therefore j is a common fixed point of $f$and g.

By taking $f = g$ in Corollary 3.9we have the following Corollary.

Corollary 3.10 Let $f: M \to M$ be two surjective mappings of a non-Newtonian complete metric space (M, d_N). Suppose that f satisfying inequality

$$d_N\big(f^2(j), f(j)\big) \dot{+} \kappa \dot{\times} d_N(f^2(j), j) \dot{\geq} a \dot{\times} d_N(f(j), j) \tag{3.24}$$

for $j \in M$ and some nonnegative real numbers a, b and k with $a \dot{-} \kappa \dot{>} \dot{1} \dot{+} \kappa$. If f is continuous, then f has a fixed point in M.

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