On integral inequalities for product and quotient of two multiplicatively convex functions

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Abstract

In this paper, we derived integral inequalities of Hermite-Hadamard type in the setting of multiplicative calculus for multiplicatively convex and convex functions. We also derived integral inequalities of Hermite-Hadamard type for product and quotient of multiplicatively convex and convex functions in multiplicative calculus.

Keywords: Convex sets, convex functions, multiplicative integrals and Hermite Hadamard inequalities.

1 Introduction

During the period 1967-1970, Grossman and Katz defined a new type of derivative and integral replacing the roles of addition and subtraction with multiplication and division, and thus established a new calculus, called multiplicative calculus or non-Newtonian calculus. However, the multiplicative calculus is not as popular as the calculus of Newton and Leibnitz despite the fact that it addresses all the problems that are expected from the subject of calculus. The multiplicative calculus has a relatively restrictive application area compared to the calculus of Newton and Leibnitz. In reality, it only covers positive functions. Therefore, one might ask whether it is reasonable to develop a new instrument with a restrictive purpose, while a well-developed instrument with a wider scope has already been created. The answer is similar to why mathematicians use a polar coordinate system while there is a system of rectangular coordinates, which well describes the points of a plane ([10, 29]).

Recall that the multiplicative integral called * integral is denoted by $\int_{u_1}^{u_2}(f(x))^{dx}$ while the ordinary integral is denoted by $\int_{u_1}^{u_2} f(x)dx$.This is due to the fact that the sum of the terms of the product in the definition of a proper Riemann integral of f on $[u_1, u_2]$ is replaced with the product of terms raised to certain powers.

It is also known that [2] if f is positive and Riemann integrable on $[u_1, u_2]$,

then it is *integrable on $[u_1, u_2]$ and

$$\int_{u_1}^{u_2} (f(x))^{dx} = e^{\int_{u_1}^{u_2} \ln(f(x))dx}.$$

Consistent with [2] , the following results and notations will be needed in the sequel.

(i) $\int_{u_1}^{u_2} ((f(x))^p)^{dx} = \int_{u_1}^{u_2} \left((f(x))^{dx}\right)^p$,

(ii) $\int_{u_1}^{u_2} (f(x)g(x))^{dx} = \int_{u_1}^{u_2} (f(x))^{dx} . \int_{u_1}^{u_2} (g(x))^{dx}$,

(iii) $\int_{u_1}^{u_2} (\frac{f(x)}{g(x)})^{dx} = \frac{\int_{u_1}^{u_2} (f(x))^{dx}}{\int_{u_1}^{u_2} (g(x))^{dx}}$,

(iv) $\int_{u_1}^{u_2} (f(x))^{dx} = \int_{u_1}^{c} (f(x))^{dx} . \int_{c}^{u_2} (f(x))^{dx}, \quad u_1 \leq c \leq u_2$.

(v) $\int_{u_1}^{u_1} (f(x))^{dx} = 1$ and $\int_{u_1}^{u_2} (f(x))^{dx} = \left(\int_{u_1}^{u_2} (f(x))^{dx}\right)^{-1}$.

The concept of convexity and its variant forms have played a fundamental role in the development of various fields. Hermite (1883) and Hadamard (1896) independently shown that the convex functions are related to an integral inequality known as Hermite-Hadamard inequality.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on interval I and $u_1, u_2 \in I$. Then following inequality holds

$$f\left(\frac{u_1 + u_2}{2}\right) \leq \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f(x)dx \leq \frac{f(u_1) + f(u_2)}{2}, \tag{1}$$

which is known as Hermite-Hadamrd integral inequality for the convex functions. By an appropriate selection of the mapping f, some classical inequalities for the mean can be derived from (1). Both inequalities in (1) holds in reverse direction if f is concave. For several recent results concerning these types of inequalities, we refer to [1, 6, 13, 15, 21, 23, 25] and references stated therein.

The main purpose of this article is to establish integral inequalities of the Hermite Hadamard type for convex functions and multiplicatively convex functions and their products and quotient in the setting of the multiplicative calculus.

2 Preliminaries

Definition 1 *A non-empty set K is said to be convex, if for every $u_1, u_2 \in K$ we have*

$$u_1 + \mu(u_2 - u_1) \in K, \quad \forall\ \mu \in [0, 1].$$

Definition 2 *A function f is said to be convex function on set K, if*

$$f(u_1 + \mu(u_2 - u_1)) \leq f(u_1) + \mu(f(u_2) - f(u_1)),\ \forall\ \mu \in [0, 1].$$

Definition 3 *A function f is said to be log or multiplicatively convex function on set K, if*

$$f(u_1 + \mu(u_2 - u_1)) \leq (f(u_1))^{1-\mu} . (f(u_2))^{\mu},\ \forall\ \mu \in [0, 1].$$

Definition 4 *A function f is said to be quasi convex function on set K, if*

$$f(u_1+\mu(u_2-u_1)) \leq \max(f(u_1), f(u_2)),\ \forall\ \mu \in [0,1].$$

From the above definitions we have a relation

$$\begin{aligned} f(u_1+\mu(u_2-u_1)) &\leq (f(u_1))^{1-\mu}.(f(u_2))^{\mu} \\ &\leq f(u_1)+\mu(f(u_2)-f(u_1)) \\ &\leq \max(f(u_1), f(u_2)). \end{aligned}$$

3 Hermite-Hadamard Integral Inequalities

In this section, we derive integral inequalities of the Hermite Hadamard type for positive functions in the framework of multiplicative calculus.

Theorem 5 *Let f be a positive and multiplicatively convex function on interval $[u_1, u_2]$, then following inequalities hold*

$$f\left(\frac{u_1+u_2}{2}\right) \leq \left(\int_{u_1}^{u_2} (f(x))^{dx}\right)^{\frac{1}{u_2-u_1}} \leq G(f(u_1), f(u_2)), \tag{2}$$

where $G(.,.)$ is a geometric mean. The above is called Hermite Hadamard Integral Inequalities for multiplicatively convex function.

Proof. Let f be a positive and multiplicatively convex function. Note that

$$\begin{aligned} \ln f\left(\frac{u_1+u_2}{2}\right) &= \ln\left(f\left(\frac{(1-\mu)u_1+\mu u_2+\mu u_1+(1-\mu)u_2}{2}\right)\right) \\ &= \ln\left(f\left(\frac{(1-\mu)u_1+\mu u_2}{2}+\frac{\mu u_1+(1-\mu)u_2}{2}\right)\right) \\ &\leq \ln\left(f\left(\frac{(1-\mu)u_1+\mu u_2}{2}\right)^{\frac{1}{2}}.f\left(\frac{\mu u_1+(1-\mu)u_2}{2}\right)^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\ln\left(f\left(\frac{(1-\mu)u_1+\mu u_2}{2}\right)\right)+\frac{1}{2}\ln\left(f\left(\frac{\mu u_1+(1-\mu)u_2}{2}\right)\right). \end{aligned}$$

Integrating the above inequality with respect to μ on $[0,1]$, we have

$$\begin{aligned} \ln f\left(\frac{u_1+u_2}{2}\right) &\leq \frac{1}{2}\int_0^1 \ln\left(f\left(\frac{(1-\mu)u_1+\mu u_2}{2}\right)\right)d\mu+\frac{1}{2}\int_0^1 \ln\left(f\left(\frac{\mu u_1+(1-\mu)u_2}{2}\right)\right)d\mu \\ &= \frac{1}{2}\left[\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln(f(x))\,dx+\frac{1}{u_1-u_2}\int_{u_2}^{u_1}\ln(f(x))\,dx\right] \\ &= \frac{1}{2}\left[\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln(f(x))\,dx+\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln(f(x))\,dx\right] \\ &= \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln(f(x))\,dx \end{aligned}$$

Thus

$$\begin{aligned} f\left(\frac{u_1+u_2}{2}\right) &\leq e^{\left(\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln(f(x))dx\right)} \\ &= \left(\int_{u_1}^{u_2}(f(x))^{dx}\right)^{\frac{1}{u_2-u_1}}. \end{aligned}$$

Hence

$$f\left(\frac{u_1+u_2}{2}\right) \leq \left(\int_{u_1}^{u_2}(f(x))^{dx}\right)^{\frac{1}{u_2-u_1}}. \tag{3}$$

Consider the second inequality

$$\begin{aligned} \left(\int_{u_1}^{u_2}(f(x))^{dx}\right)^{\frac{1}{u_2-u_1}} &= \left(e^{\left(\int_{u_1}^{u_2}\ln(f(x))dx\right)}\right)^{\frac{1}{u_2-u_1}} \\ &= e^{\frac{1}{u_2-u_1}\left(\int_{u_1}^{u_2}\ln(f(x))dx\right)} \\ &= e^{\left(\int_0^1\ln(f(u_1+\mu(u_2-u_1)))d\mu\right)} \\ &\leq e^{\left(\int_0^1\ln\left((f(u_1))^{1-\mu}(f(u_2))^{\mu}\right)d\mu\right)} \\ &= e^{\left(\int_0^1(1-\mu)\ln f(u_1)+\mu\ln f(u_2)d\mu\right)} \\ &= e^{\left(\ln(f(u_1).f(u_2))^{\frac{1}{2}}\right)} \\ &= \sqrt{f(u_1).f(u_2)} \\ &= G(f(u_1),f(u_2)), \end{aligned}$$

hence

$$\left(\int_{u_1}^{u_2}(f(x))^{dx}\right)^{\frac{1}{u_2-u_1}} \leq G(f(u_1),f(u_2)). \tag{4}$$

Combining (3) and (4), we have

$$f\left(\frac{u_1+u_2}{2}\right) \leq \left(\int_{u_1}^{u_2}(f(x))^{dx}\right)^{\frac{1}{u_2-u_1}} \leq G(f(u_1),f(u_2)).$$

■

Example 6 *Note that $f(x)=e^{x^2}$ is a multiplicatively convex function. Suppose that $u_1=1$ and $u_2=3$. Then*

$$\begin{aligned} f\left(\frac{u_1+u_2}{2}\right) &= e^{\left(\frac{1+3}{2}\right)^2} = 54.5982 \\ \left(\int_{u_1}^{u_2}(f(x))^{dx}\right)^{\frac{1}{u_2-u_1}} &= \left(\int_1^3\left(e^{x^2}\right)^{dx}\right)^{\frac{1}{2}} = 76.1979 \\ G(f(u_1),f(u_2)) &= 148.4132. \end{aligned}$$

which is true.

Theorem 7 *Let f and g be positive and multiplicatively convex functions on $[u_1, u_2]$, then following inequalities hold*

$$f\left(\frac{u_1+u_2}{2}\right) g\left(\frac{u_1+u_2}{2}\right) \leq \left(\int_{u_1}^{u_2} (f(x))^{dx} . \int_{u_1}^{u_2} (g(x))^{dx}\right)^{\frac{1}{u_2-u_1}} \leq G(f(u_1), f(u_2)).G(g(u_1), g(u_2)). \tag{5}$$

Proof. Let f and g be positive and multiplicatively convex functions. Note that

$$\begin{aligned}
\ln\left(f\left(\frac{u_1+u_2}{2}\right) g\left(\frac{u_1+u_2}{2}\right)\right) &= \ln\left(f\left(\frac{u_1+u_2}{2}\right)\right) + \ln\left(g\left(\frac{u_1+u_2}{2}\right)\right) \\
&= \ln\left(f\left(\frac{(1-\mu)u_1+\mu u_2+\mu u_1+(1-\mu)u_2}{2}\right)\right) \\
&\quad + \ln\left(g\left(\frac{(1-\mu)u_1+\mu u_2+\mu u_1+(1-\mu)u_2}{2}\right)\right) \\
&= \ln\left(f\left(\frac{(1-\mu)u_1+\mu u_2}{2}+\frac{\mu u_1+(1-\mu)u_2}{2}\right)\right) \\
&\quad + \ln\left(g\left(\frac{(1-\mu)u_1+\mu u_2}{2}+\frac{\mu u_1+(1-\mu)u_2}{2}\right)\right) \\
&\leq \ln\left(\left(f\left((1-\mu)u_1+\mu u_2\right)\right)^{\frac{1}{2}} . \left(f\left(\mu u_1+(1-\mu)u_2\right)\right)^{\frac{1}{2}}\right) \\
&\quad + \ln\left(\left(g\left((1-\mu)u_1+\mu u_2\right)\right)^{\frac{1}{2}} . \left(g\left(\mu u_1+(1-\mu)u_2\right)\right)^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\ln\left(f\left((1-\mu)u_1+\mu u_2\right)\right) + \frac{1}{2}\ln\left(f\left(\mu u_1+(1-\mu)u_2\right)\right) \\
&\quad + \frac{1}{2}\ln\left(g\left((1-\mu)u_1+\mu u_2\right)\right) + \frac{1}{2}\ln\left(g\left(\mu u_1+(1-\mu)u_2\right)\right).
\end{aligned}$$

Integrating above inequality with respect to μ on $[0,1]$, we have

$$\begin{aligned}
\ln\left(f\left(\frac{u_1+u_2}{2}\right) g\left(\frac{u_1+u_2}{2}\right)\right) &\leq \int_0^1 \left[\frac{1}{2}\ln\left(f\left((1-\mu)u_1+\mu u_2\right)\right) + \frac{1}{2}\ln\left(f\left(\mu u_1+(1-\mu)u_2\right)\right)\right] d\mu \\
&\quad + \int_0^1 \left[\frac{1}{2}\ln\left(g\left((1-\mu)u_1+\mu u_2\right)\right) + \frac{1}{2}\ln\left(g\left(\mu u_1+(1-\mu)u_2\right)\right)\right] d\mu \\
&= \frac{1}{2(u_2-u_1)}\int_{u_1}^{u_2} \ln\left(f(x)\right) dx + \frac{1}{2(u_1-u_2)}\int_{u_2}^{u_1} \ln\left(f(x)\right) dx \\
&\quad + \frac{1}{2(u_2-u_1)}\int_{u_1}^{u_2} \ln\left(g(x)\right) dx + \frac{1}{2(u_1-u_2)}\int_{u_2}^{u_1} \ln\left(g(x)\right) dx \\
&= \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \ln\left(f(x)\right) dx + \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \ln\left(g(x)\right) dx
\end{aligned}$$

$$
\begin{aligned}
f\left(\frac{u_1+u_2}{2}\right) g\left(\frac{u_1+u_2}{2}\right) &\leq e^{\left(\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln(f(x))dx+\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln(g(x))dx\right)} \\
&= \left(e^{\int_{u_1}^{u_2}\ln(f(x))dx+\int_{u_1}^{u_2}\ln(g(x))dx}\right)^{\frac{1}{u_2-u_1}} \\
&= \left(e^{\int_{u_1}^{u_2}\ln(f(x))dx}.e^{\int_{u_1}^{u_2}\ln(g(x))dx}\right)^{\frac{1}{u_2-u_1}} \\
&= \left(\int_{u_1}^{u_2}(f(x))^{dx}.\int_{u_1}^{u_2}(g(x))^{dx}\right)^{\frac{1}{u_2-u_1}}.
\end{aligned}
$$

Hence

$$
f\left(\frac{u_1+u_2}{2}\right) g\left(\frac{u_1+u_2}{2}\right) \leq \left(\int_{u_1}^{u_2}(f(x))^{dx}.\int_{u_1}^{u_2}(g(x))^{dx}\right)^{\frac{1}{u_2-u_1}}. \quad (6)
$$

Consider the second inequality:

$$
\begin{aligned}
\left(\int_{u_1}^{u_2}(f(x))^{dx}.\int_{u_1}^{u_2}(g(x))^{dx}\right)^{\frac{1}{u_2-u_1}} &= \left(e^{\int_{u_1}^{u_2}\ln(f(x))dx+\int_{u_1}^{u_2}\ln(g(x))dx}\right)^{\frac{1}{u_2-u_1}} \\
&= \left(e^{(u_2-u_1)\left\{\int_0^1\ln(f(u_1+\mu(u_2-u_1)))d\mu+\int_{u_1}^{u_2}\ln(g(u_1+\mu(u_2-u_1)))d\mu\right\}}\right)^{\frac{1}{u_2-u_1}} \\
&= e^{\int_0^1\ln(f(u_1+\mu(u_2-u_1)))d\mu+\int_{u_1}^{u_2}\ln(g(u_1+\mu(u_2-u_1)))d\mu} \\
&\leq e^{\int_0^1\ln\left((f(u_1))^{1-\mu}.(f(u_2))^{\mu}\right)d\mu+\int_0^1\ln\left((g(u_1))^{1-\mu}.(g(u_2))^{\mu}\right)d\mu} \\
&= e^{\int_0^1\ln((1-\mu)f(u_1)+\mu f(u_2))d\mu+\int_0^1\ln((1-\mu)g(u_1)+\mu g(u_2))d\mu} \\
&= e^{\ln(f(u_1).f(u_2))^{\frac{1}{2}}+\ln(g(u_1).g(u_2))^{\frac{1}{2}}} \\
&= \sqrt{(f(u_1).f(u_2))}.\sqrt{(g(u_1).g(u_2))} \\
&= G\left(f(u_1),f(u_2)\right).G(g(u_1),g(u_2)).
\end{aligned}
$$

Hence

$$
\left(\int_{u_1}^{u_2}(f(x))^{dx}.\int_{u_1}^{u_2}(g(x))^{dx}\right)^{\frac{1}{u_2-u_1}} \leq G\left(f(u_1),f(u_2)\right).G(g(u_1),g(u_2)). \quad (7)
$$

Combining (6) and (7), we have

$$
f\left(\frac{u_1+u_2}{2}\right) g\left(\frac{u_1+u_2}{2}\right) \leq \left(\int_{u_1}^{u_2}(f(x))^{dx}.\int_{u_1}^{u_2}(g(x))^{dx}\right)^{\frac{1}{u_2-u_1}} \leq G(f(u_1),f(u_2)).G(g(u_1),g(u_2)).
$$

This is called Hermite Hadamard type integral inequality for the product of multiplicatively convex functions. ■

Example 8 *Note that* $f(x) = e^{x^2}$ *and* $g(x) = e^{|x|}$ *are multiplicatively convex functions. Let* $u_1 = 1$ *and* $u_2 = 3$. *Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &= e^{\left(\frac{1+3}{2}\right)^2}.e^{\frac{1+3}{2}} = 403.4288, \\ \left(\int_{u_1}^{u_2} (f(x))^{dx} . \int_{u_1}^{u_2} (g(x))^{dx}\right)^{\frac{1}{u_2-u_1}} &= \left(\int_1^3 \left(e^{x^2}\right)^{dx} . \int_1^3 \left(e^{|x|}\right)^{dx}\right)^{\frac{1}{2}} = 563.0302, \\ G(f(u_1), f(u_2)).G(g(u_1), g(u_2)) &= \sqrt{e^1.e^{3^2}}.\sqrt{e^1.e^3} = 1096.6335. \end{aligned}$$

which is true.

Theorem 9 *Let* f *and* g *be positive and multiplicatively convex functions on* $[u_1, u_2]$. *Then the following inequalities*

$$\frac{f\left(\frac{u_1+u_2}{2}\right)}{g\left(\frac{u_1+u_2}{2}\right)} \leq \left(\frac{\int_{u_1}^{u_2} (f(x))^{dx}}{\int_{u_1}^{u_2} (g(x))^{dx}}\right)^{\frac{1}{u_2-u_1}} \leq \frac{G(f(u_1), f(u_2))}{G(g(u_1), g(u_2))} \tag{8}$$

hold.

Proof. Note that

$$\begin{aligned} \ln\left(\frac{f\left(\frac{u_1+u_2}{2}\right)}{g\left(\frac{u_1+u_2}{2}\right)}\right) &= \ln\left(f\left(\frac{u_1+u_2}{2}\right)\right) - \ln\left(g\left(\frac{u_1+u_2}{2}\right)\right) \\ &= \ln\left(f\left(\frac{(1-\mu)u_1 + \mu u_2 + \mu u_1 + (1-\mu)u_2}{2}\right)\right) \\ &\quad - \ln\left(g\left(\frac{(1-\mu)u_1 + \mu u_2 + \mu u_1 + (1-\mu)u_2}{2}\right)\right) \\ &= \ln\left(f\left(\frac{(1-\mu)u_1 + \mu u_2}{2} + \frac{\mu u_1 + (1-\mu)u_2}{2}\right)\right) \\ &\quad - \ln\left(g\left(\frac{(1-\mu)u_1 + \mu u_2}{2} + \frac{\mu u_1 + (1-\mu)u_2}{2}\right)\right) \\ &\leq \ln\left(\left(f\left((1-\mu)u_1 + \mu u_2\right)\right)^{\frac{1}{2}} . \left(f\left(\mu u_1 + (1-\mu)u_2\right)\right)^{\frac{1}{2}}\right) \\ &\quad - \ln\left(\left(g\left((1-\mu)u_1 + \mu u_2\right)\right)^{\frac{1}{2}} . \left(g\left(\mu u_1 + (1-\mu)u_2\right)\right)^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\ln\left(f\left((1-\mu)u_1 + \mu u_2\right)\right) + \frac{1}{2}\ln\left(f\left(\mu u_1 + (1-\mu)u_2\right)\right) \\ &\quad - \frac{1}{2}\ln\left(g\left((1-\mu)u_1 + \mu u_2\right)\right) - \frac{1}{2}\ln\left(g\left(\mu u_1 + (1-\mu)u_2\right)\right). \end{aligned}$$

Integrating the above inequality with respect to μ on $[0,1]$, we have

$$\begin{aligned}
\ln\left(\frac{f\left(\frac{u_1+u_2}{2}\right)}{g\left(\frac{u_1+u_2}{2}\right)}\right) &\leq \int_0^1 \left[\frac{1}{2}\ln\left(f\left((1-\mu)u_1+\mu u_2\right)\right)+\frac{1}{2}\ln\left(f\left(\mu u_1+(1-\mu)u_2\right)\right)\right]d\mu \\
&\quad -\int_0^1 \left[\frac{1}{2}\ln\left(g\left((1-\mu)u_1+\mu u_2\right)\right)+\frac{1}{2}\ln\left(g\left(\mu u_1+(1-\mu)u_2\right)\right)\right]d\mu \\
&= \frac{1}{2(u_2-u_1)}\int_{u_1}^{u_2}\ln\left(f(x)\right)dx+\frac{1}{2(u_1-u_2)}\int_{u_2}^{u_1}\ln\left(f(x)\right)dx \\
&\quad -\frac{1}{2(u_2-u_1)}\int_{u_1}^{u_2}\ln\left(g(x)\right)dx+\frac{1}{2(u_1-u_2)}\int_{u_2}^{u_1}\ln\left(g(x)\right)dx \\
&= \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln\left(f(x)\right)dx-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln\left(g(x)\right)dx \\
\frac{f\left(\frac{u_1+u_2}{2}\right)}{g\left(\frac{u_1+u_2}{2}\right)} &\leq e^{\left(\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln(f(x))dx-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\ln(g(x))dx\right)} \\
&= \left(e^{\int_{u_1}^{u_2}\ln(f(x))dx-\int_{u_1}^{u_2}\ln(g(x))dx}\right)^{\frac{1}{u_2-u_1}} \\
&= \left(\frac{e^{\int_{u_1}^{u_2}\ln(f(x))dx}}{e^{\int_{u_1}^{u_2}\ln(g(x))dx}}\right)^{\frac{1}{u_2-u_1}} \\
&= \left(\frac{\int_{u_1}^{u_2}\left(f(x)\right)^{dx}}{\int_{u_1}^{u_2}\left(g(x)\right)^{dx}}\right)^{\frac{1}{u_2-u_1}}.
\end{aligned}$$

Hence

$$\frac{f\left(\frac{u_1+u_2}{2}\right)}{g\left(\frac{u_1+u_2}{2}\right)} \leq \left(\frac{\int_{u_1}^{u_2}\left(f(x)\right)^{dx}}{\int_{u_1}^{u_2}\left(g(x)\right)^{dx}}\right)^{\frac{1}{u_2-u_1}}. \tag{9}$$

Now consider,

$$\begin{aligned}
\left(\frac{\int_{u_1}^{u_2}\left(f(x)\right)^{dx}}{\int_{u_1}^{u_2}\left(g(x)\right)^{dx}}\right)^{\frac{1}{u_2-u_1}} &= \left(\frac{e^{\int_{u_1}^{u_2}\ln(f(x))dx}}{e^{\int_{u_1}^{u_2}\ln(g(x))dx}}\right)^{\frac{1}{u_2-u_1}} \\
&= \left(e^{\int_{u_1}^{u_2}\ln(f(x))dx-\int_{u_1}^{u_2}\ln(g(x))dx}\right)^{\frac{1}{u_2-u_1}} \\
&= e^{\int_0^1\ln(f(u_1+\mu(u_2-u_1)))d\mu-\int_0^1\ln(g(u_1+\mu(u_2-u_1)))d\mu} \\
&\leq e^{\int_0^1\ln\left((f(u_1))^{1-\mu}(f(u_2))^{\mu}\right)-\int_0^1\ln\left((g(u_1))^{1-\mu}(g(u_2))^{\mu}\right)} \\
&= e^{\int_0^1((1-\mu)\ln(f(u_1))+\mu\ln f(u_2))d\mu-\int_0^1((1-\mu)\ln(g(u_1))+\mu\ln g(u_2))d\mu} \\
&= e^{\ln(f(u_1).f(u_2))^{\frac{1}{2}}-\ln(g(u_1).g(u_2))^{\frac{1}{2}}} \\
&= \frac{\sqrt{(f(u_1).f(u_2))}}{\sqrt{(g(u_1).g(u_2))}} \\
&= \frac{G\left((f(u_1),f(u_2))\right)}{G\left(g(u_1),g(u_2)\right)}.
\end{aligned}$$

Hence

$$\left(\frac{\int_{u_1}^{u_2}(f(x))^{dx}}{\int_{u_1}^{u_2}(g(x))^{dx}}\right)^{\frac{1}{u_2-u_1}} \leq \frac{G\left((f(u_1),f(u_2))\right)}{G\left(g(u_1),g(u_2)\right)}. \tag{10}$$

Combining (9) and (10), we have

$$\frac{f\left(\frac{u_1+u_2}{2}\right)}{g\left(\frac{u_1+u_2}{2}\right)} \leq \left(\frac{\int_{u_1}^{u_2}(f(x))^{dx}}{\int_{u_1}^{u_2}(g(x))^{dx}}\right)^{\frac{1}{u_2-u_1}} \leq \frac{G(f(u_1),f(u_2))}{G(g(u_1),g(u_2))}.$$

This is called Hermite Hadamard Integral inequality for quotient of multiplicatively convex functions. ■

Example 10 *Note that $f(x)=e^{x^2}$ and $g(x)=e^{|x|}$ are multiplicatively convex functions. Suppose that $u_1=1$ and $u_2=3$. Then*

$$\begin{aligned}
\frac{f\left(\frac{u_1+u_2}{2}\right)}{g\left(\frac{u_1+u_2}{2}\right)} &= \frac{e^{\left(\frac{1+3}{2}\right)^2}}{e^{\left(\frac{1+3}{2}\right)}} = 7.3890,\\
\left(\frac{\int_{u_1}^{u_2}(f(x))^{dx}}{\int_{u_1}^{u_2}(g(x))^{dx}}\right)^{\frac{1}{u_2-u_1}} &= \left(\frac{\int_1^3\left(e^{x^2}\right)^{dx}}{\int_1^3(e^x)^{dx}}\right)^{\frac{1}{2}} = 10.3123,\\
\frac{G(f(u_1),f(u_2))}{G(g(u_1),g(u_2))} &= \frac{\sqrt{e.e^{3^2}}}{\sqrt{e.e^3}} = 20.0854.
\end{aligned}$$

which is true.

Theorem 11 *Let f and g be convex and multiplicatively convex positive functions, respectively. Then we have*

$$\left(\frac{\int_{u_1}^{u_2}(f(x))^{dx}}{\int_{u_1}^{u_2}(g(x))^{dx}}\right)^{\frac{1}{u_2-u_1}} \leq \frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}}{G(g(u_1),g(u_2)).e}, \tag{11}$$

where $G(.,.)$ is a geometric mean.

Proof. Note that

$$\begin{aligned}
\left(\frac{\int_{u_1}^{u_2}(f(x))^{dx}}{\int_{u_1}^{u_2}(g(x))^{dx}}\right)^{\frac{1}{u_2-u_1}} &= \left(\frac{e^{\int_{u_1}^{u_2}\ln(f(x))dx}}{e^{\int_{u_1}^{u_2}\ln(g(x))dx}}\right)^{\frac{1}{u_2-u_1}}\\
&= \left(e^{\int_{u_1}^{u_2}\ln(f(x))dx-\int_{u_1}^{u_2}\ln(g(x))dx}\right)^{\frac{1}{u_2-u_1}}\\
&= e^{\int_0^1\ln(f(u_1+\mu(u_2-u_1)))d\mu-\int_0^1\ln(g(u_1+\mu(u_2-u_1)))d\mu}\\
&\leq e^{\int_0^1\ln(f(u_1)+\mu(f(u_2)-f(u_1)))d\mu-\int_0^1\ln((g(u_1))^{1-\mu}(g(u_2))^{\mu})d\mu}\\
&= \frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}}{G(g(u_1),g(u_2)).e}.
\end{aligned}$$

Hence

$$\left(\frac{\int_{u_1}^{u_2}(f(x))^{dx}}{\int_{u_1}^{u_2}(g(x))^{dx}}\right)^{\frac{1}{u_2-u_1}} \leq \frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}}{G(g(u_1),g(u_2)).e}.$$

This completes the proof. ■

Theorem 12 *Let f and g be multiplicatively convex and convex positive functions, respectively. Then*

$$\left(\frac{\int_{u_1}^{u_2}(f(x))^{dx}}{\int_{u_1}^{u_2}(g(x))^{dx}}\right)^{\frac{1}{u_2-u_1}} \leq \frac{G(f(u_1),f(u_2)).e}{\left(\frac{(g(u_2))^{g(u_2)}}{(g(u_1))^{g(u_1)}}\right)^{\frac{1}{g(u_2)-g(u_1)}}},$$

where $G(.,.)$ is geometric mean.

Proof. Note that

$$\begin{aligned}
\left(\frac{\int_{u_1}^{u_2}(f(x))^{dx}}{\int_{u_1}^{u_2}(g(x))^{dx}}\right)^{\frac{1}{u_2-u_1}} &= \left(\frac{e^{\int_{u_1}^{u_2}\ln(f(x))dx}}{e^{\int_{u_1}^{u_2}\ln(g(x))dx}}\right)^{\frac{1}{u_2-u_1}} \\
&= \left(e^{\int_{u_1}^{u_2}\ln(f(x))dx-\int_{u_1}^{u_2}\ln(g(x))dx}\right)^{\frac{1}{u_2-u_1}} \\
&= e^{\int_0^1\ln(f(u_1+\mu(u_2-u_1)))d\mu-\int_0^1\ln(g(u_1+\mu(u_2-u_1)))d\mu} \\
&\leq e^{\int_0^1\ln((f(u_1))^{1-\mu}.(f(u_2))^{\mu})d\mu-\int_0^1\ln(g(u_1)+\mu(g(u_2)-g(u_1)))d\mu} \\
&= e^{\ln(G(f(u_1),f(u_2)))-\ln\left(\left(\frac{(g(u_2))^{g(u_2)}}{(g(u_1))^{g(u_1)}}\right)^{\frac{1}{g(u_2)-g(u_1)}}\right)+1} \\
&= \frac{G(f(u_1),f(u_2)).e}{\left(\frac{(g(u_2))^{g(u_2)}}{(g(u_1))^{g(u_1)}}\right)^{\frac{1}{g(u_2)-g(u_1)}}}.
\end{aligned}$$

Hence

$$\left(\frac{\int_{u_1}^{u_2}(f(x))^{dx}}{\int_{u_1}^{u_2}(g(x))^{dx}}\right)^{\frac{1}{u_2-u_1}} \leq \frac{G(f(u_1),f(u_2)).e}{\left(\frac{(g(u_2))^{g(u_2)}}{(g(u_1))^{g(u_1)}}\right)^{\frac{1}{g(u_2)-g(u_1)}}}.$$

This completes the proof. ■

Theorem 13 *Let f and g be convex and multiplicatively convex positive functions, respectively. Then*

$$\left(\int_{u_1}^{u_2}(f(x))^{dx}.\int_{u_1}^{u_2}(g(x))^{dx}\right)^{\frac{1}{u_2-u_1}} \leq \frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}.G(g(u_1),g(u_2))}{e},$$

where $G(.,.)$ is geometric mean.

Proof. Note that

$$\begin{aligned}
\left(\int_{u_1}^{u_2}(f(x))^{dx}.\int_{u_1}^{u_2}(g(x))^{dx}\right)^{\frac{1}{u_2-u_1}} &= \left(e^{\int_{u_1}^{u_2}\ln(f(x))dx+\int_{u_1}^{u_2}\ln(g(x))dx}\right)^{\frac{1}{u_2-u_1}} \\
&= \left(e^{(u_2-u_1)\left\{\int_0^1\ln(f(u_1+\mu(u_2-u_1)))d\mu+\int_{u_1}^{u_2}\ln(g(u_1+\mu(u_2-u_1)))d\mu\right\}}\right)^{\frac{1}{u_2-u_1}} \\
&= e^{\int_0^1\ln(f(u_1+\mu(u_2-u_1)))d\mu+\int_{u_1}^{u_2}\ln(g(u_1+\mu(u_2-u_1)))d\mu} \\
&\leq e^{\int_0^1\ln(f(u_1)+\mu(f(u_2)-f(u_1)))d\mu+\int_0^1\ln((g(u_1))^{1-\mu}.(g(u_2))^{\mu})d\mu} \\
&= e^{\ln\left(\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}\right)-1+\ln(G(g(u_1),g(u_2)))} \\
&= \frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}.G(g(u_1),g(u_2))}{e}.
\end{aligned}$$

Hence

$$\left(\int_{u_1}^{u_2}(f(x))^{dx}.\int_{u_1}^{u_2}(g(x))^{dx}\right)^{\frac{1}{u_2-u_1}} \leq \frac{\left(\frac{(f(u_2))^{f(u_2)}}{(f(u_1))^{f(u_1)}}\right)^{\frac{1}{f(u_2)-f(u_1)}}.G(g(u_1),g(u_2))}{e}.$$

■

References

[1] M. K. Bakula and J. Pecaric, "Note on some Hadamard-type inequalities, " Journal of Inequalities in Pure and Applied Mathematics, vol. 5, no. 3, article 74, 2004.

[2] A. E. Bashirov, E. Kurpınar and A. Özyapıcı, Multiplicative calculus and its applications, Journal of Mathematical Analysis and Applications, 337 (1), (2008) 36-48.

[3] A. E. Bashirov, E. Kurpınar, Y. Tando and A. Özyapıcı, On modeling with multiplicative differential equations, Applied Mathematics, 26(4), (2011) 425-438.

[4] A. E. Bashirov and M. Riza, On complex multiplicative differentiation, TWMS Journal of Applied Engineering Mathematics, 1(1), (2011) 75-85.

[5] S. S. Dragomir, R. P. Agarwal, and N. S. Barnett, "Inequalities for beta and gamma functions via some classical and new integral inequalities," Journal of Inequalities and Applications, vol. 5, no. 2, pp. 103–165, 2000.

[6] S.S Dragomir and B.Mond, Integral inequalities of Hadamard type for log-convex functions, Demonst. Math., 31, (1998) 354-364.

[7] S. S. Dragomir and C. E. M. Pearce, Slected topics on Hermite-Hadamard type inequalities, RGMIA Monograph, Victorial University, Australia (2000).

[8] S. S. Dragomir, J. E. Pecaric and L. E. Person, Some inequalities of Hadamard type, Soochow J. Math. 21, (1995) 335-341.

[9] Y. L. Daletskii and N. I. Teterina, Multiplicative stochastic integrals, Uspekhi Matematicheskikh Nauk, 27(2:164), (1972) 167-168.

[10] M. Grossman and R. Katz, Non-Newtonian Calculus, Lee Press, Pigeon Cove, MA (1972).

[11] M. Grossman, Bigeometric Calculus: A System with a Scale-Free Derivative, Archimedes Foundation, Rockport, MA (1983).

[12] Y. Gurefe, U. Kadak, E. Misirli and A. Kurdi, A new look at the classical sequence spaces by using multiplicative calculus, U. P. B. Sci. Bull. Series A78, (2016) 9-20.

[13] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, Cambridge Mathematical Library, Cambridge, UK, 1998.

[14] R. L. Karandikar, Multiplicative decomposition of non-singular matrix valued continuous semimartingales, The Annals of Probability, 10(4), (1982) 1088-1091.

[15] U. S. Kirmaci and M. E. Özdemir, "Some inequalities for mappings whose derivatives are bounded and applications to special means of real numbers," Applied Mathematics Letters, vol. 17, no. 6, pp. 641–645, 2004.

[16] W. Kasprzak, B. Lysik and M. Rybaczuk, Dimensions, Invariants Models and Fractals, Ukrainian Society on Fracture Mechanics, SPOLOM, Wroclaw-Lviv, Poland (2004).

[17] K. Inayat Noor and M. Aslam Noor, Relaxed strongly nonconvex functions, Appl. Math. E-Notes,6, (2006) 259-267.

[18] U. Kadak and H. Efe, The construction of Hilbert spaces over the Non-Newtonian field, International Journal of Analysis (2014) 1-10.

[19] U. Kadak and Y. Gurefe, A generalization on weighted means and convex functions with respect to the Non-Newtonian calculus, International Journal of Analysis (2016) 1-9.

[20] M. Aslam Noor, Some new classes of nonconvex functions, Nonl. Funct. Anal. Appl., 11, (2006) 165-171.

[21] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, Classical and New Inequalities in Analysis, vol. 61 of Mathematics and Its Applications (East European Series), Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[22] A. Özyapici and E. Misirli, Exponential approximation on multiplicative calculus, 6th ISAAC Congress, (2007) p. 471.

[23] M. E. Özdemir and U. S. Kırmacı, "Two new theorem on mappings uniformly continuous and convex with applications to quadrature rules and means," Applied Mathematics and Computation, vol. 143, no. 2-3, pp. 269–274, 2003.

[24] B. G. Pachpatte, Inequalities for Differentiable and Integral Equations, Academic Press, Boston, Mass, USA, 1997.

[25] B.G. Pachpatte, A note on integral inequalitis involving two log convex functions, Math. Inequal. Appl., 7, (2004) 511-515.

[26] J. E. Pecaric, F Proschan and Y.L. Tong, Convex functions, Partial Orderings and Statistical Applications, Acedamic Press, New York (1992).

[27] M. Riza, A. Özyapıcı and E. Kurpınar, Multiplicative finite difference methods, Quarterly of Applied Mathematics, 67 (4), (2009) 745-754.

[28] M. Rybaczuk, A. Kedzia and W. Zielinski, The concepts of physical and fractional dimensions II. The differential calculus in dimensional spaces, Chaos Solutions Fractals, 12, (2001) 2537-2552.

[29] D. Stanley, A multiplicative calculus, Primus IX (4) (1999) 310–326.

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