SCIENCEDOMAIN *international*
www.sciencedomain.org

Initial value problems for second order neutral impulsive integro-differential equations with advanced argument

Guobing Ye[1], Xuebin Wang *[1], Xuan Ye[2] and Chen Liu [1]

[1] *School of Science, Hunan University of Technology, Zhuzhou Hunan* 412007*, PR China*
[2] *School of Design, Jianghan University, Wuhan Hubei* 430056*, PR China*

Original research papers

Received: XX January 2019
Accepted: XX February 2019
Online Ready: XX February 2019

Abstract

This paper discusses initial value problems for second order neutral impulsive integro-differential equations with advanced argument. By using the fixed point theorem of either Leray-Schaude or Banach, two existence results are obtained. By comparison, each of them has his own strong and weak points. If appropriate changes are made to some conditions for two results, the same results can be got. Two examples to illustrate our main results are given, which are compared with the existence results for impulsive differential equations from existing literature.

Keywords: Neutral impulsive integro-differential equation; Second order; Initial value; Fixed point

2010 Mathematics Subject Classification: 34A37; 45J05; 34A12

1 Introduction

Impulsive differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotic, optimal control, etc. About initial value problems for impulsive differential equations, many authors have obtained very good existence results (for example, see [1-7]). Now consider the following equation

$$\begin{cases} (u(\phi(t)))'' = f(t, u(t), u'(t), Ku(t), Hu(t)), \ t \in J = [0, a], \ t \neq \xi_k, \\ \Delta u(t_k) = I_{0k}(u(t_k)), \ \Delta u'(t_k) = I_{1k}(u'(t_k)), \ k = 1, \cdots, p, \\ u(0) = u_0, \ u'(0) = u_0', \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = a$, $\phi \in C^2(J, \mathbb{R})$, ϕ is monotone increasing with $t \leq \phi(t) \leq a$ $(t \in J)$, $\phi(0) = 0$, $\phi(a) = a$, $\phi'(t) > 0$ with $\phi^{-1} \in C^2(J, \mathbb{R})$, and let $\phi(\xi_k) = t_k$ $(k = 1, \cdots, p)$, $J^* = J \backslash \{t_1, \cdots, t_p\}$, $\bar{J} = J \backslash \{\xi_1, \cdots, \xi_p\}$, $f : J \times \mathbb{R}^4 \to \mathbb{R}$ is continuous

*Corresponding author: E-mail: yeguobing19@sina.com

everywhere except at $\{\xi_k\} \times \mathbb{R}^4$, $f(\xi_k^+, x, x', y_1, y_2)$ and $f(\xi_k^-, x, x', y_1, y_2)$ exist, $f(\xi_k^-, x, x', y_1, y_2) = f(\xi_k, x, x', y_1, y_2)$, and $Ku(t) = \int_0^t k(t,s)u(s)ds$, $Hu(t) = \int_0^T h(t,s)u(s)ds$, $k(t,s) \in C(D, \mathbb{R}^+)$, $h(t,s) \in C(J \times J, \mathbb{R}^+)$, $D = \{(t,s) \in \mathbb{R}^2, 0 \le s \le t \le a\}$, $k_0 = \max\{k(t,s) : (t,s) \in D\}$, $h_0 = \max\{h(t,s) : (t,s) \in J \times J\}$, further and $I_{0k}, I_{1k} \in C(\mathbb{R}, \mathbb{R})$, $\Delta u(t_k) = u(t_k^+) - u(t_k), \Delta u'(t_k) = u'(t_k^+) - u'(t_k)$. Denote by $PC(X,Y)$, where $X \subset \mathbb{R}, Y \subset \mathbb{R}$, the set of all functions $u : X \to Y$ which are piecewise continuous in X with points of discontinuity of the first kind at the points $t_k \in X$, i.e., there exist the limits $u(t_k^+) < \infty$ and $u(t_k^-) = u(t_k) < \infty$.

2 Preliminaries

According to the properties of ϕ, there exist positive constants m_1 and m_2 such that $m_1 \le \phi'(t) \le m_2$ for all $t \in J$.

Let $E_0 = \{u | u, u' \in PC(J, \mathbb{R})\} \bigcap C^2(J^*, \mathbb{R})$. Evidently, E_0 is a real Banach space with norm $\|u(t)\|_{E_0} = \max\{\|u(t)\|_{PC}, \|u'(t)\|_{PC}\}$, where $\|u(t)\|_{PC} = \sup_{t\in J} |u(t)|$, $\|u'(t)\|_{PC} = \sup_{t \in J} |u'(t)|$. Further, let $E = \{u(\phi(t)) | u(t) \in E_0\}$. We can check that E is also a real Banach space with norm $\|u(\phi(t))\| = \max\{\|u(\phi(t))\|_{PC}, \|(u(\phi(t)))'\|_{PC^*}\}$, where $\|u(\phi(t))\|_{PC} = \sup_{t\in J} |u(\phi(t))| = \|u(t)\|_{PC}$, $\|(u(\phi(t)))'\|_{PC^*} = \sup\limits_{\phi(t)\in J} \left|\dfrac{du(\phi(t))}{d\phi(t)}\right| \cdot \sup\limits_{t\in J} \dfrac{d\phi}{dt} = \sup\limits_{t \in J} |u'(t)| \cdot m_2 = m_2 \|u'(t)\|_{PC}$.

Define operator $B : u(t) \longmapsto u(\phi(t))$, where $u(t) \in E_0$ and $u(\phi(t)) \in E$. It is evident that B is topological linear isomorphic, which implies that E is a real Banach space.

Since $\dfrac{\phi(a) - \phi(0)}{a - 0} = \phi'(\bar{t})$ $(0 < \bar{t} < a)$, i.e., $\phi'(\bar{t}) = 1$, we get $m_2 \ge 1$, next $\|(u(\phi(t)))'\|_{PC^*} = m_2 \|u'(t)\|_{PC} \ge \|u'(t)\|_{PC}$, so

$$\|u(t)\|_{E_0} \le \|u(\phi(t))\|. \tag{2.1}$$

Lemma 2.1. *$u(t) \in E_0$ is a solution of (1.1) if and only if $u(t) \in E_0$ is a solution of the following integral equation*

$$\begin{aligned} u(\phi(t)) = u_0 + u_0' t + \int_0^t (t-s) f(s, u(s), u'(s), Ku(s), Hu(s)) ds + \\ \textstyle\sum_{0<\xi_k<t} [I_{0k}(u(t_k)) + (t - \xi_k) I_{1k}(u'(t_k))],\ t \in J. \end{aligned} \tag{2.2}$$

Proof. (i) Necessity

For $\xi_k < t \le \xi_{k+1}$ $(k = 0, 1, \cdots, p)$, by (1.1), we get

$$u(t_1) - u(0) = u(\phi(\xi_1)) - u(\xi(0)) = \int_0^{\xi_1} (u(\phi(s)))' ds,$$

$$u(t_2) - u(t_1^+) = u(\phi(\xi_2)) - u(\phi(\xi_1^+)) = \int_{\xi_1}^{\xi_2} (u(\phi(s)))' ds,$$

$$\cdots\cdots$$

$$u(t_k) - u(t_{k-1}^+) = u(\phi(\xi_k)) - u(\phi(\xi_{k-1}^+)) = \int_{\xi_{k-1}}^{\xi_k} (u(\phi(s)))' ds,$$

$$u(\phi(t)) - u(t_k^+) = u(\phi(t) - u(\phi(\xi_k^+)) = \int_{\xi_k}^{t} (u(\phi(s)))' ds.$$

Adding these together, we obtain

$$u(\phi(t)) = u(0) + \int_0^t (u(\phi(s)))' ds + \sum_{i=1}^{k} [x(t_i^+) - x(t_i)],$$

$$u(\phi(t)) = u_0 + \int_0^t (u(\phi(s)))' ds + \sum_{0<\xi_k<t} I_{0k}(u(t_k)),\ t \in J. \tag{2.3}$$

Similarly, we obtain

$$(u(\phi(t)))' = u_0' + \int_0^t (u(\phi(s)))'' ds + \sum_{0<\xi_k<t} I_{1k}(u'(t_k)),\ t \in J. \tag{2.4}$$

Substituting (2.4) into (2.3), it is easy to get (2.2).

(ii) Sufficiency

According to (2.2), it is clear that

$$u(0) = u_0,\ \ \Delta u(t_k) = I_{0k}(u(t_k)). \tag{2.5}$$

Differentiating both sides of (2.2), we have

$$(u(\phi(t)))' = u_0' + \int_0^t f(s, u(s), u'(s), Ku(s), Hu(s)) ds + \sum_{0<\xi_k<t} I_{1k}(u'(t_k)),\ t \in J. \tag{2.6}$$

Similarly, we also have

$$(u(\phi(t)))'' = f(t, u(t), u'(t), Ku(t), Hu(t)),\ \ t \in \bar{J}. \tag{2.7}$$

By(2.6), it is evident that

$$u'(0) = u_0',\ \ \Delta u'(t_k) = I_{1k}(u'(t_k)). \tag{2.8}$$

From (2.5),(2.7) and (2.8), we get that $u(t)$ is a solution of (1.1).

□

Lemma 2.2. *(Leray-Schauder [6]) Let the operator $A : X \to X$ be completely continuous, where X is a real Banach space. If the set $G = \{\|x\| \,|\, x \in X, x = \lambda A x, 0 < \lambda < 1\}$ is bounded, then the operator A has at least one fixed point in the closed ball $T = \{x | x \in X, \|x\| \le R\}$, where $R = \sup G$.*

Lemma 2.3. *(Compactness criterion [7]) $H \subset PC(J, \mathbb{R})$ is a relatively compact set if and only if $H \subset PC(J, \mathbb{R})$ is uniformly bounded and equicontinuous on every J_k $(k = 0, \cdots, p)$, where $J_0 = [t_0, t_1], J_k = (t_k, t_{k+1}]$ $(k = 1, \cdots, p)$.*

3 Main Result

Let us introduce the following conditions for later use:

(H1) There exist nonnegative constants b, c, d_i $(i = 1, 2)$, b_k, c_k $(k = 1, \cdots, p)$, and $g \in L(J, \mathbb{R}^+)$ such that $|f(t, x_2, y_2, z_{12}, z_{22}) - f(t, x_1, y_1, z_{11}, z_{21})|$
$\le g(t)\big(b\|x_2 - x_1\|_{PC} + c\|y_2 - y_1\|_{PC} + \sum_{i=1}^{2} d_i \|z_{i2} - z_{i1}\|_{PC}\big),\ t \in J,$
$|I_{0k}(x_2(t_k)) - I_{0k}(x_1(t_k))| \le b_k |x_2(t_k) - x_1(t_k)|,\ I_{0k}(0) = 0,\ k = 1, \cdots, p,$
$|I_{1k}(y_2(t_k)) - I_{1k}(y_1(t_k))| \le c_k |y_2(t_k) - y_1(t_k)|,\ I_{1k}(0) = 0,\ k = 1, \cdots, p,$
where $x_1, x_2 \in E_0$, $y_i(t) = \bar{y}_i'(t)$, $\bar{y}_i(t)$ $(i = 1, 2) \in E_0$, $z_{1i} = K\bar{z}_{1i}$, $z_{2i} = H\bar{z}_{2i}$, $\bar{z}_{1i}$, $\bar{z}_{2i}$ $(i = 1, 2) \in E_0$, $a_0 = \int_0^a g(t) dt$.

(H2) There exist positive constant M such that $|f(t, u(t), u'(t), Ku(t), Hu(t))| \le M(1 + \|u(t)\|_{E_0})$.

(H3) $l = \max\{l_1, l_2\} < 1$, where $l_1 = a^2 M + \sum_{k=1}^{p} (b_k + a c_k)$, $l_2 = \frac{m_2}{m_1}\big(aM + \sum_{k=1}^{p} c_k\big)$.

(H4) $r = \max\{r_1, r_2\} < 1$, where $r_1 = aa_0\big(b + c + ad_1k_0 + ad_2h_0\big) + \sum\limits_{k=1}^{p}(b_k + ac_k)$,

$$r_2 = \frac{m_2}{m_1}\big[a_0\big(b + c + ad_1k_0 + ad_2h_0\big) + \sum_{k=1}^{p} c_k\big].$$

Theorem 3.1. *If conditions (H1),(H2) and (H3) are satisfied, then (1.1) has at least one solution in the closed ball* $\overline{B} = \{u(\phi(t))|u(\phi(t)) \in E,\ \|u(\phi(t))\| \le R\}$*, where* $R = \sup G$, $G = \{\|u(\phi(t))\| \mid u(\phi(t)) \in E,\ u(\phi(t)) = \lambda Au(\phi(t)),\ 0 < \lambda < 1\}$.

Proof. (i) For any $u(\phi(t)) \in E$ define the operator A by

$$\begin{aligned} Au(\phi(t)) = u_0 + u_0' t + \int_0^t (t-s) f(s, u(s), u'(s), Ku(s), Hu(s))ds + \\ \sum_{0<\xi_k<t} [I_{0k}(u(t_k)) + (t - \xi_k) I_{1k}(u'(t_k))],\ t \in J. \end{aligned} \tag{3.1}$$

It is easy to see that $Au(\phi(t)) \in E_0$. According to the properties of ϕ, for any $v(t) \in E_0$, we have $v(t) = v(\phi^{-1}(\phi(t))) = v\phi^{-1}(\phi(t))$. Let $u = v\phi^{-1}$. Next, it is clear that $v(t) = u(\phi(t)) \in E$. It follows that A maps E into E. Thus $Au(\phi(t)) \in E$ with

$$(Au(\phi(t)))' = u_0' + \int_0^t f(s, u(s), u'(s), Ku(s), Hu(s))ds + \sum_{0<\xi_k<t} I_{1k}(u'(t_k)),\ t \in J. \tag{3.2}$$

A is a completely continuous operator will be verified by the following tree steps.

Step1. A is continuous.

Let any $u_n(\phi(t))$ $(n = 1, 2, \cdots)$, $u(\phi(t)) \in E$ with $\|u_n(\phi(t)) - u(\phi(t))\| \to 0$ as $n \to \infty$. By (3.1) and (H1), we have

$$\begin{aligned} |Au_n(\phi(t)) - Au(\phi(t))| \le & \int_0^t (t-s) g(s)\big[b\|u_n(s) - u(s)\|_{PC} + \\ & c\|u_n'(s) - u'(s)\|_{PC} + d_1\|Ku_n(s) - Ku(s)\|_{PC} + d_2\|Hu_n(s) - Hu(s)\|_{PC}\big]ds + \\ & \sum_{0<\xi_k<t} [b_k|u_n(t_k) - u(t_k)| + (t-\xi_k)c_k|u_n'(t_k) - u'(t_k)|] \\ \le & \big(b + c + ad_1k_0 + ad_2h_0\big)\|u_n(t) - u(t)\|_{E_0} \int_0^t (t-s)g(s)ds + \\ & \|u_n(t) - u(t)\|_{E_0} \sum_{0<\xi_k<t} [b_k + (t-\xi_k)c_k], \end{aligned}$$

$$|Au_n(\phi(t)) - Au(\phi(t))| \le \big[aa_0\big(b + c + ad_1k_0 + ad_2h_0\big) + \sum_{k=1}^{p}(b_k + ac_k)\big]\|u_n(t) - u(t)\|_{E_0},\ t \in J. \tag{3.3}$$

Then from (3.3) and (2.1), we have

$$\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \le \big[aa_0\big(b + c + ad_1k_0 + ad_2h_0\big) + \sum_{k=1}^{p}(b_k + ac_k)\big]\|u_n(t) - u(t)\|_{E_0},$$

$$\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \le \big[aa_0\big(b + c + ad_1k_0 + ad_2h_0\big) + \sum_{k=1}^{p}(b_k + ac_k)\big]\|u_n(\phi(t)) - u(\phi(t))\|. \tag{3.4}$$

Thus

$$\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Similarly, from (3.2) and (2.1), we get

$$\begin{aligned} \left|\frac{d[Au_n(\phi(t)) - Au(\phi(t))]}{d\phi(t)}\right| \frac{d\phi}{dt} &= |(Au_n(\phi(t)) - Au(\phi(t)))'| = |(Au_n(\phi(t)))' - (Au(\phi(t)))'| \\ &\le \big[a_0\big(b + c + ad_1k_0 + ad_2h_0\big) + \sum_{k=1}^{p} c_k\big]\|u_n(\phi(t)) - u(\phi(t))\|, \end{aligned}$$

$$\left|\frac{d[Au_n(\phi(t))-Au(\phi(t))}{d\phi(t)}\right|\le\frac{1}{m_1}\big[a_0\big(b+c+ad_1k_0+ad_2h_0\big)+\sum_{k=1}^{p}c_k\big]\|u_n(\phi(t))-u(\phi(t))\|,\ t\in J,$$

$$\|(Au_n(\phi(t))-Au(\phi(t)))'\|_{PC^*}\le\frac{m_2}{m_1}\big[a_0\big(b+c+ad_1k_0+ad_2h_0\big)+\sum_{k=1}^{p}c_k\big]\|u_n(\phi(t))-u(\phi(t))\|. \quad (3.6)$$

Thus

$$\|(Au_n(\phi(t))-Au(\phi(t)))'\|_{PC^*}\to 0 \text{ as } n\to\infty. \quad (3.7)$$

By (3.5) and (3.7), it is easy to see that $\|Au_n(\phi(t))-Au(\phi(t))\|\to 0$ as $n\to\infty$, that is to say, A is continuous.

Step 2. A maps any bounded subset of E into one bounded subset of E.

Let T be any bounded subset of E. Then there exist $h>0$ such that $\|u(\phi(t))\|\le h$ for all $u(\phi(t))\in J$.

By (3.1),(H1),(H2) and (2.1), we have

$$\begin{aligned}|Au(\phi(t))|&\le|u_0|+|u_0'|t+\int_0^t(t-s)M(1+\|u(s)\|_{E_0})ds+\sum_{0<\xi_k<t}[b_k|u(t_k)|+(t-\xi_k)c_k|u'(t_k)|]\\&\le|u_0|+a|u_0'|+M(1+\|u(t)\|_{E_0})\int_0^a ads+\|u(t)\|_{E_0}\sum_{0<\xi_k<t}(b_k+ac_k)\\&\le|u_0|+a|u_0'|+M(1+\|u(\theta(t))\|)\int_0^a ads+\|u(\phi(t))\|\sum_{k=1}^{p}(b_k+ac_k)\\&\le|u_0|+a|u_0'|+a^2M(1+h)+h\sum_{k=1}^{p}(b_k+ac_k),\ t\in J,\end{aligned}$$

so

$$\|Au(\phi(t))\|_{PC}\le|u_0|+a|u_0'|+a^2M(1+h)+h\sum_{k=1}^{p}(b_k+ac_k). \quad (3.8)$$

Similarly, from(3.2),(H1),(H2) and (2.1), we get

$$\left|\frac{dAu(\phi(t))}{d\phi(t)}\right|\cdot\frac{d\phi}{dt}=|(Au(\phi(t)))'|\le|u_0'|+aM(1+h)+h\sum_{k=1}^{p}c_k,\ t\in J,$$
$$\left|\frac{dAu(\phi(t))}{d\phi(t)}\right|\le\frac{1}{m_1}\big[|u_0'|+aM(1+h)+h\sum_{k=1}^{p}c_k\big],\ t\in J,$$

so

$$\|(Au(\phi(t)))'\|_{PC^*}\le\frac{m_2}{m_1}\big[|u_0'|+aM(1+h)+h\sum_{k=1}^{p}c_k\big]. \quad (3.9)$$

According to (3.8) and (3.9), we obtain

$$\|Au(\phi(t))\|\le\max\Big\{|u_0|+a|u_0'|+a^2M(1+h)+h\sum_{k=1}^{p}(b_k+ac_k),\frac{m_2}{m_1}\big[|u_0'|+aM(1+h)+h\sum_{k=1}^{p}c_k\big]\Big\}.$$

Therefore $A(T)$ is uniformly bounded.

Step 3. $A(T)$ is equicontinuous on every J_k $(k=0,\cdots,p)$, where $J_0=[0,\xi_1]$, $J_k=(\xi_k,\xi_{k+1}]$ $(k=1,\cdots,p)$.

For any $Au(\phi(t))\in A(T)$ and any $\varepsilon>0$, take $\delta=\big[|u_0'|+aM(1+h)+h\sum_{k=1}^{p}c_k\big]^{-1}\varepsilon$. Then if $t_1,\,t_2\in J_k$ and $|t_1-t_2|<\delta$ with $t_1<t_2$, from (3.1),(H1),(H2) and (2.1), we have

$$|Au(\phi(t_2)) - Au(\phi(t_1))| \leq |u_0'|(t_2 - t_1) + \int_{t_1}^{t_2} (t-s)M(1 + \|u(s)\|_{E_0})ds + \sum_{i=1}^{k}(t_2 - t_1)c_i|u'(t_i)|$$
$$\leq \big[|u_0'| + aM(1 + \|u(t)\|_{E_0}) + \|u(t)\|_{E_0} \sum_{i=1}^{k} c_i\big](t_2 - t_1)$$
$$\leq \big[|u_0'| + aM(1 + \|u(\phi(t))\|) + \|u(\phi(t))\| \sum_{k=1}^{p} c_k\big]|t_2 - t_1| \leq \big[|u_0'| + aM(1+h) + h \sum_{k=1}^{p} c_k\big]|t_2 - t_1| < \varepsilon.$$

Thus, $A(T)$ is equicontinuous on every J_k $(k = 0, \cdots, p)$.

As a consequence of Step 1-3, A is completely continuous.

(ii) For any $\|u(\phi(t))\| \in G$, similar with getting (3.8) and (3.9), we have respectively

$$\|Au(\phi(t))\|_{PC} \leq |u_0| + a|u_0'| + a^2M + \big[a^2M + \sum_{k=1}^{p}(b_k + ac_k)\big]\|u(\phi(t))\|$$
$$= |u_0| + a|u_0'| + a^2M + l_1\|u(\phi(t))\|,$$
$$\|(Au(\phi(t)))'\|_{PC^*} \leq \frac{m_2}{m_1}(|u_0'| + aM) + \frac{m_2}{m_1}\big(aM + \sum_{k=1}^{p} c_k\big)\|u(\phi(t))\| = \frac{m_2}{m_1}(|u_0'| + aM) + l_2\|u(\phi(t))\|.$$

Then $\|u(\phi(t))\| = \lambda\|Au(\phi(t))\| \leq \|Au(\phi(t))\| \leq L + l\|u(\phi(t))\|$, where $L = \max\big\{|u_0| + a|u_0'| + a^2M,\ \frac{m_2}{m_1}(|u_0'| + aM)\big\}$. It follows that $\|u(\phi(t))\| \leq \frac{L}{1-l}$, i.e., G is bounded.

From (i) and (ii), now all conditions of Lemma 2.2 are satisfied and therefore the proof is complete. □

Theorem 3.2. *If conditions (H1) ($I_{0k}(0) = 0,\ I_{1k}(0) = 0$ are not needed) and (H4) are satisfied, then (1.1) has a unique solution.*

The proof of Theorem 3.2 is similar to that of Theorem 3.1, and is omitted here.

Remark 3.1. By comparing Theorem 3.1-3.2, each of them has his own strong and weak points. The condition (H3) of Theorem 3.1 is more easily satisfied than the condition (H4) of Theorem 3.2. The condition (H2) of Theorem 3.1 is also satisfied easily, but Theorem 3.2 hasn't the condition. The result of Theorem 3.1 determines that (1.1) has at least one solution in the closed ball $\overline{B}$.

Remark 3.2. If the corresponding formulas of (1.1),(H1),(H3) and (H4) are respectively changed into $\Delta u(t_k) = I_{0k}(u(t_k), u'(t_k))$, $\Delta u'(t_k) = I_{1k}(u(t_k), u'(t_k))$ of (1.1), $|I_{0k}(x_2(t_k)) - I_{0k}(x_1(t_k))| \leq b_{1k}|x_2(t_k) - x_1(t_k)| + b_{2k}|y_2(t_k) - y_1(t_k)|$, $|I_{1k}(y_2(t_k)) - I_{1k}(y_1(t_k))| \leq c_{1k}|x_2(t_k) - x_1(t_k)| + c_{2k}|y_2(t_k) - y_1(t_k)|$ of (H1), $l_1 = a^2M + \sum_{k=1}^{p}[(b_{1k} + b_{2k}) + a(c_{1k} + c_{2k})]$, $l_2 = \frac{m_2}{m_1}\big(aM + \sum_{k=1}^{p}(c_{1k} + c_{2k})\big)$ of (H3), $r_1 = aa_0\big(b + c + ad_1k_0 + ad_2h_0\big) + \sum_{k=1}^{p}[(b_{1k} + b_{2k}) + a(c_{1k} + c_{2k})]$, $r_2 = \frac{m_2}{m_1}\big[a_0\big(b + c + ad_1k_0 + ad_2h_0\big) + \sum_{k=1}^{p}(c_{1k} + c_{2k})\big]$ of (H4), then there are also the same results as Theorem 3.1-3.2.

4 Examples

Example 4.1. *Consider the equation*

$$\begin{cases} \big(u(t + \frac{1}{2}t(1-t))\big)'' = \frac{t}{66}\Big[11\sin(u(t) + e^t) - 2u'(t) + 6\int_0^t (ts)u(s)ds + \\ \qquad 3\int_0^1 (ts^2)u(s)ds\Big],\ t \in J = [0,1],\ t \neq \xi_1 = \frac{1}{2}, \\ \Delta u(t_1) = \frac{1}{12}u(t_1),\ \Delta u'(t_1) = \frac{1}{12}u'(t_1),\ t_1 = \frac{5}{8}, \\ u(0) = u_0,\ u'(0) = u_0', \end{cases} \tag{4.1}$$

Firstly, it is easy to verify that $\phi(t)=t+\frac{1}{2}t(1-t)$, $k(t,s)=ts$, $k_0=1$, $h(t,s)=ts^2$, $h_0=1$ all satisfy the requisitions of (1.1). From $\phi'(t)=\frac{3}{2}-t$, we get $m_1=1/2$, $m_2=3/2$. Next, since $f(t,x,y,z_1,z_2)=\frac{t}{66}[11\sin(x+e^t)-2y+6z_1+3z_2]$, and $|\sin(x_2(t)+e^t)-\sin(x_1(t)+e^t)|=|(x_2(t)+e^t)-(x_1(t)+e^t)|\cdot|\cos(\bar{x}(t)+e^t)|\le|x_2(t)-x_1(t)|$ ($\bar{x}(t)$ is located between $x_1(t)$ and $x_2(t)$), we have

$$\begin{aligned}&|f(t,x_2,y_2,z_{12},z_{22})-f(t,x_1,y_1,z_{11},z_{21})|\\ &\quad\le\frac{t}{66}[11|\sin(x_2+e^t)-\sin(x_1+e^t)|+2|y_2-y_1|+6|z_{12}-z_{11}|+3|z_{22}-z_{21}|]\\ &\quad\le t\big[\frac{1}{6}|x_2-x_1|+\frac{1}{33}|y_2-y_1|+\frac{1}{11}|z_{12}-z_{11}|+\frac{1}{22}|z_{22}-z_{21}|\big]\\ &\quad\le t\big[\frac{1}{6}\|x_2-x_1\|_{PC}+\frac{1}{33}\|y_2-y_1\|_{PC}+\frac{1}{11}\|z_{12}-z_{11}\|_{PC}+\frac{1}{22}\|z_{22}-z_{21}\|_{PC}\big],\ t\in J,\end{aligned}$$

where $b=\frac{1}{6}$, $c=\frac{1}{33}$, $d_1=\frac{1}{11}$, $d_2=\frac{1}{22}$, $a=1$, $a_0=\int_0^1 tdt=\frac{1}{2}$. From $I_{01}(x)=\frac{1}{12}x$, $I_{11}(y)=\frac{1}{12}y$, we have

$$\begin{aligned}|I_{01}(x_2(t_1))-I_{01}(x_1(t_1))|&\le\frac{1}{12}|x_2(t_1)-x_1(t_1)|,\ I_{01}(0)=0,\\ |I_{11}(y_2(t_1))-I_{11}(y_1(t_1))|&\le\frac{1}{12}|y_2(t_1)-y_1(t_1)|,\ I_{11}(0)=0,\end{aligned}$$

where $b_1=c_1=\frac{1}{12}$. Further, we have

$$\begin{aligned}&|f(t,u(t),u'(t),Ku(t),Hu(t))|\\ &\quad\le\frac{1}{66}\Big[11|\sin(u(t)+e^t)|+2|u'(t)|+6\int_0^t k(t,s)|u(s)|ds+3\int_0^1 h(t,s)|u(s)|ds\Big]\\ &\quad\le\frac{1}{66}[11+2\|u(t)\|_{E_0}+6\|u(t)\|_{E_0}+3\|u(t)\|_{E_0}]=\frac{1}{6}(1+\|u(t)\|_{E_0}),\end{aligned}$$

where $M=\frac{1}{6}$. Finally, since $l_1=a^2M+(b_1+ac_1)=\frac{1}{3}$, $l_2=\frac{m_2}{m_1}(aM+c_1)=\frac{3}{4}$, we get $l=\max\{l_1,l_2\}=\frac{3}{4}<1$.

Thus (4.1) satisfies all conditions of Theorem 3.1. It follows that (4.1) has at least one solution in the closed ball $\overline{B}$.

Example 4.2. *Consider the equation*

$$\begin{cases}\big(u(t+\frac{1}{2}t(1-t))\big)''=\frac{t}{108}\Big[12\sqrt{1+u^2(t)}-6\arctan(u'(t)+e^t)+3\int_0^t(ts)u(s)ds+\\ \qquad 3\int_0^1(ts^2)u(s)ds\Big],\ t\in J=[0,1],\ t\ne\xi_1=\frac{1}{2},\\ \Delta u(t_1)=\frac{1}{18}u(t_1)+1,\ \Delta u'(t_1)=\frac{1}{18}u'(t_1)+2,\ t_1=\frac{5}{8},\\ u(0)=u_0,\ u'(0)=u_0',\end{cases}\tag{4.2}$$

Firstly, it is easy to verify that $\phi(t)=t+\frac{1}{2}t(1-t)$, $k(t,s)=ts$, $k_0=1$, $h(t,s)=ts^2$, $h_0=1$ all satisfy the requisitions of (1.1). From $\phi'(t)=\frac{3}{2}-t$, we get $m_1=1/2$, $m_2=3/2$. Next, since $f(t,x,y,z_1,z_2)=\frac{t}{108}[12\sqrt{1+x^2}-6\arctan(y+e^t)+3(z_1+z_2)]$, and $|\sqrt{1+x_2^2(t)}-\sqrt{1+x_1^2(t)}|=|x_2(t)-x_1(t))|\cdot|\frac{|\bar{x}(t)|}{1+|\bar{x}(t)|^2}\le|x_2(t)-x_1(t)|$, $|\arctan(y_2(t)+e^t)-\arctan(y_1(t)+e^t)|=|(y_2(t)+$

$e^t) - (y_1(t) + e^t)| \cdot \dfrac{1}{1 + (\bar{y}(t) + e^t)^2} \le |y_2(t) - y_1(t)|$ ($\bar{x}(t)$ is located between $x_1(t)$ and $x_2(t)$, $\bar{y}(t)$ is located between $y_1(t)$ and $y_2(t)$), we have

$$\begin{aligned} &|f(t, x_2, y_2, z_{12}, z_{22}) - f(t, x_1, y_1, z_{11}, z_{21})| \\ &\le \frac{t}{108}\Big[12\Big|\sqrt{1+x_2^2} - \sqrt{1+x_1^2}\Big| + 6|\arctan(y_2 + e^t) - \arctan(y_1 + e^t)| + 3\sum_{i=1}^{2}|z_{i2} - z_{i1}|\Big] \\ &\le t\Big[\frac{1}{9}|x_2 - x_1| + \frac{1}{18}|y_2 - y_1| + \frac{1}{36}\sum_{i=1}^{2}|z_{i2} - z_{i1}|\Big] \\ &\le t\Big[\frac{1}{9}\|x_2 - x_1\|_{PC} + \frac{1}{18}\|y_2 - y_1\|_{PC} + \frac{1}{36}\sum_{i=1}^{2}\|z_{i2} - z_{i1}\|_{PC}\Big], \ t \in J, \end{aligned}$$

where $b = \frac{1}{9}$, $c = \frac{1}{18}$, $d_1 = d_2 = \frac{1}{36}$, $a = 1$, $a_0 = \int_0^1 tdt = \frac{1}{2}$. From $I_{01}(x) = \frac{1}{18}x + 1$, $I_{11}(y) = \frac{1}{18}y + 2$, we have

$$|I_{01}(x_2(t_1)) - I_{01}(x_1(t_1))| \le \frac{1}{18}|x_2(t_1) - x_1(t_1)|, \ |I_{11}(y_2(t_1)) - I_{11}(y_1(t_1))| \le \frac{1}{18}|y_2(t_1) - y_1(t_1)|,$$

where $b_1 = c_1 = \frac{1}{18}$. Finally, since $r_1 = aa_0(b + c + ad_1k_0 + ad_2h_0) + (b_1 + ac_1) = \frac{2}{9}$, $r_2 = \frac{m_2}{m_1}[a_0(b + c + ad_1k_0 + ad_2h_0) + c_1] = \frac{1}{2}$, we get $r = \max\{r_1, r_2\} = \frac{1}{2} < 1$.

Thus (4.2) satisfies all conditions of Theorem 3.2. It follows that (4.2) has a unique solution.

5 Conclusion

We have derived some existence results for second order neutral impulsive integro-differential equations with advanced argument. Firstly, $u(t) \in E_0$ is a solution of (1.1) if and only if $u(t) \in E_0$ is a solution of the integral equation. Although the methods used are conventional, the impulsive integro-differential equations are different from the past, with that are higher-order and advanced. The difficulty of solving the problem have increased a lot, with the equations are from the first order and advanced to the higher order and advanced. Because Theorem 3.1-3.2 have their own advantages and disadvantages, we can choose according to the conditions given. Finally, two examples are given to illustrate the effectiveness and superiority of our main results, which are compared with the examples for impulsive differential equations from existing literature.

Acknowledgment

The authors are very grateful to the referee for useful remarks and interesting comments. This research is supported by the NNSF of China (no. 61703154).

References

[1]Guobing Ye, Yulin Zhao, Huang Li. Existence results for third-order impulsive neutral differential equations with deviating arguments. Advances in Difference Equations, Volume 2014, doi:10.1186/1687-1847-2014-38.

[2]Xingqiu Zhang. (2009). Existence of positive solution for second-order nonlinear impulsive singular differential equations of mixed type in Banach spaces. Nonlinear Analysis:Theory & Application, 70(4), 1620-1628.

[3]Hua Su, Lishan Liu, Xiaoyan Zhang. (2007). The solutions of initial value problems for nonlinear second-order impulsive integro-differential equations of mixed type in Banach spaces. Nonlinear Analysis:Theory & Application, 66(5), 1025-1036.

[4]Abdelghani Ouahab. (2006). Local and global exitence and uniqueness results for impulsive functional differential equations with multiple delay. Journal of Mathematical Analysis and Applications, 323, 456-472.

[5]Fei Guo, Lishan Liu, Yonghong Wu. Global solutions of initial value problems for nonlinear second-order impulsive integro-differential equations, Nonlinear Analysis:Theory & Application, 61(8), 1363-1382.

[6]Dajun Guo. (2002). Nonlinear functional analysis, Shangdong Science & Technology Press, Jinan, 2002.

[7]Xilin Fu, Baoqing Yan, Yansheng Liu. (2005). Theory of impulsive differential system, Science Press, Beijing, 2005.