An Accurate Implicit Quarter Step First Derivative Block Hybrid Method (AIQSFDBHM) for Solving Ordinary Differential Equations

Abstract

An accurate implicit quarter step first derivative blocks hybrid method for solving first order ordinary differential equations have been developed via interpolation and collocation method, for the solution of stiff systems of first order ODEs. The analysis of the method was study and it was found to be consistent, convergent, zero-stability. We further compute the region of absolute stability, which was shown to be $A_\alpha - stable$. *The numerical experiments considered, showed that the method compete favorably with existing ones. Thus, the pair of numerical methods developed in this research is computationally reliable and this new method is proposed for adoption when solving first order initial value problems.*
Keyword: Quarter-step, block hybrid, stiff ODEs, first derivative.

1. INTRODUCTION

In a bid to model real-life problems in areas of engineering, biological sciences, physical sciences, electronics and many others, initial value problems are most times encountered, (Shokri and Shokri, 2013). A sample first order initial value problem takes the form given below

$$y' = f(x, y),\ y(a) = \alpha \tag{1.1}$$

where f is a continuous function over an interval of integration. However in most cases, these initial value problems cannot be solved analytically and hence the need for numerical methods. These numerical methods are adopted to obtain an approximate solution to the initial value problem under consideration (James, Adesanya, and Fasasi, 2013). The solution of (1.1) using the known analytical methods is not always easy and in some cases cannot even be solved at all using these methods. In view of the importance of numerical methods in the solution of (1.1), numerical analysts have developed methods for the numerical solution of both stiff and non-stiff problems of the form (1.1). Ordinary differential equations (ODE's) are important tools in solving real world problems and a wide variety of natural phenomena are modeled by these ODE's. Over the years, several researchers have considered the important numerical solution of (1.1). (Sunday, Skwame and Tumba, 2015) developed a quarter-step hybrid block method for solving (1.1), (Sabo, *et al.,* 2019) constructed an A-stable uniform order six linear multi-step methods for direct integration of (1.1). Also, (Skwame, Sabo and Kyagya, 2017) construct an implicit one-step block hybrid methods with multiple off-grid points for solving (1.1). And (Omar, and Adeyeye, 2016) formed numerical solution of first order initial value problems using a self-starting implicit two-step obrechkoff-type block method. (Kumleng, et-al., 2017), construct a family of continuous block A-stable third derivative for numerical integration of (1.1). Other authors who have done considerable work on the numerical solution of (1.1) case include (Lambert, 1973; Butcher, 2008; Fatunla, 1988), to predictor-corrector methods (Kayode and Adeyeye, 2011; Adesanya, Anake and Udoh 2008; Awoyemi and Idowu, 2005) and then block methods (Omar and Kuboye, 2015; Sunday, Odekunle, and Adesanya, 2013; Hasni, Majid and Senu, 2013; Areo and Adeniyi, 2013).

This section introduced the main aim of the paper and reviews, in section 2, we shall present the construction of our proposed numerical scheme for problem (1.1), section 3 provides an analysis for derived scheme while section 4 illustrates the method using some selected test problems. Finally, the paper is ended in section 5 with some concluding remarks.

2. Derivation of Hybrid Method

In this section, we intend to construct the proposed **AIQSFDBHM** which will be used to generate the method. We consider an approximation of the form:

$$y(x) = \sum_{j=0}^{n=p+q-1} \alpha_j T_j(x) \tag{2.1}$$

$$y'(x) = \sum_{j=0}^{n=p+q-1} \alpha_j T'_j(x) \tag{2.2}$$

Where α_j is unknown coefficients and $T_j(x)$ are polynomial basis functions of degree $n = p + q - 1$, where the number of interpolation points is p and the number of distinct collocation points q are, respectively, chosen to satisfy $1 \le p \le k$ and $q > 0$. The integer $k \ge 1$ denotes the step number of the method.

To derive this, these off-step points are carefully selected to guarantee zero stability condition. For the method, the off-step points are $\left(\frac{1}{12}, \frac{1}{6}, \frac{1}{4}\right)$, Using (2.1) and (2.2) with $p = 1$, $q = 4$, we have a polynomial of degree as $p + q - 1$ follows

$$y(x) = \sum_{j=0}^{4} \alpha_j T_j(x) \tag{2.3}$$

With first derivative,

$$y'(x) = \sum_{j=0}^{4} \alpha_j T'_j(x) \tag{2.4}$$

interpolating (2.3) at x_n and collocating (2.4) at $x_n = \frac{1}{12}, \frac{1}{6}$ *and* $\frac{1}{4}$ yields,

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_n + \frac{1}{6}h & 3\left(x_n + \frac{1}{12}h\right)^2 & 4\left(x_n + \frac{1}{12}h\right)^3 \\ 0 & 1 & 2x_n + \frac{1}{3}h & 3\left(x_n + \frac{1}{6}h\right)^2 & 4\left(x_n + \frac{1}{6}h\right)^3 \\ 0 & 1 & 2x_n + \frac{1}{2}h & 3\left(x_n + \frac{1}{4}h\right)^2 & 4\left(x_n + \frac{1}{4}h\right)^3 \end{bmatrix} \tag{2.5}$$

Solving (2.5) by Gaussian Elimination method yields,

$$y(t) = \alpha_0(t)y_n + h\left(\beta_0 f_n + \beta_{\frac{1}{12}} f_{n+\frac{1}{12}} + \beta_{\frac{1}{6}} f_{n+\frac{1}{6}} + \beta_{\frac{1}{4}} f_{n+\frac{1}{4}}\right) \tag{2.6}$$

where $\alpha_0, \beta_0, \beta_{\frac{1}{12}}, \beta_{\frac{1}{6}}\, and\ \beta_{\frac{1}{4}}$ are continuous coefficients obtained as

$$\left.\begin{aligned} &\alpha_0 = 1 \\ &\beta_0 = -x_n + x - \frac{11(-x_n + x)^2}{h} + \frac{48(-x_n + x)^3}{h^2} - \frac{72(-x_n + x)^4}{h^3} \\ &\beta_{\frac{1}{12}} = \frac{18(-x_n + x)^2}{h} - \frac{120(-x_n + x)^3}{h^2} + \frac{216(-x_n + x)^4}{h^3} \\ &\beta_{\frac{1}{6}} = -\frac{9(-x_n + x)^2}{h} + \frac{96(-x_n + x)^3}{h^2} - \frac{216(-x_n + x)^4}{h^3} \\ &\beta_{\frac{1}{4}} = \frac{2(-x_n + x)^2}{h} - \frac{24(-x_n + x)^3}{h^2} + \frac{72(-x_n + x)^4}{h^3} \end{aligned}\right\} \tag{2.6}$$

In what follows, let us express (2.6) as continuous function of t by letting $th = x - x_n$, yields,

$$\left.\begin{aligned} &\alpha_0 = 1 \\ &\beta_0 = th - 11t^2h + 48t^3h - 72t^4h \\ &\beta_{\frac{1}{12}} = 18t^2h - 120t^3h + 216t^4h \\ &\beta_{\frac{1}{6}} = -9t^2h + 96t^3h - 216t^4h \\ &\beta_{\frac{1}{4}} = 2t^2h - 24t^3h + 72t^4h \end{aligned}\right\} \tag{2.7}$$

Evaluating (2.7) at $x_{n+\frac{1}{12}}, x_{n+\frac{1}{6}}, x_{n+\frac{1}{4}}$ yields the following discrete schemes

$$
\left.
\begin{aligned}
y_{n+\frac{1}{12}} &= y_n + \frac{1}{32}hf_n + \frac{19}{288}hf_{n+\frac{1}{12}} - \frac{5}{288}hf_{n+\frac{1}{12}} + \frac{1}{288}hf_{n+\frac{1}{4}} \\
y_{n+\frac{1}{6}} &= y_n + \frac{1}{36}hf_n + \frac{1}{9}hf_{n+\frac{1}{12}} + \frac{1}{36}hf_{n+\frac{1}{6}} \\
y_{n+\frac{1}{6}} &= y_n + \frac{1}{32}hf_n + \frac{3}{32}hf_{n+\frac{1}{12}} + \frac{3}{32}hf_{n+\frac{1}{6}} + \frac{1}{32}hf_{n+\frac{1}{4}}
\end{aligned}
\right\} \qquad (2.8)
$$

3. Analysis of the Method

In this section, the analysis of quarter step block hybrid method shall be analyzed.

3.1 Order of the Methods

Following (Fatunla 1991) and (Lambert 1973), we define a linear operator $\mathcal{L}$ defined by

$$\mathcal{L}[y(x):h] = \sum_{j=0}^{k} \alpha_j y(x_n + jh) - h\beta_j y'(x_n + jh) \qquad (3.1)$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a, b]$. Expanding $y(x_n + jh)$ *and* $y'(x_n + jh)$ in Taylor series about x_n and collecting like terms in h *and* y gives:

$$\mathcal{L}\,[y(x):h] = C_0 y(x) + C_1^{(1)} hy'(x) + C_2^{(1)} + \cdots + C_p h^p y^p \qquad (3.2)$$

Definition 3.1

The differential operator (3.1) and the associated are said to be of order p if (2.8) are said to be of order p if

$$C_0 = C_1 = C_2 = \cdots C_p, C_{p+1} \neq 0 \qquad (3.3)$$

The term C_{p+1} is called error constant and it implies that the local truncation error is given by

$$E_{n+k} = C_{p+1} h^{p+1} y^{p+1}(x_n) + 0(h^{p+2}) \qquad (3.4)$$

Following Definition 3.1 above, the quarter step block method (2.8) is of uniform order four with error constant, $C_5 = \left[-1.0605\times10^{-7}, -4.4653\times10^{-8}, -1.5070\times10^{-7}\right]^T$

3.2. Consistency

Following Fatunla (1991) and Lambert (1973), the block method (2.8) is consistent if it has order greater or equal to one (that is $p \geq 1$), that is

i. $\rho(1) = 0$

ii. $\rho'(1) = \sigma(1)$

where, ρ *and* σ and are the first and second characteristic polynomials of the method.

3.3. Zero-Stability

the block method (2.8) is said to be zero stable if no roots of the first characteristic polynomial $\rho(\xi)$ has modulus greater than one and every root with modulus one is distinct, (Lambert (1973, 1991).

3.4. Convergence.

Definition 3.3 Convergence (Lambert, 1973)

A continuous linear multistep method is said to convergent if, for all IVPs (1.1) satisfying the hypothesis of Lipchitz condition. That the main aim of numerical method is to produce solution that have similar to the theoretical solution at all times. The convergence of (2.8) is considered in the light of the basic properties discussed earlier in conjunction with the fundamental theorem of (Dahlquist, 1956) for linear multistep method. We state Dahlquist theorem without proof.

Theorem 3.3.1: (Dahlquist, 1956)

The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero-stable.

$$\begin{bmatrix} w \quad -\frac{19}{288}wh & \frac{5}{288}wh & -1-\frac{1}{13}h-\frac{1}{288}wh \\ -\frac{1}{9}wh & w-\frac{1}{36}wh & -1-\frac{1}{36}w \\ \frac{3}{32}wh & -\frac{3}{32}wh & w-1-\frac{3}{32}wh+\frac{3}{32}wh \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{12}} \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{4}} \end{bmatrix} \qquad (3.5)$$

3.5. Region of Absolute Stability (RAS)

Definition 3.3.6: Region of Absolute Stability (Yan, 2011)

Region of absolute stability is a region in the complex z plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solutions of $y' = \lambda h$ satisfy $y_j \to 0, as\ j = 0$ for any initial condition.

To determine the regions of absolute stability of the computational method, a method that requires neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. This method according to Lambert (1973) is called the Boundary Locus Method (BLM). The stability polynomial for the (2.8) is given by,

$$\left(-\frac{1}{6912}w^2+\frac{5}{55296}w^3\right)h^3+\left(-\frac{11}{1728}w^2+\frac{7}{13824}w^3\right)h^2+\left(-\frac{1}{8}w^2-\frac{1}{16}w^3\right)h+w^3-w^2$$

The region of absolute stability of (2.8) is shown below

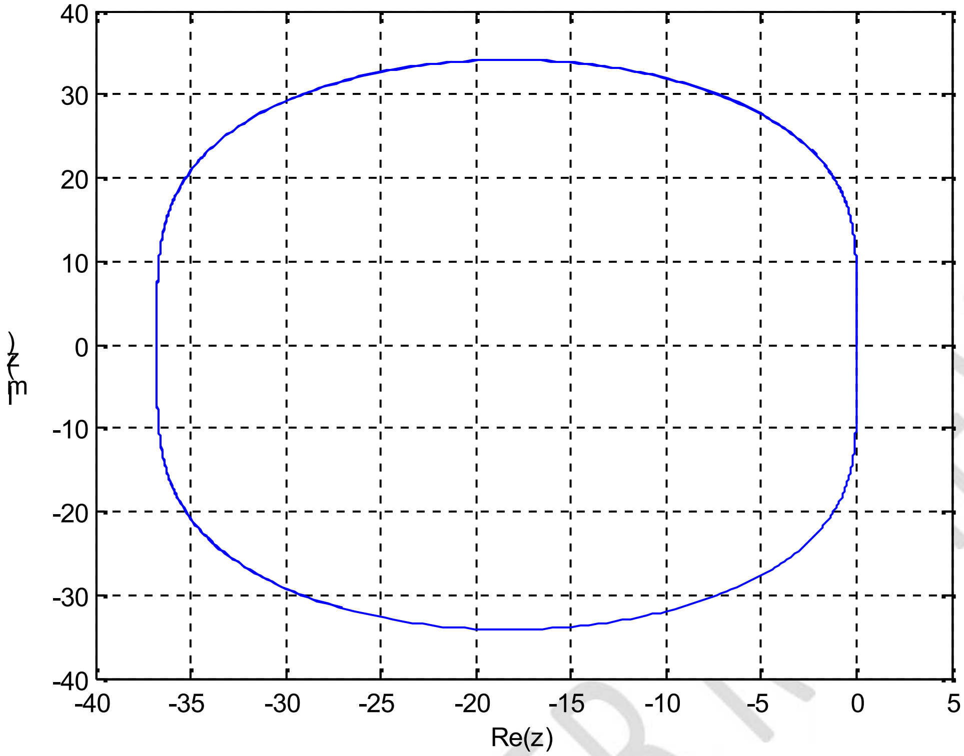

Figure 3.1: Stability Region for quarter step block hybrid method and the RAS obtained is $A_{\alpha} - stable$.

4 The Implementation of Method

We shall apply the newly developed pair of quarter step on some first order ordinary differential equation of the form (1.1) and we shall display our result with existing once as displayed below.

4.1 Numerical Examples (SIR Model)

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled equations relating the number of susceptible people $S(t)$number of people infected $I(t)$ and the number of people who have recovered $R(t)$. This is a good and simple model for many infectious diseases including measles, mumps and rubella. It is given by the following three coupled equations

$$\frac{dS}{dt} = \mu(1-S) - \beta IS \qquad (4.1)$$

$$\frac{dI}{dt} = \mu I - \gamma I + \beta IS \qquad (4.2)$$

$$\frac{dR}{dt} = \mu R + \gamma I \qquad (4.3)$$

where, $\mu, \gamma \ and \ \beta$ and β are positive parameters. Define y to be

$Y = S + I + R$

and adding Equations (4.1)-(4.3) we obtain the following evolution equation for

$y' = \mu(1 - y)$

Taking $\mu = 0.5$, $y(0) = 0.5$ and attaching an initial condition (for a particular closed population), we obtain,

$y' = (t) = 0.5(1 - y)$, $y(0) = 0.5$, $h = 0.1$

with exact solution:

$y(t) = 1 - 0.5e^{-0.5x}$

Source: (Omar and Adeyeye, 2016).

4.2 Numerical Examples

Consider the ODE

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad 0 \le x \ge 0.8, \quad h = 0.1$$

$$y(x) = \frac{1}{2}\begin{pmatrix} e^{-2x} + e^{-40x}(\cos(40x) + \sin(40x)) \\ e^{-2x} - e^{-40x}(\cos(40x) + \sin(40x)) \\ 2e^{-40x}(\sin(40x) - \cos(40x)) \end{pmatrix}$$

(**Source:** Skwame, Sabo and Kyagya, 2017)

4.3 Numerical Examples

Consider the ODE

$$y_1' = -100y_1 + 9.901y_2; \quad y_1(0) = 1$$
$$y_2' = 0.1y_1 - y_2; \quad y_2(0) = 10, \quad h = 0.1$$

With Exact Solution

$$y_1(x) = e^{-0.99x}$$

$$y_2(x) = 10e^{-0.99x}$$

$$x \in [0, 1]$$

(Source, Sabo, *et-al*, 2019)

Table 4.1: Comparison of error for solving numerical example 4.1

X	Error in Omar & Adeyeye, (2016).	Error in our method
0.1	4.96×10^{-6}	8.00×10^{-10}
0.2	8.73×10^{-6}	1.50×10^{-9}
0.3	8.98×10^{-6}	2.20×10^{-9}
0.4	8.55×10^{-6}	1.30×10^{-9}
0.5	1.27×10^{-5}	3.80×10^{-9}
0.6	1.16×10^{-5}	4.60×10^{-9}
0.7	1.47×10^{-5}	5.20×10^{-9}
0.8	1.40×10^{-5}	5.90×10^{-9}
0.9	1.66×10^{-5}	6.50×10^{-9}
1.0	1.58×10^{-5}	7.20×10^{-9}

Table 4.2: Comparison of error for solving numerical example 4.2

X	Error in Skwame, *et-al*., (2017)			Error in our method		
	y_1	y_2	y_3	y_1	y_2	y_3
0.1	2.23×10^{-2}	2.23×10^{-2}	2.53×10^{-2}	2.21×10^{-2}	2.21×10^{-2}	2.01×10^{-2}
0.2	1.06×10^{-4}	9.14×10^{-5}	1.68×10^{-4}	2.45×10^{-5}	2.45×10^{-5}	4.5×10^{-5}
0.3	8.23×10^{-6}	9.10×10^{-6}	1.33×10^{-5}	2.16×10^{-6}	2.16×10^{-6}	1.47×10^{-5}
0.4	9.60×10^{-6}	9.30×10^{-6}	1.60×10^{-7}	1.25×10^{-7}	1.28×10^{-7}	3.51×10^{-8}
0.5	9.67×10^{-6}	9.67×10^{-6}	1.68×10^{-9}	1.60×10^{-9}	1.20×10^{-9}	1.80×10^{-9}
0.6	9.50×10^{-6}	9.50×10^{-6}	9.12×10^{-11}	1.20×10^{-9}	1.60×10^{-10}	2.38×10^{-10}
0.7	9.08×10^{-6}	9.08×10^{-6}	1.05×10^{-10}	1.20×10^{-10}	1.00×10^{-11}	1.01×10^{-11}
0.8	8.49×10^{-6}	8.49×10^{-6}	6.84×10^{-11}	1.10×10^{-11}	0.00×10^{0}	0.00×10^{0}

Table 4.3: Comparison of error for solving numerical example 4.3

X	Error in Sabo, *et-al.*, (2018)		Error in new method	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	1.80×10^{-9}	1.40×10^{-8}	5.00×10^{-10}	3.00×10^{-9}
0.2	2.70×10^{-9}	2.30×10^{-8}	8.00×10^{-10}	7.00×10^{-9}
0.3	3.70×10^{-9}	3.30×10^{-8}	1.10×10^{-9}	1.00×10^{-8}
0.4	4.40×10^{-9}	3.90×10^{-8}	1.70×10^{-9}	1.40×10^{-8}
0.5	5.00×10^{-9}	470×10^{-8}	1.80×10^{-9}	1.70×10^{-8}
0.6	5.20×10^{-9}	5.00×10^{-8}	2.00×10^{-9}	1.80×10^{-8}
0.7	5.40×10^{-9}	5.20×10^{-8}	2.30×10^{-9}	2.10×10^{-8}
0.8	5.70×10^{-9}	5.40×10^{-8}	2.40×10^{-9}	2.20×10^{-8}
0.9	5.60×10^{-9}	5.50×10^{-8}	2.50×10^{-9}	2.30×10^{-8}
1.0	5.70×10^{-9}	5.50×10^{-8}	2.50×10^{-9}	2.30×10^{-8}

5. CONCLUSION

The new accurate implicit quarter step first derivative blocks hybrid method for solving ordinary differential equations have been introduced via interpolation and collocation method for the solution of stiff systems of ODEs. The analysis of the method was study and it was found to be consistent, convergent, zero-stability. We further compute the region of absolute stability region and it was found to be $A_\alpha - stable$. It is obvious that, the numerical experiments considered showed that the methods compete favorably with existing ones. Thus, the pair of numerical methods developed in this research is computationally reliable in solving first order initial value problems.

REFFERENCE

Adesanya, A. O., Anake, T. A. & Udoh, M.O., (2008). Improved continuous method for direct solution of general second order ordinary differential equation. *J. Nigerian Assoc. Math. Phys.*, 13: 59-62.

Areo, E. A. & Adeniyi, R. B., (2013). A self-starting linear multistep method for direct solution of initial value problems of second order ordinary differential equations. *Int. J. Pure Applied Math.,* 82: 345-364.

Awoyemi, D. O. & Idowu, O. M., 2(005). A class of hybrid collocation methods for third order ordinary differential equations. *Int. J. Comput. Math*., 82:1287-1293. DOI: 10.1080/00207160500112902

Butcher, J. C., (2008). Numerical methods for ordinary differential equations. 2nd Edn., *John Wiley and Sons, Chichester*, ISBN-10: 0470753757: 482.

Dahlquist G. G., (1956). Convergence and stability in the numerical integration of ordinary differential equations. *Math. Scand*. 4:33-50.

Fatunla, S. O., (1991). Block methods for second order IVP's. *Inter.J.Comp.Maths*. 41: 55-63.

Fatunla, S.O., (1988). Numerical Methods for Initial Value Problems in Ordinary Differential Equations. 1st Edn., *Academic Press, Boston*,: 295.

Hasni, M. M., Majid, Z. A. & Senu, N., (2013). Numerical solution of linear dirichlet two-point boundary value problems using block method. *Int. J. Pure Applied Math.,* 85: 495-506. DOI: 10.12732/ijpam.v85i3.6

International Journal of Innovative Mathematics, Statistics & Energy Policies 5(1):1-9.

James, A. A., Adesanya, O. A. & Fasasi, K. M., (2013). Starting order seven method accurately for the solution of first initial value problems of first order ordinary differential equations. *Progress Applied Math.,* 6:30-39. DOI: 10.3968/j.pam.1925252820130601.5231

Kayode, S. J. & Adeyeye, O., (2011). Two-step two point hybrid methods for general second order differential equations. *Afr. J. Math. Comput. Sci. Res*., 6: 191-196.

Kumleng, G. M, Kutchin, S. Y, Omagwu, S. & Nyam, I. A. (2017). A family of continuous block A-stable third derivative linear multistep methods for stiff initial value problems

Lambert, J. D., (1973). Computational Methods in Ordinary Differential Equations. 1st Edn., Wiley, Chichester, ISBN-10: 0471511943: 278.

Lambert, J. D., (1991). Numerical methods for ordinary differential systems. *John Wiley, New York*

Omar, Z. & Adeyeye, O., (2016). Numerical solution of first order initial value problems using a self-starting implicit two-step obrechkoff-type block method. *Journal of Mathematics and Statistics.* 12 (2): 127.134.

Omar, Z. & Kuboye, J. O., (2015). Computation of an accurate implicit block method for solving third order ordinary differential equations directly. *Global J. Pure Applied Math*., 11: 177-186.

Sabo, J., Raymond, D., Kyagya, T. Y. & Bambur, A. A., (2019). An A-stable Linear Multistep method for testing stiffly first order ordinary differential equations. *Invention Journal of Research Technology in Engineering & Management,* 3(1): 23-30

Shokri, A. & Shokri, A. A., (2013). The new class of implicit L-stable hybrid Obrechkoff method for the numerical solution of first order initial value problems. *Comput. Phys. Commun*., 184: 529-531.

Skwame, Y., Sabo, J. & Kyagya, T. Y., (2017). The construction of implicit one-step block hybrid methods with multiple off-grid points for the solution of stiff differential equations. *Journal of Scientific Research & Reports*. 16(1): 1-7, 2017.

Sunday, J., Odekunle, M. R. & Adesanya, A. O., (2013). Order six block integrator for the solution of first order ordinary differential equations. *Int. J. Math. Soft Comput*., 3: 87-96.

Sunday, J., Skwame, Y. & Tumba, P., (2015). A quarter-step hybrid block method for first-order ordinary differential equations. *Journal of Mathematics & Computer Science*. 6(4): 269-278.

Yan, Y. L., (2011). Numerical methods for differential equations. *Kowloon: City University of Hong Kong*.