# Modules Whose Endomorphism Rings Are Right Rickart

**Thoraya Abdelwhab**[1,2] **and Xiaoyan Yang**[*1]
[1] *Department of Mathematics, Northwest Normal University, Lanzhou, Gansu, P.R. China*
[2] *Faculty of Mathematical Sciences, University of Khartoum, Khartoum, Khartoum, Sudan*

## Abstract

In this paper, we study modules whose endomorphism rings are right Rickart (or right p.p.) rings, which we call R-endoRickart modules. We provide some characterizations of R-endoRickart modules. Some classes of rings are characterized in terms of R-endoRickart modules. We prove that an R-endoRickart module with no infinite set of nonzero orthogonal idempotents in its endomorphism ring is precisely an endoBaer module. We show that a direct summand of an R-endoRickart modules inherits the property, while a direct sum of R-endoRickart modules does not. Necessary and sufficient conditions for a finite direct sum of R-endoRickart modules to be an R-endoRickart module are provided.

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# 1 Introduction

It is well known that Baer rings and Rickart rings (also known as p.p. rings ) play an important role in providing a rich supply of idempotents and hence in the structure theory for rings. Rickart rings and Baer rings have their roots in functional analysis with close links to $C^*$-algebras and von Neumann algebras. Kaplansky [**?**] introduced the notion of Baer rings, which was extended to Rickart rings in ([**?**],[**?**]), and to quasi-Baer rings in [**?**], respectively. A number of research papers have been devoted to the study of Baer, quasi-Baer, and Rickart rings (see e.g [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]). A ring $R$ is said to be Baer if the right annihilator of any nonempty subset of $R$ is generated by an idempotent as a right ideal of $R$. The notion of Baer rings was generalized to a module theoretic version and studied in recent years (see [**?**],[**?**]). An $R$-module $M$ is called a Baer module if for each left ideal $I$ of $S = \mathrm{End}_R(M)$, $r_M(I) = eM$ for $e^2 = e \in S$. A more general notion of a Baer ring is that of a right Rickart ring. A ring $R$ is called a right Rickart ring if the right annihilator of any element in $R$ is generated by an idempotent as a right ideal of $R$. It is clear that any Baer ring is a right Rickart ring. A module $M_R$ is called Rickart if the right annihilator of each left principal ideal of $\mathrm{End}_R(M)$ is generated by an idempotent, i.e, for each $\varphi \in S = \mathrm{End}_R(M)$, there exists $e = e^2$ in $S$ such that $r_M(\varphi) = eM$. In this paper, we introduce the notion of R-endoRickart module, investigate some basic properties of these modules.

In section 2, we introduce the notion of R-endoRickart module, investigate some basic properties of these modules. It is shown that a direct summand of an R-endoRickart modules inherits the

**Corresponding author: E-mail: yangxy@nwnu.edu.cn*
*E-mail: thorayah_33@hotmail.com*

property. The classes of hereditary rings and von Neumann regular rings are characterized in terms of R-endoRickart $R$-modules.

In Section 3, we investigate when a direct sum of R-endoRickart modules is also R-endoRickart. We obtain necessary and sufficient conditions for a finite direct sum of copies of R-endoRickart modules to be R-endoRickart.

In Section 4, We show that if the endomorphism ring $\mathrm{End}_R M$ of an R-endoRickart module $M$ has no infinite set of nonzero orthogonal idempotents, then $M$ is an endoBaer module (a module whose endomorphism ring is a Baer), and obtain that every R-endoRickart module with only countably many direct summands is an endoBaer module. We also prove that a module $M$ is an R-endoRickart with the endomorphism ring $\mathrm{End}_R M$ has the $SSIP$ if and only if $M$ is an endoBaer module.

Throughout this paper, all rings are associative with unity. All modules are unital right $R$-modules unless otherwise indicated and $S = \mathrm{End}_R(M)$ is the ring of endomorphisms of $M_R$. Mod-$R$ denotes the category of all right $R$-modules, and $M_R$ a right $R$-module. By $N \subseteq M$, $N_R \leq M_R$ and $N_R \leq^{\oplus} M_R$ denote that $N$ is a subset, submodule and direct summand of $M$, respectively. By $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ we denote the ring of real, integer and natural numbers, respectively. $\mathbf{Z}_n$ denotes $\mathbf{Z}/n\mathbf{Z}$, $M^{(n)}$ denotes the direct sum of $n$ copies of $M$. The notations $r_R(.)$ and $r_M(.)$ denote the right annihilator of a subset of $M$ with elements from $R$ and the right annihilator of a subset of $R$ with elements from $M$, respectively.

# 2 R-endoRickart Modules

In this section, we introduce the notion of R-endoRickart module, investigate some basic properties of these modules. It is shown that a direct summand of an R-endoRickart modules inherits the property. The classes of hereditary rings and von Neumann regular rings are characterized in terms of R-endoRickart $R$-modules.

**Definition 2.1.** An $R$-module $M$ is called R-endoRickart if $\mathrm{End}_R(M)$ is a right Rickart ring.

Recall that $R$ is a hereditary ring if all submodules of projective modules over $R$ are again projective. If this is required only for finitely generated submodules, it is called semihereditary. Also recall $R$ is a von Neumann regular ring if for every $a \in R$ there exists an $x \in R$ such that $a = axa$.

*Remark* 2.1. (1) Obviously, $R_R$ is an R-endoRickart module if $R$ is a right Rickart ring, a Baer ring, a von Neumann regular ring or a hereditary ring.

(2) Every semisimple module is an R-endoRickart module.

(3) Any Rickart module is an R-endoRickart since the endomorphism ring of a Rickart module is right Rickart [**?**, Proposition 3.2].

(4) Any Baer module is R-endoRickart since the endomorphism ring of a Baer module is a Baer. (see [**?**, Theorem 4.1]).

Recall that a sequence $(a_0, a_1, a_2, ...)$ is a p-adic number where $p$ is a prime, if for all $n \geq 0$ we have $a_n \in \mathbb{Z}/p^{n+1}\mathbb{Z}$ and $a_{n+1} \equiv a_n$ (mod $p^n$). The set of p-adic numbers is denoted $\mathbb{Z}_p$ and is called the ring of p-adic integers. In the next example we show that not every R-endoRickart module is a Rickart ( i.e, the converse of Remark **??** (3) does not hold in general).

**Example 2.1.** *Consider the module $M = \mathbb{Z}_{p^\infty}$, as a $\mathbb{Z}$-module. We know that the endomorphism ring $S = \mathrm{End}_{\mathbb{Z}}(M)$ is the ring of $p$-adic integers (see [**?**, Example 3, p. 216]). Since $S$ is a Baer ring, it is a Rickart ring, and then $M = \mathbb{Z}_{p^\infty}$ is an R-endoRickart module. However $M$ is not a Rickart module.*

Recall that a module $M$ is $k$-local retractable if $r_M(\varphi) = r_S(\varphi)(M)$ for any $\varphi \in S = \mathrm{End}_R(M)$.

**Proposition 2.1.** *Let $M$ be a $k$-local retractable module and $S = \mathrm{End}_R(M)$. Then the following conditions are equivalent:*

(i) $M$ *is an Rickart module.*
(ii) $M$ *is an R-endoRickart module.*

Proof. (i) $\Rightarrow$ (ii) follows from Remark **??**.

(ii) $\Rightarrow$ (i) Let $M$ be an R-endoRickart module, since $S = \mathrm{End}_R(M)$ is a right Rickart ring and $M$ is $k$-local retractable module, then $M$ is an Rickart module by [**?**, Theorem 3.9]. □

Recall that a module $M$ is said to have $D_2$ condition if for any $N \leq M$ with $M/N \cong M' \leq^{\oplus} M$, we have $N \leq^{\oplus} M$.

**Corollary 2.1.** *The following conditions are equivalent for a $k$-local retractable module $M$ and $S = \mathrm{End}_R(M)$:*
(i) $M$ *is an R-endoRickart module.*
(ii) $M$ *is an Rickart module.*
(iii) $M$ *satisfies the $D_2$ condition, and $\mathrm{Im}\varphi$ is isomorphic to a direct summand of $M$ for any $\varphi \in S$.*

Proof. Follows from Proposition **??** and [**?**, Proposition 2.11] . □

If $M$ is an $R$-module, $N$ a direct summand of $M$, and $e$ the projection of $M$ onto $N$, then it is easy to see that $e$ is an idempotent of $S = \mathrm{Hom}_R(M, M)$ and $\mathrm{Hom}_R(N, N) = eSe$. This fact will be used in the next proposition.

**Proposition 2.2.** *Every direct summand of an R-endoRickart module is R-endoRickart.*

Proof. Let $M$ be an R-endoRickart module, $N$ a direct summand of $M$, $S = \mathrm{Hom}_R(M, M)$, and $e$ the projection onto $N$. Then $\mathrm{Hom}_R(N, N) = eSe$. But for any right Rickart ring $S$ and any idempotent $e \in S$, $eSe$ is a right Rickart ring by [**?**, Corollary 3.3]. Thus $N$ is R-endoRickart. □

Recall that a morphism $f : M \to N$, ($M$ and $N$ are right $R$-modules) is a regular morphism (or regular map) if there exists $g : N \to M$ such that $f = fgf$.

*Remark* 2.2. If $M$ is an R-endoRickart module, then so are $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ for every regular $\varphi \in \mathrm{End}_R(M)$.

Proof. This follows from the fact that $\varphi \in \mathrm{End}_R(M)$ is regular if and only if $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are direct summands of $M$ by [**?**, Theorem 16]. □

**Corollary 2.2.** *If $R$ is a right Rickart ring, then $eR$ is an R-endoRickart $R$-module for every $e^2 = e \in R$.*

Corollary **??** also follows from the fact that if $R$ is a right Rickart ring then so is $eRe$ for every $e^2 = e \in R$ by [**?**, Corollary 3.3].

The next example shows an application of Proposition **??**.

**Example 2.2.** *(Example 1.7, [**?**]) Let $A = \prod_{n=1}^{\infty} \mathbb{Z}_2$ . Consider $T = \{(a_n)_{n=1}^{\infty} \in A | a_n$ is eventually constant$\}$, $I = \{(a_n)_{n=1}^{\infty} \in A | a_n$ = 0 is eventually $\}$ = $\bigoplus_{n=1}^{\infty} \mathbf{Z}_2$. Now, consider the ring $R = \begin{pmatrix} T & T/I \\ 0 & T/I \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} (1,1,...) & 0+I \\ 0 & 0+I \end{pmatrix}$ in $R$. Note that $R$ is a right hereditary ring, but $R$ is not a Baer ring. Since $R$ is a right Rickart ring (being right hereditary), $M = R_R$ is an R-endoRickart module, and the modules $M_1 = eR$ and $M_2 = (1-e)R$ are endoRickart $R$-modules by Proposition **??**.*

The next example shows that the submodule of a module can be an R-endoRickart however the module is not.

**Example 2.3.** *The $\mathbb{Z}$-module $\mathbb{Z}_4$ is not R-endoRickart since $S = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_4)$ is not right Rickart ring. However, the submodule $2\mathbb{Z}_4$ of $\mathbb{Z}_4$ is an R-endoRickart $\mathbb{Z}$-module because $2\mathbb{Z}_4 \cong_{\mathbb{Z}} \mathbb{Z}_2$ ( $\mathbb{Z}_2$ is a Rickart module).*

**Proposition 2.3.** *If $\mathrm{End}_R(M)$ is a von Neumann regular ring, then $M$ is an R-endoRickart module.*

Proof. Since $\mathrm{End}_R(M)$ is a von Neumann regular ring, then it is a right Rickart ring. Hence $M$ is an R-endoRickart module. □

Recall that a right $R$-module $M$ is retractable if $\mathrm{Hom}_R(M, N) \neq 0$ whenever $N$ is a non-zero submodule of $M$. Also recall that a module $M$ is quasi-retractable if $\mathrm{Hom}_R(M, r_M(I)) \neq 0$ for every $I \leq S_S$ with $r_M(I) \neq 0$.

**Proposition 2.4.** *Let $M$ be a (quasi-) retractable module and $S = \mathrm{End}_R(M)$. Then the following conditions are equivalent:*

(i) $M$ *is an Rickart module.*

(ii) $M$ *is an R-endoRickart module.*

Proof. (i) ⇒ (ii) follows from Remark **??**.

(ii) ⇒ (i) Let $M$ be an R-endoRickart module, since $S = \mathrm{End}_R(M)$ is a right Rickart ring and $M$ is (quasi-) retractable module, then $M$ is an Rickart module by [**?**, Proposition 3.5]. □

Recall that a module $M$ is said to have $C_2$ condition if any submodule $N$ of $M$ which is isomorphic to a direct summand of $M$ is a direct summand of $M$.

**Proposition 2.5.** *Let $M$ be either a (quasi-) retractable or a $k$-local retractable module and $S = \mathrm{End}_R(M)$. Then the following conditions are equivalent:*

(i) $M$ *is an R-endoRickart module with $C_2$ condition.*

(ii) *S is a von Neumann regular ring.*

(iii) *For each $\varphi \in S$, $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are direct summands of $M$.*

Proof. Follows from [**?**, Theorem 3.17], Proposition **??**, Proposition **??** and Proposition **??**. □

**Corollary 2.3.** *Let $M$ be either a (quasi-) retractable or a $k$-local retractable module with $C_2$ condition. If $M$ is an R-endoRickart module, then $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are R-endoRickart for each $\varphi \in S$.*

Proof. $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are direct summands of $M$ for each $\varphi \in S$ by Proposition **??**. Thus they are R-endoRickart modules by Proposition **??**. □

Next, we characterize several classes of rings in terms of R-endoRickart modules.

**Theorem 2.1.** *The following conditions are equivalent for a ring $R$:*

(i) *Every free module $M_R$ is an R-endoRickart module.*

(ii) *Every free module $M_R$ is a Rickart module.*

Proof. (i) ⇒ (ii) This follows from the fact that the endomorphism ring of a free module $M_R$ is a right Rickart ring if and only if $M_R$ is a Rickart module by [**?**, Corollary 5.3].

(ii) ⇒ (i) It is clear. □

Recall that a module $M$ is endoregular if $\mathrm{End}_R(M)$ is a von Neumann regular ring.

**Proposition 2.6.** *Every endoregular module $M$ is an R-endoRickart module.*

Proof. Let $M$ be an endoregular module. Then $\mathrm{End}_R(M)$ is a von Neumann regular ring, thus $M$ is an R-endoRickart module by Proposition **??**. □

**Proposition 2.7.** *Let $M$ be either a (quasi-) retractable or a $k$-local retractable module with $C_2$ condition and $S = \mathrm{End}_R(M)$, Then the following conditions are equivalent:*

(i) $M$ *is an endoregular module.*

(ii) $M$ *is an R-endoRickart module.*

(iii) *For each $\varphi \in S$, $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are direct summands of $M$.*

Proof. (i) $\Rightarrow$ (ii) Follows from Proposition **??**.

(ii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) Follow from Proposition **??**. □

Recall that a module $M$ has the (strong) summand intersection property, $SIP$ $(SSIP)$, if the intersection of any two (any family of) direct summands is a direct summand of $M$. $M$ is said to have the (strong) summand sum property, $SSP$ $(SSSP)$, if the sum of any two (any family of) direct summands is a direct summand of $M$.

**Corollary 2.4.** *Let $M$ be either a (quasi-) retractable or a $k$-local retractable module with $C_2$ condition, then the following statements hold:*

(i) *Every R-endoRickart module $M$ satisfies the $SIP$ and the $SSP$.*

(ii) *For every R-endoRickart module $M$, $\bigcap_{i=1}^{n} \mathrm{Ker}\varphi_i$ and $\sum_{i=1}^{n} \mathrm{Im}\varphi_i$ are R-endoRickart modules for every finite set $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ in $\mathrm{End}_R(M)$.*

Proof. (i) Note that every R-endoRickart module is an endoregular by Proposition **??**. This is a direct consequence of [**?**, Proposition 2.28].

(ii) For each $\varphi_i \in \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$, $\mathrm{Ker}\varphi_i$ and $\mathrm{Im}\varphi_i$ are direct summands of $M$ by Proposition **??**. Then $\bigcap_{i=1}^{n} \mathrm{Ker}\varphi_i$ and $\sum_{i=1}^{n} \mathrm{Im}\varphi_i$ are direct summands of $M$ by (i). Thus R-endoRickart modules by Proposition **??**. □

**Proposition 2.8.** *Let $M$ be an $R$-module and $S = End_R(M)$, if for every $0 \neq \varphi \in S$, $\varphi$ is a monomorphism, then $M$ is an indecomposable R-endoRickart module.*

Proof. Assume that $M$ is not indecomposable. Then $M = N_1 \oplus N_2$ with $N_1, N_2 \neq 0$. Take $\varphi = \pi_1$ the canonical projection of $M$ onto $N_1$. Then $\mathrm{Ker}(\varphi) = N_2 \neq 0$, a contradiction (as $\varphi$ is a monomorphism), and so $M$ is indecomposable. It is clear that for every $\varphi \in S$, $Ker\varphi \leq^{\oplus} M$, $M$ is a Rickart module, and hence an R-endoRickart module. □

**Proposition 2.9.** *If the $End(M)$ is a domain, then a module $M$ is an indecomposable R-endoRickart.*

Proof. Every domain is trivially a right Rickart ring, then $M$ is an R-endoRickart module. Since there are no idempotents other than 0 and 1 in a domain, $M$ is also indecomposable. □

**Proposition 2.10.** *If $M$ is an R-endoRickart module, with only countably many direct summands, then $M$ contains no infinite direct sums of disjoint summands.*

Proof. Since $M$ has only countably many direct summands, $S$ has no infinite set of nonzero orthogonal idempotents, hence there exist no infinite sets of mutually disjoint direct summands in $M$. □

**Corollary 2.5.** *If $M$ is an R-endoRickart module, with only countably many direct summands, then $M$ is a finite direct sum of indecomposable summands.*

Proof. By Proposition **??**, $S$ has no infinite sets of orthogonal idempotents, hence any direct sum decomposition of $M$ must be finite, thus $M$ is a finite direct sum of indecomposable submodules. □

Recall that a ring is regular in the sense of commutative algebra if it is a commutative unit ring such that all its localizations at prime ideals are regular local rings.

**Corollary 2.6.** *Let $M$ be an R-endoRickart module with only countably many direct summands and the endomorphism ring $S = End_R(M)$ is a regular. Then $M$ is a semisimple Artinian.*

Proof. $S$ is a regular Baer ring with only countably many idempotents by [**?**, Theorem 7.55]. Then $S$ is a semisimple Artinian ring, by [**?**, Theorem 2 and Theorem 3]. It is easy to check that $M$ is also a semisimple Artinian module. □

**Corollary 2.7.** *Let $M$ be $R$-module with only countably many direct summands and $S = End_R(M)$ is a regular ring. Then $M$ is an R-endoRickart module if and only if $M$ is a semisimple Artinian.*

Proof. The proof follows directly from Remark **??** and Corollary **??**. □

**Proposition 2.11.** *The following conditions are equivalent for a ring $R$:*
(i) *Every free $R$-module $M$ is an R-endoRickart module.*
(ii) *$R$ is a right hereditary ring.*

Proof. Since that a free module is a retractable, $M$ is R-endoRickart module if and only if it is a Rickart by Proposition **??**. Thus every free $R$-module $M$ is an R-endoRickart module if and only if $R$ is a right hereditary ring by [**?**, Theorem 2.26] and Remark **??**. □

**Corollary 2.8.** *Let $R$ be a right hereditary ring, then every projective right $R$-module is an R-endoRickart module.*

Proof. From Proposition **??** every free $R$-module is an R-endoRickart module, since that every projective module is a direct summand of a free module, then every projective module is an R-endoRickart by Proposition **??**. □

**Proposition 2.12.** *Let $R$ be a von Neumann regular ring. Then a free module $R^{(n)}$ is an R-endoRickart $R$-module for some $n \in \mathbb{N}$.*

Proof. This follows from the well-known fact that $R$ is von Neumann regular if and only if so is $Mat_n(R)$. since $Mat_n(R) = End_R(R^n)$ is a von Neumann regular ring. Thus $R^n$ is R-endoRickart by Proposition **??**. □

Recall that a ring $R$ is a principal ideal domain or $PID$ if $R$ is an integral domain in which every ideal is principal, i.e., can be generated by a single element.

**Proposition 2.13.** *Let $M$ be a free module $M$ of countable rank over a principal ideal domain $(PID)$ $R$, then $M$ is an R-endoRickart and has the $SSIP$.*

Proof. Since $R$ is a principal ideal domain $(PID)$, then $M$ has the $SSIP$ (see [**?**, Exercise 51(c)], and it is a Rickart $R$-module by [**?**, Theorem 2.26]. Thus it is an R-endoRickart by Remark **??**. □

**Corollary 2.9.** *Let $M$ be a projective module. Then the following statements hold:*
(i) *Every submodule of $M$ over a hereditary ring is an R-endoRickart module.*
(ii) *Every finitely generated submodule of $M$ over a von Neumann regular ring is an R-endoRickart module.*

Proof. (i) Since all submodules of projective modules over a hereditary ring $R$ are again projective. Thus they are R-endoRickart modules by Corollary **??**.

(ii) Let $I$ be a finitely generated submodule of $M$. It is well-known that a von Neumann regular ring is left and right semihereditary, and every finitely generated submodule of a projective module over a von Neumann regular ring $R$ is isomorphic to a direct summand of a finitely generated free $R$-module by [**?**]. Hence $I \cong K \leq^{\oplus} R^{(n)}$. Therefore, $I$ is an R-endoRickart module by Proposition **??** and Propositions **??** . □

# 3 Direct Sums Of R-endoRickart Modules

It is shown that a direct sum of R-endoRickart modules may not be R-endoRickart. In this section, we investigate when a direct sum of R-endoRickart modules is also R-endoRickart. We obtain necessary and sufficient conditions for a finite direct sum of copies of ($k$-local) retractable R-endoRickart module to be R-endoRickart.

The next example shows that a direct sum of R-endoRickart modules may not inherit the R-endoRickart property.

**Example 3.1.** *A finite direct sum of R-endoRickart modules is not necessarily an R-endoRickart module. For example, the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$ is not R-endoRickart while $\mathbb{Z}$ and $\mathbb{Z}_2$ are both R-endoRickart $\mathbb{Z}$-modules ($\mathbb{Z}$ and $\mathbb{Z}_2$ are both Rickart modules). We note that the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$ is a retractable module (Any direct sum of $\mathbb{Z}_{p^i}$ is retractable, where $p$ is a prime number). For the endomorphism $f(x, \bar{y}) = \bar{x}$ where $x \in \mathbb{Z}$ and $y \in \mathbb{Z}_2$, $\mathrm{Ker} f = 2\mathbb{Z} \oplus \mathbb{Z}_2$ which is not a direct summand of $\mathbb{Z} \oplus \mathbb{Z}_2$. So $\mathbb{Z} \oplus \mathbb{Z}_2$ is not a Rickart module [see ([**?**], Example 2.24)]. Thus $\mathbb{Z} \oplus \mathbb{Z}_2$ is not an R-endoRickart module by Proposition **??**.*

Recall that a module $M$ is a quasi-continuous if every complement in $M$ is a direct summand of $M$, and for any direct summands $M_1$ and $M_2$ of $M$ such that $M_1 \cap M_2 = 0$, the submodule $M_1 \oplus M_2$ is also a direct summand of $M$.

**Proposition 3.1.** *Let $M_i$ be a direct summand of a quasi-continuous R-endoRickart module $M$ for all $i = 1, .., n$, such that $M_i \cap M_j = 0$ for $i \neq j$. Then $M_i$ is an R-endoRickart module for all $i$ and $\bigoplus_{i=1}^{n} M_i$ is an R-endoRickart module.*

Proof. Since $M$ is a quasi-continuous module and $M_i \cap M_j = 0$ for all $i \neq j$, $\bigoplus_{i=1}^{n} M_i$ is a direct summand of $M$, Therefore, it is an R-endoRickart module by Proposition **??**. □

**Proposition 3.2.** *Let $M$ be an artinian R-endoRickart module. Then there exists a decomposition*

$$M = N_1 \oplus N_2 \oplus N_3 \oplus \cdots \oplus N_n,$$

*where $N_i$ is an indecomposable R-endoRickart module for each $i$.*

Proof. From [**?**, Proposition 19.20] Since $M$ is artinian, there exists a decomposition

$$M = N_1 \oplus N_2 \oplus N_3 \oplus \cdots \oplus N_n,$$

where each $N_i$ is an indecomposable. Also, each $N_i$ is an R-endoRickart module by Proposition **??**. □

**Proposition 3.3.** *Let $R$ be a commutative ring and $M = \bigoplus_{i \in I} M_i$ a direct sum of cyclic R-endoRickart modules $M_i$ over an arbitrary index set $I$. If $S = \mathrm{End}_R(M)$ is a domain, then $M$ is an R-endoRickart module.*

Proof. Note that $M$ is a $k$-local retractable Rickart module by [**?**, Proposition 4.9] and [**?**, Proposition 5.1]. Thus $M$ is an R-endoRickart module by Proposition **??**. □

The following result study finite direct sums of copies of an arbitrary R-endoRickart module $M$.

**Theorem 3.1.** *Let $M$ be a finitely generated R-endoRickart module and $S = \mathrm{End}(M)$, then the following conditions are equivalent:*

*(i) The arbitrary direct sum of copies of $M$ is an R-endoRickart module.*

*(ii) $S$ is a hereditary ring.*

Proof. (i)⇒(ii) For a finitely generated module $M$ and $S = End(M)$, we have that $End(M^{(f)}) \cong End(S^{(f)})$ as rings, where $f$ is an arbitrary set. Hence, if an arbitrary direct sum of copies of $M$ is R-endoRickart, its endomorphism ring $End(M^{(f)})$ is a right Rickart ring, hence $End(S^{(f)})$ is also a right Rickart ring, thus $S^{(f)}$ is an R-endoRickart module. Since $S^{(f)}$ is a free $S$-module, Hence By Proposition **??**, $S$ is hereditary.

(ii)⇒(i) let $S = \mathrm{End}(M)$ is hereditary, for an arbitrary set $f$, Since $S^{(f)}$ is a free $S$-module, we obtain that $S^{(f)}$ is an R-endoRickart $S$-module By Proposition **??**, hence $End(S^{(f)})$ is a right Rickart ring, thus $End(M^{(f)})$ is a right Rickart ring, and $M^{(f)}$ is an R-endoRickart module. □

The following result studies finite direct sums of copies of an arbitrary ($k$-local) retractable R-endoRickart module $M$.

**Proposition 3.4.** *Let $M$ be a ($k$-local) retractable R-endoRickart module with $C_2$ condition. Then any finite direct sum of copies of $M$ is an R-endoRickart module.*

Proof. Since a finite direct sum of copies of $M$ is a Rickart module by [**?**, Corollary 2.31], Proposition **??** and Proposition **??**. Thus it is an R-endoRickart by Remark **??**. □

The next example shows an application of Proposition **??**.

Recall that an element $m \in M$ is singular if $r_R(m) \leq^{ess} R_R$. We denote the set of all singular elements of $M$ by $Z(M)$. Then we say a module $M$ nonsingular if $Z(M) = 0$ and singular if $Z(M) = M$. A ring $R$ is right nonsingular if $R_R$ is nonsingular.

**Example 3.2.** *Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and the $R$-module $M = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$ . Since $M$ is a nonsingular quasi-injective $R$-module, $M$ is a Rickart module with $C_2$ condition(see [**?**], Example 2.32), thus $M$ is an R-endoRickart module with $C_2$ condition. Thus $M^{(n)}$ is an R-endoRickart module by Proposition **??**.*

Recall that a ring $R$ is a Pr$\ddot{u}$fer domain if $R$ is a commutative ring without zero divisors in which every non-zero finitely generated ideal is invertible.

**Theorem 3.2.** *([**?**, Corollary 15]). If $R$ is a commutative integral domain, then $M_n(R)$ is a Baer ring (for some $n > 1$) if and only if every finitely generated ideal of $R$ is invertible, i.e., if $R$ is a Pr$\ddot{u}$fer domain.*

**Theorem 3.3.** *Let $M$ be a free $R$-module of finite $rank > 1$ with only countably many direct summands. Then the following conditions are equivalent for a commutative integral domain $R$ :*

*(i)$M$ is R-endoRickart.*

*(ii) $R$ is a Pr$\ddot{u}$fer domain.*

Proof. Consider $R$ is a Pr$\ddot{u}$fer domain, then $M_n(R)$ is a Baer ring by Theorem **??**. but $End(M) \cong M_n(R)$ is a Baer ring, thus $End(M)$ is a right Rickart ring, so we obtain that $M$ is an R-endoRickart module.

Conversely, if $M$ is an R-endoRickart module, $End(M)$ is a right Rickart ring has no infinite set of nonzero orthogonal idempotents (as $M$ is $R$-module with only countably many direct summands), then it is a Baer ring by [**?**, Theorem 7.55], hence $M_n(R)$ for $n > 1$ is a Baer ring, thus $R$ must be a Pr$\ddot{u}$fer domain. □

Recall that a module over a ring is torsion free if $0$ is the only element annihilated by a regular element (nonzero divisor) of the ring.

**Proposition 3.5.** *Let $M$ be a finite direct sum of copies of some finite rank, torsion-free module and $S = End(M)$ is a $PID$. Then $M$ is R-endoRickart module.*

Proof. By [**?**] $Ker\varphi \leq^{\oplus} M, \forall \varphi \in S$, hence $M$ is a Rickart module, thus it is an R-endoRickart by our Remark **??**. □

Recall that a ring $R$ is a right $n$-fir if any right ideal that can be generated with $\leqslant n$ elements is free of unique rank (i.e., for every $I \leq R_R$, $I \cong R^k$ for some $k \leqslant n$, and if $I \cong R^l \Rightarrow k = l$) (for alternate definitions see , [**?**, Theorem 1.1]).

The definition of (right) $n$-fir ring is left-right symmetric, thus we will call such rings simply $n$-firs.

**Proposition 3.6.** *Let $M$ be a module with endomorphism ring $S$ is $n$-fir, then $M$ is an R-endoRickart module and $S^n$ is a Baer module. Consequently, $M_n(S)$ is a Baer ring*

Proof. Since $S$ is an $n$-fir, it is in particular an integral domain (see page 45, [**?**]), then trivially a right Rickart ring. Thus $M$ is an R-endoRickart module. $S^n$ is a Baer module by [**?**, Theorem 3.16]. Consequently, $M_n(S)$ is a Baer ring. □

Next we study finite direct sums of copies of a finitely generated R-endoRickart module $M$.

**Proposition 3.7.** *Let $M$ be a finitely generated module with endomorphism ring $S$ is $n$-fir, then $M$ is an R-endoRickart module and a finite direct sum of copies of $M$ is an R-endoRickart module.*

Proof. We note that, for a finitely generated module $M$ and $S = End(M)$, we have that $End(M^n) \cong End(S^n)$ as rings, where $n \in \mathbf{N}$. Since $S$ is $n$-fir, then $M$ is an R-endoRickart module and $S^n$ is a Baer module by Proposition **??**, and so $End(S^n)$ is a Baer ring ( the endomorphism ring of a Baer module is a Baer ). Thus $S^n$ is an R-endoRickart $S$-module by Remark **??**, hence $End(S^n)$ is a right Rickart ring (being a Baer ring), thus $End(M^n)$ is a right Rickart ring, and $M^n$ is an R-endoRickart. □

# 4 R-endoRickart Modules Versus EndoBaer Modules

In this section, we show that if the endomorphism ring $\mathrm{End}_R M$ of an R-endoRickart module $M$ has no infinite set of nonzero orthogonal idempotents, then $M$ is an endoBaer module, and obtain that every R-endoRickart module with only countably many direct summands is an endoBaer module. We also prove that a module $M$ is R-endoRickart with the endomorphism ring $\mathrm{End}_R M$ has the $SSIP$ if and only if $M$ is an endoBaer module.

**Definition 4.1.** An $R$-module $M$ is called endoBaer if $\mathrm{End}_R(M)$ is a Baer ring.

*Remark* 4.1. Any Baer module is an endoBaer, since the endomorphism ring of a Baer module is a Baer. (see [**?**, Theorem 4.1]).

**Proposition 4.1.** *Let $M$ be a (quasi-) retractable module. Then the following conditions are equivalent:*
(i) $M$ *is an endoBaer module.*
(ii) $M$ *is a Baer module.*

Proof. (i) ⇒ (ii) Since $M$ is an endoBaer module, $S = \mathrm{End}_R(M)$ is a Baer ring, Also $M$ is a (quasi-) retractable, thus $M$ is a Baer module by [**?**, Proposition 4.6] and [**?**, Theorem 2.5].
(ii) ⇒ (i) follows from Remark **??**. □

*Remark* 4.2. It is clear any endoBaer module is an R-endoRickart, since that any Baer ring is a right Rickart ring. But the converse does not hold in general.

The following examples exhibit an R-endoRickart module which is not an endoBaer module with the property that its endomorphism ring has an infinite set of nonzero orthogonal idempotents.

**Example 4.1.** *Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$ be a commutative ring, $R$ is a von Neumann regular, and Baer. Consider $T = \{(a_n)_{n=1}^{\infty} \in R | a_n$ is eventually constant$\}$, a subring of $R$. Then $T$ is a right Rickart ring, while $T$ is not a Baer ring by ([23, Example 7.54] and it has an infinite set of nonzero orthogonal idempotents, $\{\alpha_i = (a_k) \in T \mid a_k = 1$ if $k = i,$ otherwise, $a_k = 0\}$. Consider $M = T_T$. Then $M$ is an R-endoRickart module, which is not an endoBaer module.*

**Example 4.2.** *From example **??**, note that $R$ is a right hereditary ring, but $R$ is not a Baer ring. Since $R$ is a right Rickart ring (being right hereditary), $M = R_R$ is an R-endoRickart module, which is not an endoBaer module.*

**Example 4.3.** *([**?**], Example 1.6). Let $A$ be a field, take $A_n = A$ for $n = 1, 2, ....$ and let*

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} A_n & \bigoplus_{n=1}^{\infty} A_n \\ \bigoplus_{n=1}^{\infty} A_n & \langle \bigoplus_{n=1}^{\infty} A_n, 1 \rangle \end{pmatrix}$$

*which is a subring of the $2 \times 2$ matrix ring over the ring $\prod_{n=1}^{\infty} A_n$, where $\langle \bigoplus_{n=1}^{\infty} A_n, 1 \rangle$ is the $A$-algebra generated by $\bigoplus_{n=1}^{\infty} A_n$ and $1$. Then $R$ is a von Neumann regular ring which is not a Baer ring. thus*

*$M = R_R$ is an R-endoRickart module, which is not an endoBaer module. Denote the idempotent $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $M = eR$ is a R-endoRickart $R$-module by Proposition **??**. However, $M$ is not an endoBaer $R$-module because $End_R(M) \cong \langle \bigoplus_{n=1}^{\infty} A_n, 1\rangle$ is not a Baer ring (see ([**?**], Example 2.19)).*

**Example 4.4.** *Since that a free modules $\mathbf{Z}^{\mathbf{N}}$ and $\mathbf{Z}^{\mathbf{R}}$ are R-endoRickart $\mathbf{Z}$-modules ($\mathbf{Z}^{\mathbf{N}}$ and $\mathbf{Z}^{\mathbf{R}}$ are both Rickart modules,see Example 2.2.12 in [**?**]), then $End_Z(\mathbf{Z}^{\mathbf{N}})$ and $End_Z(\mathbf{Z}^{\mathbf{R}})$ are right Rickart rings. Note that $End_Z(\mathbf{Z}^{\mathbf{N}})$ is also a Baer ring, but $End_Z(\mathbf{Z}^{\mathbf{R}})$ is not a Baer ring. This, because $\mathbf{Z}^{\mathbf{R}}$ is retractable but is not a Baer $\mathbf{Z}$-module (see [**?**, Proposition 2.5]. Thus $\mathbf{Z}^{\mathbf{N}}$ is an endoBaer module, but $\mathbf{Z}^{\mathbf{R}}$ is not.*

**Proposition 4.2.** *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of finitely generated R-endoRickart modules $M_i$, where $I$ is a countable index set over a principal ideal domain $R$. Then the following conditions are equivalent:*

(i) *$M$ is a semisimple module.*
(ii) *$M$ is an R-endoRickart module.*
(iii) *$M$ is an endoBaer module.*

Proof. (i)$\Rightarrow$ (ii) By Remark **??** (1).
(iii) $\Rightarrow$ (ii) It is clear.
(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) Follow from [**?**, Corollary 5.8]. □

**Proposition 4.3.** *The following conditions are equivalent for a (quasi-) retractable module $M$:*

(i) *$M$ is an indecomposable R-endoRickart module.*
(ii) *$M$ is an endoBaer module.*

Proof.(i) $\Rightarrow$ (ii) Since $M$ is an indecomposable R-endoRickart module, then $M$ is a Baer module by [**?**, Corollary 4.6] and Proposition **??**. Thus an endoBaer module by Remark **??**.
(ii) $\Rightarrow$ (i) $M$ is a Baer module by Propsition **??** and indecomposable Rickart module by [**?**, Corollary 4.6]. Thus an R-endoRickart module by Remark **??**. □

**Theorem 4.1.** *Let $M$ be a right $R$-module, and let $S = \mathrm{End}_R M$ have no infinite set of nonzero orthogonal idempotents. Then the following conditions are equivalent:*

(i) *$M$ is an R-endoRickart module.*
(ii) *$M$ is an endoBaer module.*

Proof.(i) $\Rightarrow$ (ii) Since $M$ is an R-endoRickart module, $R$ is a right Rickart ring has no infinite set of nonzero orthogonal idempotents. Thus $R$ is a right Rickart ring if and only if $R$ is a Baer ring by [**?**, Theorem 7.55].
(ii) $\Rightarrow$ (i) It is clear. □

**Proposition 4.4.** *Let $M$ be a right $R$-module with only countably many direct summands. Then the following conditions are equivalent:*

(i) *$M$ is an R-endoRickart module.*
(ii) *$M$ is an endoBaer module.*

Proof. (i) $\Rightarrow$ (ii) Since $M$ has only countably many direct summands, $\mathrm{End}_R(M)$ has no infinite set of nonzero orthogonal idempotents. Hence $M$ is an endoBaer module by Theorem **??**.
(ii) $\Rightarrow$ (i) It is clear. □

**Theorem 4.2.** *An $R$-module $M$ is an R-endoRickart and $S = \mathrm{End}_R(M)$ has the $SSIP$ if and only if $M$ is an endoBaer module.*

Proof. Let $N$ be any submodule of $S$. Since $M$ is R-endoRickart, $S$ is a right Rickart ring and for each $n \in N$, there exists $e_n^2 = e_n \in S$ such that $r_S(n) = e_n S$. Thus, there exists $e^2 = e \in S$ such that $r_S(N) = \bigcap_{n \in N} r_S(n) = \bigcap_{n \in N} e_n S = eS$ by the $SSIP$. Thus, $S$ is a Baer ring and $M$ is an endoBaer module. Conversely, suppose $M$ is an endoBaer module. Hence $M$ is an R-endoRickart module by Remark **??**, and $S$ is a Baer ring. Thus, $S$ has the $SSIP$. □

**Corollary 4.1.** *Let $M$ be a retractable module and $S = \mathrm{End}_R(M)$ has the $SSIP$. Then the following conditions are equivalent:*

(i) *$M$ is an R-endoRickart module.*

(ii) *$M$ is an endoBaer module.*

(iii) *$\varphi$ splits in $M$ for any $\varphi \in \mathrm{End}_R(M)$.*

Proof. $(i) \Leftrightarrow (ii)$ Follows from Theorem **??**.

(ii) ⇒ (iii) For $\varphi \in \mathrm{End}_R(M)$, consider the short exact sequence

$$0 \to \mathrm{Ker}\varphi = r_M(\varphi) \to M \to \varphi M \to 0.$$

Since $M$ is a retractable module and $S$ is a Baer ring, $M$ is a Baer module by [**?**, Proposition 4.6]. Thus $M$ is a Rickart module and $\mathrm{Ker}\varphi \leq^{\oplus} M$. So the short exact sequence splits.

(iii) ⇔ (i) $\varphi$ splits in $M$ for any $\varphi \in \mathrm{End}_R(M)$ if and only if $\mathrm{Ker}\varphi \leq^{\oplus} M$ if and only if $M$ is a Rickart module if and only if $M$ is an R-endoRickart module by Proposition **??**. □

**Proposition 4.5.** *Let $M$ be a (quasi-) retractable module and $S = \mathrm{End}_R(M)$ with only two idempotents, 0 and 1. Then the following conditions are equivalent:*

(i) *$M$ is an R-endoRickart module.*

(ii) *$M$ is an endoBaer module.*

Proof. $(i) \Rightarrow (ii)$ Since $S$ is a right Rickart ring with only two idempotents, 0 and 1, then $S$ is a domain by [**?**, Remark 4.10]. and then $M$ is an indecomposable R-endoRickart module by [**?**, Proposition 4.9] and Remark **??**. Thus $M$ is an endoBaer module by Proposition **??**.

(ii) ⇒ (i) It is clear. □

Recall that a ring $R$ is a right (left) self injective ring if it is injective over itself as a right (left) module. If a von Neumann regular ring $R$ is also right or left self injective, then $R$ is Baer.

**Proposition 4.6.** *Let $M$ be an $R$-module and $S = \mathrm{End}_R(M)$ be any right self-injective ring. Then the following conditions are equivalent:*

(i) *$M$ is an R-endoRickart module.*

(ii) *$M$ is an endoBaer module.*

Proof. $(i) \Rightarrow (ii)$ Let $M$ be an R-endoRickart module, $S$ is a right Rickart ring. Since $S$ is right self-injective ring, then $S$ is a right Rickart ring if and only if it is a Baer ring by [**?**, Theorem 7.52]. Thus $M$ is an endoBaer module.

(ii) ⇒ (i) It is clear. □

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