Multiple Exact Travelling Solitary Wave Solutions of Nonlinear Evolution Equations

ABSTRACT:

An extended Tanh-function method with Riccati equation is presented for constructing multiple exact travelling wave solutions of some nonlinear evolution equations which are particular cases of a generalized equation. The results of solitary waves are general compact forms with non-zero constants of integration. Taking the full advantage of the Riccati equation improves the applicability and reliability of the Tanh method with its extended form.

Keywords: Extended Tanh method, Riccati equation, Solitary waves, Evolution Equations.

1. Introduction

Nonlinear partial differential equations (NLPDEs) play a major role in the study of nonlinear science. In recent decades, constructing the exact travelling solitary wave solutions and solitons of NLPDEs have become an important research subject due to the constant proposing of analytical methods, say, [1]–[14]. Among these methods, the powerful Hyperbolic Tangent (Tanh) method [2], [15], which has been tremendously developed in the literature – for instance [7], [8], [16]. More precisely, the Extended Tanh method (later known as Tanh-coth method) and its modified form was introduced by [7]–[9] and has been successfully utilized to obtain the solutions of NLPDEs. The Modified Extended Tanh method with Riccati equation [9], [16], [17] is widely recognized as one of the most powerful tools used in a favor of obtaining the explicit travelling solitary wave solutions of NLPDEs.

The following NLPDE is proposed as a generalization of the equations under study, which involves nonlinear dispersion and dissipation effects [18]:

$$u_t + \alpha u u_x + \beta u^2 u_x + \nu u_{xx} + \mu u_{px} = 0, \qquad (1)$$

Where $\alpha\beta \neq 0, \nu\mu \neq 0$ and p are all arbitrary constants. Considering the setting of these parameters to be equal to special values, with $\beta = 0$, equation (1) is reduced to KdV-Burgers equation ($p = 3, \quad \alpha\nu\mu \neq 0$), and to Kuramoto-Sivanshinsky $\left(p = 4, \alpha\nu\mu \neq 0\right)$. The governing NLPDEs take the following well-known forms (respectively):

$$u_t + \alpha u u_x + \nu u_{xx} + \mu u_{3x} = 0, \qquad (2)$$

$$u_t + \alpha u u_x + \nu u_{xx} + \mu u_{4x} = 0, \qquad (3)$$

However, the class of this NLPDE when $\beta \neq 0$ is considered in [19]. This paper is organized to fully present the algorithm of the considered method in Section 2. The analytical solution in the form of travelling solitary wave solutions of equation (1), with its special parameters' values are obtained in Section 3. Finally, in Section.4 concluding remarks are presented.

2. The Methodology of the method

The travelling solitary wave solution of a NLPDE in two variables x, t :

$$\Psi_1\left(u, u_t, u_x, u_{xt}, u_{xx}, \ldots\right)=0, \tag{4}$$

is the solution of the nonlinear ordinary differential equation NLODE:

$$\Psi_2\left(U, U', U'', U''', \ldots\right)=0, \tag{5}$$

Which is obtained by using the travelling wave transformation $u(x,t)=U(\zeta)=U(x-\omega t)$, and the prime denotes the ordinary derivative with respect to ζ. Introducing a new variable $\psi=\psi(\zeta)$, that satisfies the Riccati equation of the form:

$$\frac{d}{d\zeta}\psi(\zeta)=k+\psi(\zeta)^2, \tag{6}$$

where k is a real constant. The modified Extended Tanh method with Riccati equation admits that the solution of (5) can be expressed by a polynomial in ψ^j :

$$\begin{aligned} u(x,t)=U(\zeta)=&\, a_N\psi^N+a_{N-1}\psi^{N-1}+\ldots+a_1\psi+a_0 \\ &+b_1\psi^{-1}+\ldots+b_{N-1}\psi^{-N-1}+b_N\psi^{-N}, \end{aligned} \tag{7}$$

where N is the balancing integer. Substituting (6) along with (7) into (5), then setting the coefficients of all powers of $\psi(\zeta)^j$ to zero, a nonlinear algebraic system is generated with respect to parameters a_0, a_j, b_j, k, ω. By the sign test of k, the Riccati equation (6) has the well-known general solutions:

$$\psi(\zeta)=-\frac{1}{\zeta} \qquad ,k=0 \tag{8}$$

$$\psi(\zeta)=\begin{cases}-\sqrt{-k}\tanh\left(\sqrt{-k}\left(x-\omega t\right)\right) \\ -\sqrt{-k}\coth\left(\sqrt{-k}\left(x-\omega t\right)\right)\end{cases} \qquad k<0 \tag{9}$$

$$\psi(\zeta)=\begin{cases}\sqrt{k}\tan\left(\sqrt{k}\left(x-\omega t\right)\right) \\ -\sqrt{k}\cot\left(\sqrt{k}\left(x-\omega t\right)\right)\end{cases} \qquad k>0 \tag{10}$$

3. The solitary travelling wave solutions

3.1 Explicit solution of KdV-Burgers equation

Using the wave transformation prescribed in the previous section gives rise to the NLODE:

$$-\omega U'+\alpha UU'+\nu U''+\mu U'''=0, \tag{11}$$

Integrating (11) with respect to ζ, to get:

$$-\omega U+\frac{\alpha}{2}U^2+\nu U'+\mu U''+\eta_0=0, \tag{12}$$

where η_0 is an arbitrary constant. With $N=2$ (by balancing U^2 and U'' using the homogeneous balance principle); therefore, equation (7) admits the ansätz:

$$U(\zeta)=a_0+a_1\psi(\zeta)+a_2\psi^2(\zeta)+b_1\psi^{-1}(\zeta)+b_2\psi^{-2}(\zeta), \tag{13}$$

Substituting (13) into (12) and with the use of (6), we obtain the following algebraic system by setting all the coefficients of $\psi^j, j=0,\pm1,\pm2$ to zero:

$$\begin{aligned}
&6k^2\mu b_2+\frac{\alpha b_2^2}{2}=0,\\
&2k^2\mu b_1-2k\nu b_2+\alpha b_1 b_2=0,\\
&-k\nu b_1+\frac{\alpha b_1^2}{2}+8k\mu b_2-\omega b_2+\alpha a_0 b_2=0,\\
&2k\mu b_1-\omega b_1+\alpha a_0 b_1-2\nu b_2+\alpha a_1 b_2=0,\\
&\eta-\omega a_0+\frac{\alpha a_0^2}{2}+k\nu a_1+2k^2\mu a_2-\nu b_1+\alpha a_1 b_1+2\mu b_2+\alpha a_2 b_2=0,\\
&2k\mu a_1-\omega a_1+\alpha a_0 a_1+2k\nu a_2+\alpha a_2 b_1=0,\\
&\nu a_1+\frac{\alpha a_1^2}{2}+8k\mu a_2-\omega a_2+\alpha a_0 a_2=0,\\
&2\mu a_1+2\nu a_2+\alpha a_1 a_2=0,\\
&6\mu a_2+\frac{\alpha a_2^2}{2}=0
\end{aligned} \tag{14}$$

The system in (14) is solved by the aid of Mathematica, and taking into consideration the solution of Riccati equation (8) - (10), we obtain the following families of solutions:

Family1.

$$k=-\frac{\nu^2}{100\mu^2},\ \alpha=\frac{144k\nu^2+25\omega^2}{50\eta},\ a_0=\frac{-12k\mu+\omega}{\alpha},\ a_1=a_2=0, b_1=\frac{12k\nu}{5\alpha}, b_2=-\frac{12k^2\mu}{\alpha} \tag{15}$$

η and ω are an arbitrary

As it is noted the value of $k<0$ whenever $(\nu\mu)^2>0$, thus the corresponding travelling wave solution is:

$$u_1(x,t)=\frac{1}{\alpha}(\frac{6\nu^2}{25\mu}+\omega)-\frac{3\nu^2}{25\alpha\mu}(\coth(\frac{\nu}{10\mu}(x-\omega t))-1)^2 \tag{16}$$

Family2.

$$k=-\frac{\nu^2}{100\mu^2},\alpha=\frac{144k\nu^2+25\omega^2}{50\eta},a_0=\frac{-12k\mu+\omega}{\alpha},a_1=-\frac{12\nu}{5\alpha},a_2=-\frac{12\mu}{\alpha},b_1=b_2=0 \tag{17}$$

ω is an arbitrary.

Since $k<0$ whenever $(\nu\mu)^2>0$, thus the corresponding travelling wave solution is:

$$u_2(x,t)=\frac{1}{\alpha}\left(\frac{6\nu^2}{25\mu}+\omega\right)-\frac{3\nu^2}{25\alpha\mu}\left(\tanh(\frac{\nu}{10\mu}(x-\omega t))-1\right)^2 \tag{18}$$

Family3.

$$k=-\frac{\nu^2}{400\mu^2},\alpha=\frac{576k\nu^2+25\omega^2}{50\eta},a_0=\frac{-24k\mu+\omega}{\alpha},a_1=-\frac{12\nu}{5\alpha},a_2=-\frac{12\mu}{\alpha}, \tag{19}$$

$b_1=-ka_1, b_2=k^2a_2$, ω **is an arbitrary**)

Since $k<0$ whenever $(\nu\mu)^2>0$, thus the corresponding travelling wave solution is:

$$u_3(x,t)=\frac{1}{\alpha}\left(\frac{3\mu q^2}{50}+\omega\right)+\frac{3q}{25\alpha}\tanh(z)\left(\nu-\frac{q\mu}{4}\tanh(z)\right)+\frac{3q}{25\alpha}\coth(z)\left(\nu-\frac{\mu q}{4}\coth(z)\right)$$

$$q=\frac{\nu}{\mu}, z=\frac{1}{20}\frac{\nu}{\mu}(x-\omega t) \tag{20}$$

This solution can be reduced to obtain the travelling solitary wave solution in equation (16).

Family4.

$$\mu=\mp\frac{6\nu^2}{25\omega},\eta=0,k=-\frac{\nu^2}{100\mu^2},a_0=\frac{-12k\mu+\omega}{\alpha},a_1=a_2=0,$$

$$b_1=\frac{12k\nu}{5\alpha},b_2=-\frac{12k^2\mu}{\alpha},\ \omega \text{ is an arbitrary}) \tag{21}$$

Since $k<0$ whenever $(\nu\mu)^2>0$, thus the corresponding travelling wave solution is:

$$u_{4,5}(x,t)=\frac{1}{\alpha}(\frac{6\nu^2}{25\mu}+\omega)-\frac{3\nu^2}{25\alpha\mu}\left(\coth(\frac{\nu}{10\mu}(x-t\omega))-1\right)^2 \tag{22}$$

Family5.

$$\mu=\mp\frac{6\nu^2}{25\omega},\eta=0,k=-\frac{\nu^2}{100\mu^2},a_0=\frac{-12k\mu+\omega}{\alpha},a_1=-\frac{12\nu}{5\alpha},a_2=-\frac{12\mu}{\alpha}, \tag{23}$$

$b_1=0, b_2=0,\quad \omega$ **is an arbitrary**

Since $k<0$ whenever $(\nu\mu)^2>0$, thus the corresponding travelling wave solution is:

$$u_{6,7}(x,t)=\frac{1}{\alpha}\left(\frac{6\nu^2}{25\mu}+\omega\right)-\frac{3\nu^2}{25\alpha\mu}\left(\tanh(\frac{\nu}{10\mu}(x-\omega t))-1\right)^2 \tag{24}$$

Family6.

$$\mu=\mp\frac{6\nu^2}{25\omega},\eta=0,k=-\frac{\nu^2}{400\mu^2},a_0=\frac{-24k\mu+\omega}{\alpha},a_1=-\frac{12\nu}{5\alpha},a_2=-\frac{12\mu}{\alpha}, \tag{25}$$

$b_1=-ka_1, b_2=k^2a_2$ α **and** ω **is an arbitrary**)

Since $k<0$ whenever $(\nu\mu)^2>0$ thus the corresponding travelling wave solution is:

$$u_{8,9}(x,t)=\frac{1}{\alpha}(\frac{3\nu^2}{10\mu}+\omega)-\frac{3\nu^2}{100\alpha\mu}\left(\tanh(t)-2\right)^2-\frac{3\nu^2}{100\alpha\mu}\left(\coth(z)-2\right)^2$$

$$q=\frac{\nu}{\mu},\;\; z=\frac{1}{20}q(x-\omega t) \tag{26}$$

which are reduced to obtain the travelling solitary wave solution in equation (15) .

The graphical representation of some travelling solitary wave solutions of (2) is illustrated as follows:

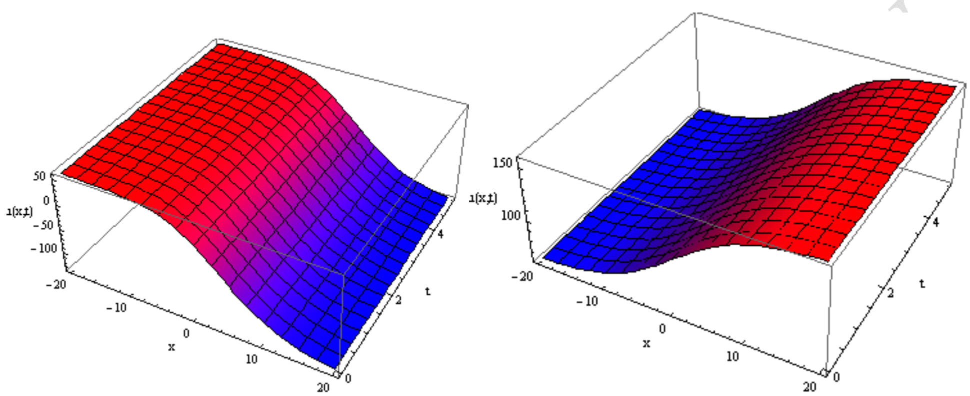

Figure 1 The plots of travelling solitary wave solutions (18) $(\eta=10)$ **and (24) when** $\nu=1, \mu=-1, \omega=0.1$.

3.2 Explicit solution of Kuramoto-Sivashinsky equation

Making the wave transformation prescribed in Section 2, the KS equation (3) is reduced to the following NLODE:

$$-\omega U'+\alpha UU'+\nu U''+\mu U^{(4)}=0, \tag{27}$$

Integrating (27) with respect to ζ, once yields:

$$-\omega U+\frac{\alpha}{2}U^2+\nu U'+\mu U'''+\varepsilon_0=0, \tag{28}$$

where ε_0 is an arbitrary constant. With $N=3$ (by balancing U''' and U^2 using the homogeneous balance principle); therefore, equation (7) admits the ansätz:

$$U(\zeta)=a_0+a_1\psi(\zeta)+a_2\psi^2(\zeta)+a_3\psi^3(\zeta)+b_1\psi^{-1}(\zeta)+b_2\psi^{-2}(\zeta)+b_3\psi^{-3}(\zeta), \tag{29}$$

Substituting (29) into (28) and with the use of (6), we obtain the following algebraic system by setting all the coefficients of $\psi^j, j=0, \pm1, \pm2, \pm3$ to zero:

$$
\begin{aligned}
&-60k^3\mu b_3+\frac{\alpha b_3^2}{2}=0,\\
&-24k^3\mu b_2+\alpha b_2 b_3=0,\\
&-6k^3\mu b_1+\frac{\alpha b_2^2}{2}-114k^2\mu b_3-3k\nu b_3+\alpha b_1 b_3=0,\\
&-40k^2\mu b_2-2k\nu b_2+\alpha b_1 b_2-\omega b_3+\alpha a_0 b_3=0,\\
&-8k^2\mu b_1-k\nu b_1+\frac{\alpha b_1^2}{2}-\omega b_2+\alpha a_0 b_2-60k\mu b_3-3\nu b_3+\alpha a_1 b_3=0,\\
&\\
&-\omega b_1+\alpha a_0 b_1-16k\mu b_2-2\nu b_2+\alpha a_1 b_2+\alpha a_2 b_3=0,\\
&-\omega a_0+\frac{\alpha a_0^2}{2}+2k^2\mu a_1+k\nu a_1+6k^3\mu a_3-2k\mu b_1-\nu b_1+\alpha a_1 b_1\\
&\qquad+\alpha a_2 b_2-6\mu b_3+\alpha a_3 b_3+\epsilon_0=0,\\
&-\omega a_1+\alpha a_0 a_1+16k^2\mu a_2+2k\nu a_2+\alpha a_2 b_1+\alpha a_3 b_2=0,\\
&8k\mu a_1+\nu a_1+\frac{\alpha a_1^2}{2}-\omega a_2+\alpha a_0 a_2+60k^2\mu a_3+3k\nu a_3+\alpha a_3 b_1=0,\\
&40k\mu a_2+2\nu a_2+\alpha a_1 a_2-\omega a_3+\alpha a_0 a_3=0,\\
&6\mu a_1+\frac{\alpha a_2^2}{2}+114k\mu a_3+3\nu a_3+\alpha a_1 a_3=0,\\
&24\mu a_2+\alpha a_2 a_3=0,\\
&60\mu a_3+\frac{\alpha a_3^2}{2}=0
\end{aligned}
\tag{30}
$$

The system in (30) is solved by the aid of Mathematica and by taking into consideration the solution of Riccati equation (8) - (10), we obtain the following families of solutions:

Family 1.

$$
\begin{aligned}
&k=-\frac{11\nu}{76\mu},\ \alpha=\frac{3600k\nu^2+361\omega^2}{722\epsilon_0},\ a_0=\frac{\omega}{\alpha},\ a_1=0,\ a_2=0,\ a_3=0,\\
&b_1=\frac{60(38k^2\mu+k\nu)}{19\alpha},\ b_2=0,\ b_3=\frac{120k^3\mu}{\alpha},\ \omega\ \textbf{and}\ \epsilon_0\ \textbf{are arbitraries}
\end{aligned}
\tag{31}
$$

As $\nu\mu>0$, we see that $k<0$. Consequently, we obtain:

$$
u_1(x,t)=\frac{\omega}{\alpha}-\frac{15\nu}{19\alpha}\sqrt{\frac{11\nu}{19\mu}}\coth(z)\left[9-11\coth^2(z)\right],\quad z=\frac{1}{2}\sqrt{\frac{11\nu}{19\mu}}(x-\omega t)
\tag{32}
$$

As $\nu\mu<0$, we see that $k>0$, the corresponding solution is:

$$u_2(x,t)=\frac{\omega}{\alpha}-\frac{15\nu}{19\alpha}\sqrt{-\frac{11\nu}{19\mu}}\cot(z)\left[9+11\cot^2(z)\right],\ \ z=\frac{1}{2}\sqrt{-\frac{11\nu}{19\mu}}(x-\omega t) \tag{33}$$

Family 2.

$$k=\frac{\nu}{76\mu},\alpha=\frac{3600k\nu^2+361\omega^2}{722\grave{o}_0},a_0=\frac{\omega}{\alpha},a_1=a_2=a_3=0,$$
$$b_1=\frac{60(38k^2\mu+k\nu)}{19\alpha},b_2=0,b_3=\frac{120k^3\mu}{\alpha},\ \ \omega\ \textbf{and}\ \grave{o}_0\ \textbf{are arbitararies} \tag{34}$$

If $\nu\mu>0$, then $k<0$. Consequently, we obtain:

$$u_3(x,t)=\frac{\omega}{\alpha}+\frac{15}{19\alpha}\nu\sqrt{\frac{-\nu}{19\mu}}\coth(z)(\left[3-\coth^2(z)\right]),\ \ z=\frac{1}{2}\sqrt{\frac{-\nu}{19\mu}}(x-\omega t) \tag{35}$$

If $\frac{\nu}{\mu}<0$, then $k>0$. Consequently, we obtain:

$$u_4(x,t)=\frac{\omega}{\alpha}+\frac{15\nu}{19\alpha}\sqrt{\frac{\nu}{19\mu}}\cot(z)\left[3+\cot^2(z)\right],\ \ z=\frac{1}{2}\sqrt{\frac{\nu}{19\mu}}(x-\omega t) \tag{36}$$

Family 3.

$$k=-\frac{11\nu}{76\mu},\alpha=\frac{3600k\nu^2+361\omega^2}{722\grave{o}_0},a_0=\frac{\omega}{\alpha},a_1=\frac{60(38k\mu+\nu)}{19\alpha},a_2=0,a_3=-\frac{120\mu}{\alpha},$$
$$b_1=b_2=b_3=0,\ \ \omega\ \textbf{and}\ \grave{o}_0\ \textbf{are arbitrarairs} \tag{37}$$

If $\nu\mu>0$, then $k<0$. Consequently, we obtain:

$$u_5(x,t)=\frac{\omega}{\alpha}-\frac{15}{19\alpha}\nu\sqrt{\frac{11\nu}{19\mu}}\tanh(z)\left[9-11\tanh^2(z)\right],\ \ z=\frac{1}{2}\sqrt{\frac{11\nu}{19\mu}}(x-\omega t) \tag{38}$$

If $\nu\mu<0$, then $k>0$. Consequently, we obtain:

$$u_6(x,t)=\frac{\omega}{\alpha}+\frac{15\nu}{19\alpha}\sqrt{\frac{-11\nu}{19\mu}}\tan(z)\left[9+11\tan(z)\right],\quad z=\frac{1}{2}\sqrt{\frac{-11\nu}{19\mu}}(x-\omega t) \tag{39}$$

Family 4.

$$k=\frac{\nu}{76\mu},\alpha=\frac{3600k\nu^2+361\omega^2}{722\grave{o}_0},a_0=\frac{\omega}{\alpha},a_1=-\frac{60(38k\mu+\nu)}{19\alpha},a_2=0,a_3=-\frac{120\mu}{\alpha},$$
$$b_1=b_2=b_3=0,\omega\ \textbf{and}\ \grave{o}_0\ \textbf{are arbitraries} \tag{40}$$

If $\nu\mu<0$, then $k<0$ and vice versa . Respectively, we obtain:

$$u_7(x,t)=\frac{\omega}{\alpha}+\frac{15\nu}{19\alpha}\sqrt{\frac{-\nu}{19\mu}}\tanh(z)\left[3-\tanh(z)^2\right],\ z=\frac{1}{2}\sqrt{\frac{-\nu}{19\mu}}(x-\omega t) \tag{41}$$

$$u_8(x,t)=\frac{\omega}{\alpha}-\frac{15\nu}{19\alpha}\sqrt{\frac{\nu}{19\mu}}\tan(z)[3+\tan^2(z)],\quad z=\frac{1}{2}\sqrt{\frac{\nu}{19\mu}}(x-\omega t) \tag{42}$$

Family 5.

$$k=-\frac{11\nu}{304\mu},\alpha=\frac{14400k\nu^2+361\omega^2}{722\grave{o}_0},a_0=\frac{\omega}{\alpha},a_1=-\frac{60(38k\mu+\nu)}{19\alpha},a_2=0,a_3=-\frac{120\mu}{\alpha},$$

$b_1=-ka_1,b_2=0,b_3=-k^3a_3$, ω **and** $\grave{o}_0$ **are arbitraries**

(43)

If $\nu\mu<0$, then $k<0$ and vice versa. Respectively, we obtain:

$$\begin{aligned}u_9(x,t)=&\frac{\omega}{\alpha}+\frac{15q}{19\alpha}(-\frac{19}{8}\mu q^2+\nu)\tanh(z)+\frac{15\mu q^3}{8\alpha}\tanh^3(z)\\&+\frac{15q}{19\alpha}(-\frac{19}{8}\mu q^2+\nu)\coth(z)+\frac{15\mu q^3}{8\alpha}\coth^3(z)\\q=&\sqrt{\frac{11\nu}{19\mu}},\quad z=\frac{1}{4}\sqrt{\frac{11\nu}{19\mu}}(x-\omega t)\end{aligned} \tag{44}$$

$$\begin{aligned}u_{10}(x,t)=&\frac{\omega}{\alpha}-\frac{15q}{19\alpha}(\frac{19}{8}\mu q^2+\nu)\tan(z)-\frac{15\mu q^3}{8\alpha}\tan^3(z)\\&+\frac{15q}{19\alpha}(\frac{19}{8}\mu q^2+\nu)\cot(z)+\frac{15\mu q^3}{8\alpha}\cot^3(z)\\q=&\sqrt{\frac{-11\nu}{19\mu}},\ z=\frac{1}{4}\sqrt{\frac{-11\nu}{19\mu}}(x-\omega t)\end{aligned} \tag{45}$$

The travelling solitary wave solutions (44) and (45) can be simplified so that $u_1(x,t)$ and $u_2(x,t)$ are obtained respectively.

Family 6.

$$k=\frac{\nu}{304\mu},\alpha=\frac{14400k\nu^2+361\omega^2}{722\grave{o}_0},a_0=\frac{\omega}{\alpha},a_1=-\frac{60(38k\mu+\nu)}{19\alpha},a_2=0,a_3=-\frac{120\mu}{\alpha}, \tag{46}$$

$b_1=-ka_1,b_2=0,b_3=-k^3a_3$, ω **and** $\grave{o}_0$ **are arbitraries**

If $\nu\mu<0$, then $k<0$ and vice versa. Respectively, we obtain:

$$u_{11}(x,t)=\frac{\omega}{\alpha}+\frac{15q}{19\alpha}(-\frac{19}{8}\mu q^2+\nu)\tanh(\frac{1}{4}q(x-\omega t))+\frac{15\mu q^3}{8\alpha}\tanh^3(z)^3$$

$$+\frac{15q}{19\alpha}(-\frac{19}{8}\mu q^2+\nu)\coth(z)+\frac{15\mu q^3}{8\alpha}\coth^3(z) \tag{47}$$

$$q=\sqrt{\frac{-\nu}{19\mu}},\ z=\frac{1}{4}q(x-\omega t)$$

$$u_{12}(x,t)=\frac{\omega}{\alpha}-\frac{15q}{19\alpha}(\frac{19}{8}\mu q^2+\nu)\tan(z)-\frac{15\mu q^3}{8\alpha}\tan^3(z)$$

$$+\frac{15q}{19\alpha}(\frac{19}{8}\mu q^2+\nu)\cot(z)+\frac{15\mu q^3}{8\alpha}\cot^3(z) \tag{48}$$

$$q=\sqrt{\frac{\nu}{19\mu}},\ z=\frac{1}{4}q(x-\omega t)$$

The travelling solitary wave solutions (47) and (48) can be simplified so that $u_3(x,t)$ and $u_4(x,t)$ are obtained respectively.

Family 7.

$$\grave{o}_0=0,\mu=-\frac{900\nu^3}{6859\omega^2},k=-\frac{361\omega^2}{3600\nu^2},a_0=\frac{\omega}{\alpha},a_1=a_2=a_3=0,$$

$$b_1=\frac{60(38k^2\mu+k\nu)}{19\alpha},b_2=0,b_3=\frac{120k^3\mu}{\alpha},\quad \omega \text{ and } \alpha \text{ are arbitaraies} \tag{49}$$

Since $k<0$, it follows that:

$$u_{13}(x,t)=\frac{\omega}{\alpha}+\frac{\omega}{2\alpha}\coth(\frac{19\omega}{60\nu}(x-\omega t))\left(3-\coth^2(\frac{19\omega}{60\nu}(x-\omega t)\right) \tag{50}$$

Family 8.

$$\grave{o}_0=0,\mu=\frac{9900\nu^3}{6859\omega^2},k=-\frac{361\omega^2}{3600\nu^2},a_0=\frac{\omega}{\alpha},a_1=a_2=a_3=0,$$

$$b_1=\frac{60(38k^2\mu+k\nu)}{19\alpha},b_2=0,b_3=\frac{120k^3\mu}{\alpha},\quad \omega \text{ and } \alpha \text{ are arbitaraies} \tag{51}$$

Since $k<0$, it follows that:

$$u_{14}(x,t)=\frac{\omega}{\alpha}-\frac{\omega}{2\alpha}\coth(\frac{19\omega}{60\nu}(x-\omega t))\left(9-11\coth^2[\frac{19\omega}{60\nu}(x-\omega t)]\right) \tag{52}$$

Family 9.

$$\grave{o}_0 = 0, \mu = -\frac{900\nu^3}{6859\omega^2}, k = -\frac{361\omega^2}{3600\nu^2}, a_0 = \frac{\omega}{\alpha}, a_1 = -\frac{60(38k\mu+\nu)}{19\alpha},$$
$$a_2 = 0, a_3 = -\frac{120\mu}{\alpha}, b_1 = b_2 = b_3 = 0 \tag{53}$$

Since $k < 0$, it follows that:

$$u_{15}(x,t) = \frac{\omega}{\alpha} + \frac{\omega}{2\alpha}\tanh\left(\frac{19\omega}{60\nu}(x-\omega t)\right)\left(3 - \tanh^2\left(\frac{19\omega}{60\nu}(x - t\omega)\right)\right) \tag{54}$$

Family 10.

$$\grave{o}_0 = 0, \mu = \frac{9900\nu^3}{6859\omega^2}, k = -\frac{361\omega^2}{3600\nu^2}, a_0 = \frac{\omega}{\alpha}, a_1 = -\frac{60(38k\mu+\nu)}{19\alpha}, a_2 = 0,$$
$$a_3 = -\frac{120\mu}{\alpha}, b_1 = b_2 = b_3 = 0 \tag{55}$$

Since $k < 0$, it follows that:

$$u_{16}(x,t) = \frac{\omega}{\alpha} - \frac{\omega}{2\alpha}\tanh\left(\frac{19\omega}{60\nu}(x-\omega t)\right)\left(9 - 11\tanh^2\left(\frac{19\omega}{60\nu}(x - \omega t)\right)\right) \tag{56}$$

Family 11.

$$\grave{o}_0 = 0, \mu = -\frac{900\nu^3}{6859\omega^2}, k = -\frac{361\omega^2}{14400\nu^2}, a_0 = \frac{\omega}{\alpha}, a_1 = -\frac{60(38k\mu+\nu)}{19\alpha}, a_2 = 0,$$
$$a_3 = -\frac{120\mu}{\alpha}, b_1 = -ka_1, b_2 = 0, b_3 = -k^3 a_3 \tag{57}$$

Since $k < 0$, it follows that:

$$u_{17}(x,t) = \frac{\omega}{\alpha} + \frac{q}{2\alpha}\left(\frac{-19(361)}{7200}\mu q^2 + \nu\right)\left(\tanh(z) + \coth(z)\right)$$
$$+\frac{6859\mu q^3}{14400\alpha}\left(\tanh^3(z) + \coth^3(z)\right), \tag{58}$$
$$q = \frac{\omega}{\nu}, z = \frac{19}{120}q(x - \omega t)$$

Simplifying (58) the travelling solitary wave solution in (50) is obtained.

Family 12.

$$\grave{o}_0 = 0, \mu = \frac{9900\nu^3}{6859\omega^2}, k = -\frac{361\omega^2}{14400\nu^2}, a_0 = \frac{\omega}{\alpha}, a_1 = -\frac{60(38k\mu+\nu)}{19\alpha}, a_2 = 0,$$
$$a_3 = -\frac{120\mu}{\alpha}, b_1 = -ka_1, b_2 = 0, b_3 = -k^3 a_3 \tag{59}$$

Since $k < 0$, it follows that:

$$u_{18}(x,t) = \frac{\omega}{\alpha} + \frac{q}{2\alpha}\left(\frac{-19(361)}{7200}\mu q^2 + \nu\right)\left(\tanh(z) + \coth(z)\right)$$

$$+ \frac{6859\mu q^3}{14400\alpha}\left(\tanh^3(z) + \coth^3(z)\right) \quad (60)$$

$$q = \frac{\omega}{\nu},\ z = \frac{19}{120}q(x - \omega t)$$

By simplifying (60) the travelling solitary wave solution (52) is obtained.

The graphical representation of some travelling solitary wave solutions of (3) is illustrated as follows:

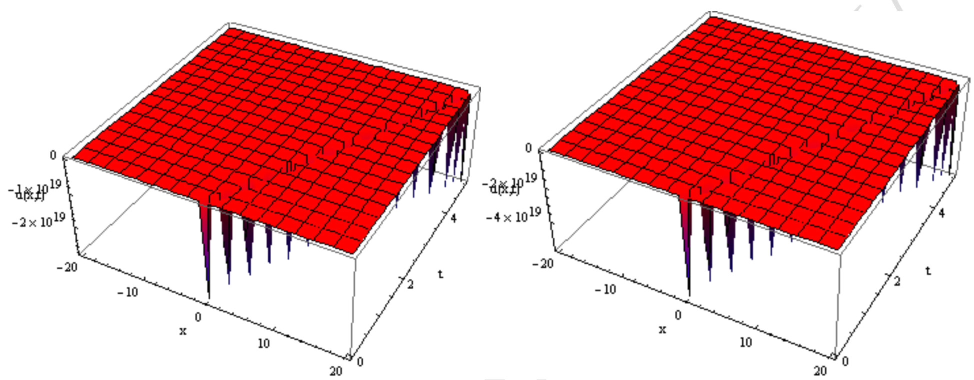

Figure 2 The plots of travelling solitary wave solutions (32) and (33) when
$\nu = 1, \mu = 1, \omega = 4; (\grave{o}_0 = -10)$.

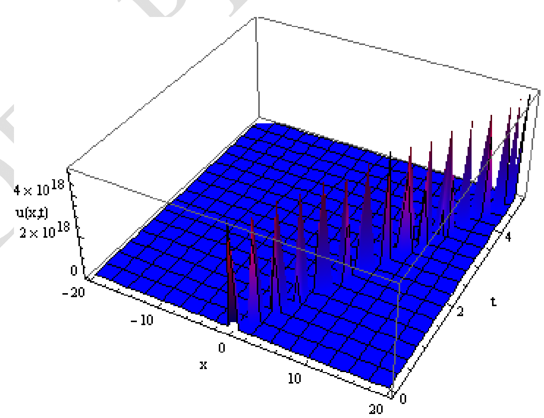

Figure 3 The plot of travelling solitary wave solutions (52) when $\nu = 1, \mu = 1, \omega = 4$.

Remark: All solutions are tested to satisfy their related PDEs and found to be more generalized compact forms with nonzero constants of integration; as mentioned in [20].

4. Conclusion

In this presented work, we have established and successfully employed the modified Extended Tanh method with Riccati equation for obtaining the solitary travelling wave solutions for a given class of NLPDEs. The method has the advantage of being direct and concise. In addition, an enormous variety of solutions was obtained with the aid of Mathematica software.

5. References

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