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Proof of Collatz conjecture

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Abstract

Collatz conjecture (stated in 1937 by Collatz and also named Thwaites conjecture, or Syracuse, 3n+1 or oneness problem) can be described as follows:
Take any positive whole number N. If N is even, divide it by 2. If it is odd, multiply it by 3 and add 1. Repeat this process to the result over and over again. Collatz conjecture is the supposition that for any positive integer N, the sequence will invariably reach the value 1.
The main contribution of this paper is to present a new approach to Collatz conjecture. The key idea of this new approach is to clearly differentiate the role of the division by two and the role of what we will name here the jump: $a = 3n + 1$.
With this approach, the proof of the conjecture is given as well as informations on generalizations for jumps of the form $qn + r$ and for jumps being polynomials of degree m >1.
The proof of Collatz algorithm necessitates only 5 steps:
1- to differentiate the main function and the jumps;
2- to differentiate branches as well as their first and last terms a and n;
3- to identify that left and irregular right shifts in branches can be replaced by regular shifts in 2^m-type columns;
4- to identify the key equation $a_i = 3n_{i-1} + 1 = 2^m$ as well as its solutions;
5- to reduce the problem to compare *the number of jumps* to *the number of divisions* in a trajectory.

Keywords: Collatz; 3n+1; Syracuse; Thwaites; oneness; conjecture; even; odd; jumps; integer

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1 Introduction

A Collatz [1] sequence is obtained from a start integer N to which one applies the infinitely iterative function f defined by:

- $f_0 = N$ with the integer $N \neq 0$;
- $f_{i+1} = f_i/2$ if f_i is even;
- $f_{i+1} = 3f_i + 1$ if f_i is odd.

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For example, if we start with N = 7 and run the iterative function, we obtain the infinite list of numbers: 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, ...

The sequence falls into an endless loop (1, 4, 2) but it is arbitrarily accepted that the sequence is considered done when the first number 1 is reached. This means that when the symbolic test ”if (f_i=1) done” is added to the function f, it transforms it into an algorithm that will be denoted here by ” the Collatz algorithm”.

Collatz conjecture [2][3] is now the supposition that for any positive integer N, Collatz algorithm will always end up at 1.

Hundreds of mathematicians have tried to crack this conjecture and, as far as I know, many of their results to date are based on probabilities, cycles or other criteria. As the problem seems still open and as I wanted to attack it with a new vision, all these results are consciously ignored here.

Let’s just notice two points:

- No information is known about the way Collatz became interested in the function $f_{i+1} = 3f_i{+}1$;
- The sequence of numbers obtained for any N, also named a trajectory, is often considered in the literature as a list of undifferentiated numbers as in [4] and many other references. This was the reason to choose an approach based on the differentiation of these numbers.

2 Preliminary notes

2.1 New terms: main function and jump

In order to obtain a list of differentiated numbers we first introduce two new terms:

- **the main function**: for Collatz algorithm, the main function is the division by two of even values of function f: $f_{i+1} = f_i/2$;
- **the jump**: for Collatz algorithm, the jump is the special treatment $f_{i+1} = 3f_i + 1$ that is used to replace odd values $f_i = n_i$ by an even value $a = f_{i+1}$ usable by the main function and that we will write for convenience and from now on as: $a = 3n + 1$.

  Let’s notice that jumps assure the continuity of the main function.

2.2 The new approach: series of numbers S$_i$

From now on, the new approach on Collatz problem is given by representing the use of the main function by commas and the use of jumps by semi-colons in Collatz sequence.

Then, for $N = 7$, Collatz algorithm gives the list of six “*series of numbers*”S_i:

7 ; 22,11 ; 34,17 ; 52,26,13 ; 40,20,10,5 ; 16,8,4,2,1

This approach highlights also the fact that jumps $a = 3n + 1$ differentiate the last number n of each series by the fact it is odd.

As this display, as far as I know, has never been used, no more references can be given.

We notice, still for $N = 7$, that if another algorithm is used, based by instance on the jump $a = n + 17$ instead of $a = 3n + 1$, this new algorithm falls into an endless loop as we have:

7 ; 24,12,6,3 ; 20,10,5 ; 22,11 ; 28,14,7 ;
24,12,6,3 ; 20,10,5 ; 22,11 ; 28,14,7 ;
...

where the last sequence (24,12,6,3 ; 20,10,5 ; 22,11 ; 28,14,7 ;) is looping on itself *without reaching 1.*

So, the question that has to be answered to prove Collatz conjecture is:

Does Collatz algorithm using the jump $a = 3n + 1$, always ends up at 1 whatever is the start number N and why?

To answer this question, we must first remind a general property of natural numbers and put forward three new ones.

2.3 Property 1 of natural integers N

From the fundamental theorem of arithmetic given in 1801 by Gauss [5], any natural number N can be factorized in only one way when the factorization is ordered by increasing primes, as:

$$N = 2^w p_1^{\alpha_1} p_2^{\alpha_2} \mathrm{p}_3^{\alpha_3} \ldots$$

where the exponents w and α_i are positive or null integers and p_i are increasing odd primes or:

Property 1: Any natural number N can be factorized as: $N = n2^w$
where $n > 0$ is an *odd* integer, composite or prime
and w is a positive or null integer.

2.4 Property 2 of series of numbers S$_i$

Using property 1, we can prove

Property 2: For any given natural number N, the series of numbers S_i
are parts of branches B_i of general form: $B_i = B(n, w) = n2^w$
with the odd integer $n > 0$ and the natural integer $w \geqslant 0$

Proof. Using Property 1 let's build the following table filled bottom to top by an odd n in column 0 and adding numbers left to right by recurrently multiplying these ones by 2.

Table 1. S_i are parts of branches $B(n, w) = n2^w$
for odd $n > 0$ and $w \geqslant 0$

Br\Cols:	C$_0$	C$_1$	C$_2$	C$_3$	C$_4$	C$_5$	C$_6$	C$_7$	...
...	...	...	...	...	...	...	...	...	...
$B(17, w)$:	17	34	68	136	272	544	1088	2176	...
$B(15, w)$:	15	30	60	120	240	480	960	1920	...
$B(13, w)$:	13	26	52	104	208	416	832	1664	...
$B(11, w)$:	11	22	44	88	176	352	704	1408	...
$B(9, w)$:	9	18	36	72	144	288	576	1152	...
$B(7, w)$:	7	14	28	56	112	224	448	896	...
$B(5, w)$:	5	10	20	40	80	160	320	640	...
$B(3, w)$:	3	6	12	24	48	96	192	384	...
$B(1, w)$:	1	2	4	8	16	32	64	128	...
$B(n, w) \uparrow$	$B(n, 0) =$ *odd* $n \uparrow$	$w \rightarrow 1$	2	3	4	5	6	7	...

Now, reading from right to left, each line is a list of numbers that are divided by 2 until they reach an odd number: this is exactly the first part of the definition of the main function. ☐

2.5 Property 3 of branches B(n,w)

Property 3: The infinite set of branches $B(n, w)$ is
a covering system of the natural number set $\mathbb{N}$
or:
Any positive integer (even or odd) is present in Table 1.

Proof. This is because all branches $B(n, w)$ are of the form $B(n, w) = n2^w$ where n is odd, which is exactly the general definition of natural numbers according to the fundamental theorem of arithmetic. □

In Table 1, property 3 is true only for numbers up to 17 as odd numbers are limited to 17, but it suffices to expand the table upwards to reach any odd number $2^w - 1$.

For $N = 7$, the result given by Collatz algorithm can then be represented as follows, with a last column indicating the part of branch used by each series of numbers:

Table 2. Collatz trajectory of $N = 7$
using parts of branches $B(n, w = list)$

a_i	, ... ,	n_i	branch
$a_1 = N = 7$	, ... ,	$n_1 = 7$;	$B_1 = B(7, w = 0)$
$a_2 = 3n_1 + 1 = 22$	, ... ,	$n_2 = 11$;	$B_2 = B(11, w = 1, 0)$
$a_3 = 3n_2 + 1 = 34$	, ... ,	$n_3 = 17$;	$B_3 = B(17, w = 1, 0)$
$a_4 = 3n_3 + 1 = 52$	, ... ,	$n_4 = 13$;	$B_4 = B(13, w = 2, 1, 0)$
$a_5 = 3n_4 + 1 = 40$	, ... ,	$n_5 = 5$;	$B_5 = B(5, w = 3, 2, 1, 0)$
$a_6 = 3n_5 + 1 = 16$	, ... ,	$n_6 = 1$	$B_6 = B(1, w = 4, 3, 2, 1, 0)$

The trajectory for $N = 7$ can thus be summed up by the list of parts of branches:

$$B(7, w = 0) \ ; B(11, w = 1, 0) \ ; B(17, w = 1, 0) \ ;$$
$$B(13, w = 2, 1, 0) \ ; B(5, w = 3, 2, 1, 0) \ ; B(1, w = 4, 3, 2, 1, 0)$$

where we notice that, for a given N, the n_i's of the branches $B(n_i, w)$ are somewhat erratic.

2.6 Condition enabling Collatz algorithm to end up at 1

For any N but *if the series S_i is supposed to be the last one* which gives i=last, we deduce that it is always the last jump from n_{last-1} to $a_{last} = 3n_{last-1} + 1$ that leads to $n_{last} = 1$. Therefore we have

Condition: Collatz algorithm ends up at 1
if there exists a triplet solution (i, n_{i-1}, m) to the equation:
$$3n_{i-1} + 1 = 2^m$$

2.7 Solutions n(m) to reach 1

To solve the last equation where $i = last = m$, we can consider that either n_{i-1} (or n_i) is a function $n_{i-1}(m)$ or the converse, m is the function $m = m(n_{i-1})$.

We can look for solutions $m(n_i)$ but as n_i can be known only by running the algorithm to its end, this is not a mathematical solution.

We have then to look for solutions $n_i(m)$. As the first a_i of each branch (except maybe in the first branch when N is odd) is of the general form:

$$a_i = 3n_{i-1} + 1$$

we have first to check if the equation:

$$3n + 1 = 2^m \tag{2.1}$$

where n and m are independent of i (and thus of N), has always at least one solution $n(m)$ or not. From equation (2.1), the solutions $n(m)$ always verify:

$$n(m) = \frac{2^m - 1}{3}$$

In the two following subsections we will prove that they can be either integer or fractional (but, in the last case, invisible because Collatz algorithm works only with integer n's).

2.7.1 The integer solution n(m)

To study this integer solution is equivalent to study the factorization of the numbers $2^m - 1$ as equation (2.1) implies that:

$$2^m - 1 = 3n$$

We know that when m is even ($m = 2k$), we algebraically have

$$2^m - 1 = 2^{2k} - 1 = 4^k - 1^k$$
$$4^k - 1^k = (4-1)(4^{k-1} + 4^{k-2} + 4^{k-3} + ... + 4 + 1)$$

so that:

$$2^{2k} - 1 = 3(4^{k-1} + 4^{k-2} + 4^{k-3} + ... + 16 + 4 + 1)$$
$$2^{2k} = 3(4^{k-1} + 4^{k-2} + 4^{k-3} + ... + 16 + 4 + 1) + 1$$

Therefore, whatever is N, the integer solution $n(m)$ of the equation $3n + 1 = 2^m$ when $m = 2k$ is:

$$m = 2k$$
$$n(m) = 4^{k-1} + 4^{k-2} + 4^{k-3} + ... + 16 + 4 + 1$$

2.7.2 The fractional solution n(m)

We know that when m is odd $(m = 2k + 1)$, algebraically we have

$$2^m - 1 = 2^{2k+1} + (-1)^{2k+1}$$
$$2^{2k+1} + (-1)^{2k+1} = (2 + (-1))(2^{2k} + 2^{2k-1} + ... + (4 + 2) + 1)$$

but as:

$$2^{2k} + 2^{2k-1} = (2+1)2^{2k-1} = 3 \times 2^{2k-1}$$

we have:

$$2^{2k+1} - 1 = 3(2^{2k-1} + 2^{2k-3} + ... + 2^3 + 2^1) + 1$$

and with $n = (2^{2k-1} + 2^{2k-3} + ... + 2^3 + 2^1)$:

$$2^{2k+1} = (3n + 1) + 1 = 3n + 2$$

which can be written:

$$2^{2k+1} = 3(n + 1/3) + 1$$

This shows that when $m = 2k + 1$ is odd, the equation $2^m = 3n + 1$ has no integer solution $n(m)$ but always a fractional one that verifies:

$$n(m) = \frac{2^m - 1}{3}$$
$$n(m) = (2^{2k-1} + 2^{2k-3} + ... + 2^3 + 2^1) + 1/3 = n + 1/3$$

which always gives:

$$3n(m) + 1 = 3\left((2^{2k-1} + 2^{2k-3} + ... + 2^3 + 2^1) + 1/3\right) + 1 = 3\left(n + 1/3\right) + 1 = 3n + 2 = 2^{2k+1} = 2^m$$

2.7.3 Local conclusion on solutions n(m)

We have therefore the important result

Local conclusion. Independently of N and for any $m \geqslant 0$, the general equation $3n + 1 = 2^m$ has always a solution for n. This solution is:

either the integer solution of $3n + 1 = 2^m$ for any even m
or, for any odd m:
the fractional solution of $3n + 1 = 2^m$
or the integer solution of $3n + 2 = 2^m$

Remark. The fractional solution happens by instance for $N = 7$ as Collatz algorithm ends up at f=1 for $n = 682 + 1/3$ because:

$$3(682 + 1/3) + 1 = 3 \times 682 + 2 = 2048 = 2^{11}$$

Thus, after $m = 11$ divisions (or commas), the full trajectory for $N = 7$ being:

7 ; 22,11 ; 34,17 ; 52,26,13 ; 40,20,10,5 ; 16,8,4,2,1

we get the following Table by retaining only the values f^* that follow a division or comma:

Table 3. Details for $N = 7$

variables	val	val	val	val	val	val	val	val	val	val	val
m	1	2	3	4	5	6	7	8	9	10	11
f^*	11	17	26	13	20	10	5	8	4	2	1
2^m	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}
$n(m)$	1/3	1	7/3	5	31/3	21	$\dots$	$\dots$	$\dots$	$\dots$	$(2^{11} - 1)/3$

This shows that jumps (semi-colons) have not to be taken into account in the calculation of m contradicting the usual way to consider Collatz sequence as a series of undifferentiated numbers $C(n) = s^k(n)_{k=0}^{\infty}$ as in [6]. This can be illustrated by placing each division in a column and the jumps a_i *under* the last odd n_{i-1}. This creates a 2-dimensional table for the trajectory where each branch is isolated in a line, as follows:

Table 4. Trajectory of $N = 7$ in 11 divisions (commas)

Br\\m	0	1	2	3	4	5	6	7	8	9	10	11
1	7											
2	22,	11										
3	(68)	34,	17									
4	(208)	(104)	52,	26,	13							
5	(640)	(320)	(160)	(80)	40,	20,	10,	5				
6	(2^{11})	(2^{10})	(2^9)	(2^8)	(2^7)	(2^6)	(2^5)	16,	8,	4,	2,	1.

The numbers in parenthesis are not part of the trajectory but show the prolongation of each branch on the left. The observation of this table gives three facts:

- All n_i's of the trajectory of $N = 7$ have a trajectory that ends up at 1.
- As each a_i is even, at least one division is always possible after a jump and

Each $a_i = 2^{\alpha_i} n_i$

- The number B of branches is equal to the number of lines in Table 4. We then have:

$B = 1 + J$ where J is the number of jumps

Now, let's consider the product P of the first a_i of each branch i (let's notice that for $N = 7$, in the first line of Table 4, 7 is both an a_i and a n_i):

$$P = \prod_{i=1}^{B} a_i = \prod_{i=1}^{B} 2^{\alpha_i} n_i = 2^{\sum_{i=1}^{B} \alpha_i} \prod_{i=1}^{B} n_i$$

As the number of divisions m to go from N to 1 by Collatz algorithm is the same as the number of multiplications by 2 to go back from 1 to N, we have:

$$m = \sum_{i=1}^{B} \alpha_i \tag{2.2}$$

so that:

$$\prod_{i=1}^{B} a_i = 2^m \prod_{i=1}^{B} n_i$$

and:

$$m = log_2 \frac{\prod_{i=1}^{B} a_i}{\prod_{i=1}^{B} n_i} \tag{2.3}$$

Verification for our case $N = 7$:

$$m = log_2 \frac{7.22.34.52.40.16}{7.11.17.13.5.1} = log_2 2048 = log_2 2^{11} = 11$$

But due to the erratic values n_i ending the successive branches, we are still not sure that one of these erratic values will verify equation (2.1). The next section examines this problem.

2.8 Capability of the algorithm to reach 1

We have seen with Property 3 that the branches $B(n, w) = n2^w$ with odd integers n are a covering system of the set $\mathbb{N}$ of natural numbers. But we have also seen that *the sequence of branches* used by a trajectory is somewhat erratic, so that it cannot be mathematically expressed.

Fortunately, there is another set of mathematical objects, different from the set of branches $B(n, w)$, that give another way to cover the set $\mathbb{N}$ of natural numbers and that can be mathematically expressed.

2.8.1 Cut-out of $\mathbb{N}$ by numbers 2^m

An old result that Greeks philosophers may have known is that if we cut out the set $\mathbb{N}$ of the natural integers using the successive powers of 2, we can write the whole set in 2^m-type columns as follows:

Table 5. Cut out of $\mathbb{N}$ by powers of 2

.	.	.	.	.	.	.	$2^{m+1} - 1$
...	...	...	...	...	...	...	.
.	.	.	15	23	39	...	.
.	.	.	14	22	38	...	.
.	.	.	13	21	37	...	.
.	.	.	12	20	36	...	.
.	.	7	11	19	35	...	.
.	.	6	10	18	34	...	.
.	3	5	9	17	33	...	2^m+1
1	2	4	8	16	32	...	2^m
0	1	2	3	4	5	...	m

where each column $m \geqslant 0$ begins at 2^m, ends at $(2^{m+1} - 1)$ and contains:

$$(2^{m+1} - 1) - (2^m - 1) = 2^m \text{ numbers.}$$

2.8.2 Right shifts implied by jumps

With this cut out of $\mathbb{N}$, we can see that:
• the second term $(n = 2^m + 1)$ in column m is always transferred by the jump $a = 3n+1$ into column $m+1$ because for $n = 2^m + 1$, we always have:

$$a = 3n + 1 = 3(2^m + 1) + 1 = (2 + 1)(2^m) + 4 = 2^{m+1} + 2^m + 2^2$$

which proves that this number a is in column $m+1$ of Table 5.
• the upper term $(n = 2^{m+1} - 1)$ of a column $m \geqslant 0$ is always transferred by $a = 3n + 1$ into column $m+2$ because for $n = 2^{m+1} - 1$, we always have:

$$a = 3n + 1 = 3(2^{m+1} - 1) + 1 = (2 + 1)(2^{m+1}) - 2 = 2^{m+2} + 2^{m+1} - 2 \tag{2.4}$$

which proves that this number a is in column $m+2$ of Table 5.
These two points prove that the first (even) number a_i of a series $i > 1$, produced by a jump $a = 3n+1$, is always obtained by *an always existing right shift of 1 or 2 2^m-type columns in Table 5*.

2.8.3 Left shifts implied by the main function

All the even terms t of a 2^m-type column with $m > 0$ can always be written as:

$$t = 2^m + 2s \quad \text{with } 0 \leqslant s \leqslant 2^{m-1} - 1$$
$$\text{so that:}$$
$$t/2 = 2^{m-1} + s \quad \text{with } 0 \leqslant s \leqslant 2^{m-1} - 1$$

This means that a division by two of an even number in column m always places the result in column $m-1$, producing *an always existing left shift of 1 column in Table 5*.

2.8.4 A critical property of Collatz algorithm

The critical property. Collatz algorithm has the capability to end up at 1.

Proof. For Collatz algorithm, we have seen that:
• each jump $a = 3n + 1$ between two branches corresponds to an *always existing right shift* of 1 or 2 columns in $\mathbb{N}$;
• each division by two corresponds to an *always existing left shift* of 1 column in Table 5;
These two points prove that the right and left shifts of 1 column are always possible so that

Collatz algorithm provides *a continuous screening of the 2^m-type columns in Table 5*,
these columns being a covering system of $\mathbb{N}$.

or, in other words:

No 2^m-type column of Table 5 is left unreachable by Collatz algorithm,
particularly column C0 and its number 1.

This does not proves that Collatz algorithm always ends up at 1 but it proves the capability of Collatz algorithm to end up at 1. ☐

2.9 Reducing the problem

As the jump $a_{i+1} = 3n_i + 1$ is a step that moves the number n_i away from 1 and corresponds to a shift of 1 *or two* 2^m-type column in Table 5 while a division corresponds only to a shift of *one* 2^m-type column in Table 5, there is still a possibility for Collatz algorithm not to reach 1. With this possibility, two problems appear.

A first problem is with modules: a jump $a_{i+1} = 3n_i + 1$ is roughly three times greater than n_i and the next division $a_{i+1}/2$ is only $(3n_i + 1)/2 \sim 3n_i/2$, so that if divisions are at least as numerous as jumps, the trajectory is moving away from 1.

A second problem is with counting: we do not know how many jumps and divisions will occur in the trajectory.

But we can notice that, according to (2.4), the first division after a jump gives the maximum number

$$a/2 = 2^{m+1} + 2^m - 1$$

which proves that the new type of jump $a/2 = (3n+1)/2$, already being a standard vertical jump from one branch to another *in Table 4* as it contains a, *is also* a right shift of *only one* 2^m-type column *in Table 5* as it contains the division by 2.

If we adopt this new jump $a/2$ corresponding to a right shift of *only one* 2^m-type column *in Table 5*, *the problem is now reduced* to compare *the number of jumps* J to *the number of divisions* m in a trajectory as, if $J \geqslant m$ the trajectory will never reach 1 as in that case it gets further and further away from 1, but if $J < m$ the trajectory will always reach 1.

3 Main Result: Proof of Collatz conjecture

Proof. Using Table 4 and generalizing it *without supposing that the algorithm ends up at 1*, we get the *basic* result already used in many papers

Table 6. Generalization of Table 4

$i(Br)\backslash j(divs)$	0	...	...	...	j	j+1	...	j+t$_i$	
1	N	...							
2	...	...	...	...					
3=i-1	...	...	...	...	$n_{i-1,j}$				
4=i	...	...	...	...	$a_{i,j}$	$a_{i,j}/2^1$	...	$n_i = a_{i,j}/2^{t_i}$	
5	...	...	...	...	...	...	...	...	...
6	...	...	...	...	...	...	...	...	...

hence in the current line i

$$a_{i,j} = n_i 2^{t_i}$$

with n_i odd.

Now, we look up at lines 1, i and i=last of Table 6.

Line 1 of Table 6. Here, the number of possible divisions depends only on the parity of N. Let's name this number m_1 with $m_1 \geqslant 0$.

Line i of Table 6. Here, $\alpha_{i,j}$'s are unknown. All we can say is that only the first division is always possible as $a_{i,j} = 3n_{i-1,j} + 1$ is always even as $n_{i-1,j}$ is odd to trigger a jump.

From now on, to keep the notation $\alpha_{i,j}$ *for unknown values*, we choose to replace $\alpha_{i,j}$ by $1 + \alpha_{i,j}$ to highlight the first always possible division by the constant value 1. Then, considering lines $i = 1$ to $i = B - 1 = J$, we have

$$m_i = m_1 + \sum_{i=2}^{J} (1 + \alpha_{i,j})$$

Last line of Table 6 (i=last). Here, we get that the total number of divisions up to any line i=last is

$$m_{tot} = m_1 + \sum_{i=2}^{J}(1+\alpha_{i,j}) + t_{last}$$
$$= m_1 + (J-1) + \sum_{i=2}^{J}(\alpha_{i,j}) + t_{last}$$

Now, as odd $n_{last-1} \geqslant 3$ (if =1 the algorithm is already at end), $a_{last} = 3n_{last-1}+1 \geqslant 10 > 2^3$ so that the minimum exponent of 2 to come down to 1 is $t_{last} \geqslant 4$. This, in turn, always makes $(J-1)+t_{last} > J$ so that the last centered equation proves that $m_{tot} > J$ and in turn that $m_{tot} (> J)$ divisions are always more numerous than J jumps whatever are the number of lines in Table 6, m_1 and the values of $\alpha_{i,j}$'s. This finally proves, according to section (2.9), that Collatz algorithm *always* ends up at 1. ☐

4 On the generalization for even jumps

A more general approach on Collatz problem is obtained by keeping the division by 2 as main function but by considering the general even jump $a = qn + r$ where q and r verify $\gcd(q,r) = 1$.

As in Collatz algorithm a jump is used only when n is odd, we choose to have only odd n's. As this makes a to be even, this implies that q and r have to be of same parity. For simplicity, we will use hereafter only odd q's and odd r's with $\gcd(q,r) = 1$.

We will now look for the conditions that odd q's and r's have to verify to make the general algorithm end up at 1 and show that Collatz algorithm verifies them. This almost mimics what has been done for the jumps $a = 3n + 1$ but it enables us to prove the uniqueness of Collatz algorithm and other results.

4.1 Condition 1 to end up at 1

To reach the branch $B(1,w)$ from a given N and so end up at 1, we know from section 2.6 that for a given N and at the end of the branch $B_{i-1}(n_{i-1},w)$, the general algorithm has to verify the condition

$$qn_{i-1} + r = 2^m$$

To solve this condition is equivalent to study the factorization of $2^m - r$ as, ignoring the index of n_{i-1}, we must have:

Condition 1: $2^m - r = qn$

This shows that the condition that makes the general algorithm reach the branch $B(1,w)$ and end up at 1 for a given N is that q must be a divisor of $2^m - r$, which then implies that $n = (2^m - r)/q$ is an integer. It appears that only two cases have to be differentiated.

4.1.1 Case where $q = 1$ with any odd $r > 0$

With $q = 1$, $a = qn + r$ can be written $a = n + r$ and, with odd n and r, condition 1 can be written:

$$2^m - r = n$$

We see now that the problem of the factorization of $2^m - r$ is transferred from its factorization qn to the factorization of n only. As, according to the fundamental theorem of arithmetic, any positive *odd* number n can be written:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \text{ ... all } p_i\text{'s being odd}$$

we must have:

$$2^m - r = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \text{ ... all } p_i\text{'s being odd}$$

This makes appear the following result.

When the trajectory from N to n is possible and j is the number of divisors d_j of n, for each divisor d_j of $2^m - r$, condition 1 is verified and j couples (m_j, r_j) determine j couples (q_j, n_j), or j couples $(q_j, r_j = 1)$ determine j couples (m_j, n_j). This is because for j values, we have

$$2^{m_j} - r_j = d_j(n/d_j) = n$$

Then, the last two series always appear as

$$2^k n, \text{ ..., } 4n, \, 2n, \, n \text{ ; } n + r = 2^m, \, 2^{m-1}, \text{ ..., } 4, 2, 1$$

and the j algorithms based on j couples $(q_j, r_j = 1)$ always end up at 1.

4.1.2 Case with odd q$>$1 and odd r$>$0

This case is identical to the first case when we change n into qn.

This makes appear the result.

When the trajectory from N to n is possible and j is the number of divisors d_j of qn, for each divisor d_j of $2^m - r$, condition 1 is verified and j couples (m_j, r_j) determine j couples (q_j, n_j), or j couples (m_j, n_j) determine j couples (q_j, r_j). This is because for j values, we always have

$$2^{m_j} - r_j = d_j(qn/d_j) = qn$$

Again, the last two series always appear as

$$2^k \, n, \text{ ..., } 4n, \, 2n, \, n \text{ ; } qn + r = 2^m, \, 2^{m-1}, \text{ ..., } 4, 2, 1$$

and the j algorithms based on j couples $(q_j, r_j = 1)$ always end up at 1.

As this case includes the couple $(q = 3, r = 1)$, it includes Collatz algorithm and we have the following result.

For a given N, if the trajectory from N to n is possible, condition 1 is verified for j couples (m_j, r_j) that determine j couples (q_j, n_j), or j couples (q_j, r_j) whatever is m, and always among them the couple $(q_1 = 3, r_1 = 1)$, as Collatz conjecture has been proved.

4.2 Condition 2 on the trajectory from N to n$_i$

The two last results are still conditional to the fact that:

Condition 2: the trajectory from N to $n = n_i$ has to be possible.

Proof. This condition is always verified by Collatz algorithm because:

- independently of N, the main function (the division by 2 applied only to even numbers) and the jumps $a = qn + r$, are always defined functions;
- and because Collatz algorithm verifies the critical condition in section 2.8.4. □

The last remaining point is to generalize Condition 1 from one given N to all N's, which will give the final result.

4.3 Uniqueness of Collatz algorithm

Proof. We know from 4.1.2 that for a given N, condition 1 is verified for j couples (q_j, r_j) that always include $(q = 3, r = 1)$ which defines Collatz jump.

But for different N's, the *number of couples* j is generally different from one N to another. This is because the number of divisors j that divide qn is generally different from one n to another.

The *involved couples* in the lists of couples are also generally different from one list to another.

As we have seen in 4.1.2 that the couple $(q = 3, r = 1)$ is always present in these lists, independently of N, it proves that for all N's, *the unique couple* (q, r) *with odd* r *common to all lists that make a general algorithm end up at 1*, is the couple $(q = 3, r = 1)$ which defines Collatz jump and algorithm. ☐

Examples:

Table 7. Different jumps $a = qn + 1$ that make the algorithm end up at 1 for different N's with q checked up to 199

N	jumps	nb divs = m	jump #
N=7	$a = 3n + 1$	11	1
N=7	$a = 9n + 1$	6	2
N=7	$a = 17n + 1$	11	3
N=7	$a = 73n + 1$	9	4
...	...	...	...
N=11	$a = 3n + 1$	10	1
N=11	$a = 3.31n + 1$	10	2
...	...	...	...
N=24	$a = 3n + 1$	8	1
N=24	$a = 5n + 1$	7	2
N=24	$a = 9n + 1$	11	3
N=24	$a = 3.7n + 1$	9	4
N=24	$a = 5.17n + 1$	11	5
...	...	...	...
N=1000	$a = 3n + 1$	72	1
...	...	...	...

5 On the generalization for general even jumps

5.1 A fast check of a general even jump a = qn+r

A fast method to check if an algorithm using $a = qn + r$ ends up at 1 is as follows:

- 1- Factorize $2^m - 1$ for all m's up to any wanted limit;
- 2- All factors appearing in these factorizations are possible q's but the only true solutions are those for which no loop happens for a given N.

  If q appears in the factorizations generated by $2^m - 1$, the given algorithm will potentially end up at 1.

Example: Check of Collatz algorithm where r=1. For $2^m \leq 1000$, we have:

Table 8. Check of Collatz algorithm

m	$2^m - 1$	factorization
9	512-1=511	7.73
8	256-1=255	3.5.17
7	128-1=127	127
6	64-1=63	3^2.7
5	32-1=31	31
4	16-1=15	3.5
3	8-1=7	7
2	4-1=3	3
1	2-1=1	

As $q = 3$ appears in the factorizations generated by $2^m - 1$ for any m, Collatz algorithm potentially ends up at 1.
Table 8 also confirms the results of Table 7 as, by instance for $N = 7$, the incomplete list of jumps making Collatz algorithm end up at 1 are obtained with $q = 3, 9, 17, 73$ which are values of Table 8.

5.2 The fastest algorithm based on divisions by 2

On one hand, when $a = qn + r > n$, the jump is a ”rear jump” with respect to 1 as the distance from a to 1 is greater than that of n to 1.

On the other hand, when $a = qn + r < n$, the jump is a ”front jump” towards 1. Therefore, with $q = 1$, a front jump $a = n + r$ is obtained if and only if $r < 0$.

As in this case we have $q = 1$ and $a = n + r$ with $n = (2^m - r)/q$ depending on m, for some small values of m (the column in $\mathbb{N}$ where n is located) it may happen, if r is too much negative, that $a = n + r$ becomes a big front jump that skips one or several 2^m-type columns of $\mathbb{N}$, leaving them unreachable and making the algorithm a *not continuous screening* of the columns.

As the smallest odd n_i that does not stop the algorithm is 3, it thus appears that the only acceptable negative odd value of r that makes the jump $a = n + r$ to be an acceptable front jump, is $r = -1$. It gives the exceptional jump:

$$a = n - 1$$

the unique and fastest algorithm that contains only front jumps and so, the fastest decreasing sequence towards 1.

For $N = 1000$, this jump $a = n - 1$ gives:
1000,500,250,125 ; 124,62,31 ; 30,15 ; 14,7 ; 6,3 ; 2,1
with only 9 divisions, much less than the 72 divisions necessary for Collatz algorithm with jump $3n+1$ as mentioned in Table 7. As a comparison:
• the jump $a = n + 1$ gives:
1000,500,250,125 ; 126,63 ; 64,32,16,8,4,2,1 with 10 divisions,
• the jump $a = n + 3$ gives:
1000,500,250,125 ; 128,64,32,16,8,4,2,1 with 10 divisions,
• the jumps $a = n + 5$ and $a = n + 7$ give loops on 5,
• the jump $a = n + 9$ gives:
1000,500,250,125 ; 134,67 ; 76,38,19 ; 28,14,7 ; 16,8,4,2,1 with 12 divisions,
• the jump $a = n + 11$ gives:
1000,500,250,125 ; 136,68,34,17 ; 28,14,7 ; 16,8,4,2,1 with 12 divisions,
• the jump $a = n + 13$ gives:
1000,500,250,125 ; 138,69 ; 82,41 ; 54,27 ; 40,20,10,5 ; 18,9 ;
22,11 ; 24,12,6,3 ; 16,8,4,2,1 with 18 divisions.

6 On polynomial jumps of degree $m >$1

As we have seen that Collatz algorithm is made of an integer main function f such that $f_{i+1} = f_i/2$ and an integer jump function $a_i = 3f_i + 1$ used to replace odd f_i values, a full generalization would have to take into account any combination of any two functions.

Here, we will only consider main functions f that are divisions by any integer polynomial g_i:

$$f_{i+1} = f_i/g_i$$

and jumps are integer polynomials:

$$a_{i+1} = a(f_i)$$

used to replace f_{i+1} when this value is less than 1. To prove the method in a simple way, we will do it first on an instance where g_i and a_{i+1} are known.

6.1 A first step

In a first step, let's choose the divisor function:

$$g_i = i^2 + 1$$

where i is an integer (not a complex number). If we choose that this algorithm ends up at 1 when $m = 4$, we consider the four first values of g_i: $g_{1,4} = \{2, 5, 10, 17\}$ whose product is 1700. Let's generate the sequence with $f_0 = N = 1700$. We get the sequence with no jumps:

$$f_0 = N, f_1 = f_0/g_1, f_2 = f_1/g_2, f_3 = f_2/g_3, f_4 = f_3/g_4$$

which gives:

$$f_0 = 1700, f_1 = 1700/2 = 850, f_2 = 850/5 = 170,$$
$$f_3 = 170/10 = 17, f_4 = 17/17 = 1$$

and we get that the sequence ends up at 1 with f_4 as expected.

This proves that there always exists an algorithm beginning with any number N and ending at 1 when the divisor function $g(i)$ is an integer polynomial that generates the exact list of the factors g_i of N.

6.2 A possible second step

A possible second step can be to find which jumps can be associated with f that can allow to start the sequence with an f_0 different from $N = 1700$.

To do that, we have to choose a value of i that makes the jump a_{i+1} replace a disqualified $f_{i+1} = f_i/g_{i+1} < 1$ coming from an integer N' different of N. By instance, let's choose $i = 2$ such that $a_2 = f_2 = 170$ replaces a disqualified value $f_2 = f_1/g_2 < 1$. Here, the divisor function $g_i = i^2 + 1$ is already defined but not the jump. Let's choose by instance the jump:

$$a_{i+1} = (f_i)^2 + b$$

which fixes b to the odd complements to $f_2 = 170$ of these squares:

$$b = 170 - (f_i)^2$$

All the possible jumps are then:

$$a_{i+1} = (f_i)^2 + 161 \text{ for } f_i = 3$$
$$a_{i+1} = (f_i)^2 + 145 \text{ for } f_i = 5$$
$$a_{i+1} = (f_i)^2 + 121 \text{ for } f_i = 7$$
$$a_{i+1} = (f_i)^2 + 89 \text{ for } f_i = 9$$
$$a_{i+1} = (f_i)^2 + 49 \text{ for } f_i = 11$$
$$a_{i+1} = (f_i)^2 + 1 \text{ for } f_i = 13$$

Choosing $f_i = f_1 = 3$, we have $b = 161$ and the sequence is:

$$f_0 = 2f_1 = 6, f_1 = 3, f_2 = 3/5 < 1 \text{ replaced by } a_2 = 3^2 + 161 = 170,$$
$$f_3 = 170/10 = 17, f_4 = 17/17 = 1$$

For all the possible odd values of b above and their associated values f_i, the sequences are:

for $f_i = 3,$	$b = 161$:	$a_{i+1} = (f_i)^2 + 161$	f_i = 6, 3; 170, 17, 1
for $f_i = 5,$	$b = 145$:	$a_{i+1} = (f_i)^2 + 145$	f_i = 10, 5; 170, 17, 1
for $f_i = 7,$	$b = 121$:	$a_{i+1} = (f_i)^2 + 121$	f_i = 14, 7; 170, 17, 1
for $f_i = 9,$	$b = 89$:	$a_{i+1} = (f_i)^2 + 89$	f_i = 18, 9; 170, 17, 1
for $f_i = 11,$	$b = 49$:	$a_{i+1} = (f_i)^2 + 49$	f_i = 22, 11; 170, 17, 1
for $f_i = 13,$	$b = 1$:	$a_{i+1} = (f_i)^2 + 1$	f_i = 26, 13; 170, 17, 1

This proves that for all the odd values 3 to 13 and all the even values $\{$3.to.13$\}2^1$, all different of $N = 1700$, the algorithm defined by:

$$f_{i+1} = f_i/g_{i+1}, \quad g_i = i^2 + 1 \quad \text{and } a_i = i^2 + b$$

ends up at f=1.

6.3 Proof of the generalization

Proof. The above proof has been built upon chosen instances of N, $g(i)$ and $a(i)$. It does not allow, at this stage, to generalize to all combinations of integer polynomials $f(i)$, $g(i)$ and $a(i)$.

But, as according to the fundamental theorem of arithmetic, any integer number $f(i)$ generated by an integer function f can be factorized in only one way when the factorization is ordered by increasing primes, it is also true for any number:

$$N = \prod_{i=1}^{m} f(i)$$

So, as it is always possible by a system of m equations to find a rational polynomial function $g(i)$ that generates the list of divisors of N, it is always possible to find an algorithm ending up at 1 for any value of m and N, which proves the generalization. □

7 Conclusion

The proof of Collatz algorithm necessitates only 5 steps:

1- to differentiate the main function and the jumps;

2- to differentiate branches as well as their first and last terms a and n;

3- to identify that left and irregular right shifts in branches can be replaced by regular shifts in 2^m-type columns;

4- to identify the key equation $a_i = 3n_{i-1} + 1 = 2^m$ as well as its solutions;

5- to reduce the problem to compare *the number of jumps* J to *the number of divisions* m in a trajectory.

Remark. The present proof is one among many others published nowadays. I am aware that there are five categories of articles on conjectures and on Collatz conjecture in particular: 1- those who makes contributions to the solution [7]..., 2- those that (try to) prove it [8]..., 3- those who suggest that the conjecture is false [9]..., 4- those that (try to) prove that it is false [10]... and 5- those who describe the historical and state-of-the-art situations at different times [11][12][13]...

The goal of the present article was not to write on the state-of-the-art of Collatz conjecture nor to reference the very basics of arithmetic progessions, divisibility, congruences,... . As I am an independent and part-time hobbyist in mathematics, with no free access to libraries of universities or other interesting websites, it was not possible for me to read a maximum of publications on the subject. Anyway, these last are too numerous and many of them are intellectually out of reach for me. As I refuse to list references that I have not read, it explains the small number of references of this article.

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