Two Step Trigonometrically Fitted Method for Numerical Solution of Initial Value Problems with Oscillating Solutions

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Abstract

A continuous two step method using trigonometric function basis is developed and used to produce two discrete methods which are simultaneously applied as numerical integrators by assembling them into a block method with trigonometric basis for solving oscillatory initial value problems(IVPs). The stability property of the method is well discussed and the performance of the method is demonstrated on some numerical examples to show accuracy and efficiency advantages.
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1 Introduction

This paper considered the class of second order differential equation of the form

$$y' = f(x, y), \quad y(a) = y_0, \quad x \in [a, b] \tag{1}$$

with oscillatory solutions where $f : \Re \times \Re^m \to \Re^m$, y, $y_0 \in \Re.^m$

In the field of ordinary differential equations, a nontrival solution to an ordinary differential equation

$$F(x, y, y', \cdots, y^{(n-1)}) = y^{(n)}$$

is called oscillating if it has an infinite number of roots.The differential equation is oscillating if it has an oscillating solution. The number of roots carries information on the spectrum of the associated boundary value problems. For example the differential equation $y''+y = 0$ is oscillating as $\sin x$ is a solution.

Oscillatory Initial Value Problems usually occur in area such as quatum mechanics, biological sciences, classical mechanics and celestial mechanics. A lot of numerical methods based on polynomial and non polynomial basis have been developed for solving this important class of problems(Odejide and Adeniran[1], Adeniran, Odejide and Ogundare[2], Adeniran and Ogundare [3], Hairer and Wanner[4], Hairer[5], Sommeijer[6], Jator et. al[7], Ngwane and Jator[9]).

This paper construct two step method with trigonometric basis, which provides two discrete methods that are combined and applied as block two step method with trigonometric basis which take the frequency of the solution as a priori knowledge.

We adopt the approach given in Nwagene and Jator [9] and Jator, Swindle and French[7], where the continuous two step method with trigonometric basis is used to generated the main and one additional method which are combined and used as a two step block method to simultaneously produce approximations.

$$\{y_{n+1}, y_{n+2}\} \text{ at block of points } \{x_{n+1}, x_{n+2}\},$$

$h = x_{n+1} - x_n$, $n = 0, \cdots, N-1$ on a partition $[a, b]$, where h is the step size, n is the grid index and $N > 0$ is the number of steps. The block methods generates approximations $\{y_{n+1}, y_{n+2}\}$ to the exact solution $\{y(x_{n+1}), y(x_{n+2})\}$

The paper is organized as follows. In section 2, we derive a trigonometric basis representation $U(x)$ for the exact solution $y(x)$ which are used to generate two discrete methods which are combined to solve (1). The analysis and implementation of the new method are given in section 3. Numerical examples are given in section 4 to show accuracy and efficiency of the new block method. The conclusion the the paper is discussed in section 5.

2 Derivation of the Method

We derive a two step with trigonometric basis which produces two discrete methods as by-products.
The main method has the form

$$y_{n+1} = y_n + h\{\alpha_0(u)f_n + \alpha_1(u)f_{n+1} + \alpha_2(u)f_{n+2}\} \tag{2}$$

and the additional method is given by

$$y_{n+2} = y_n + h\{\hat{\alpha}_0(u)f_n + \hat{\alpha}_1(u)f_{n+1} + \hat{\alpha}_2(u)f_{n+2}\} \tag{3}$$

where $u = wh$, α_j, $\hat{\alpha_1}$, $j = 0(1)2$, are coefficients that depend on the step-size and frequency. y_{n+1} is the numerical approximation to analytical solution $y(x_{n+1})$.

In order to obtain (2) and (3), we proceed by seeking to approximate the exact solution y(x) on the interval $[x_n, x_{n+h}]$ by interpolation function $U(x)$ of the form:

$$U(x) = a_0 + a_1 x + a_2 \sin(wx) + a_3 \cos(wx), \tag{4}$$

where a_i, $i = 0(1)3$, are coefficients that must be uniquely determined. We then impose that the interpolating function (4) coincides with the analytical solution at the end point x_n to obtain the equation

$$U(x_n) = y_n, \tag{5}$$

It is also demanded that the function (4) satisfies the differential equation (1) at points x_{n+j}, $j = 0, 1, 2$ to obtain the following set of three equations:

$$U'(x_n) = f_n, \quad U'(x_{n+1}) = f_{n+1}, \quad U'(x_{n+2}) = f_{n+2} \tag{6}$$

The system of equation in (5) and (6) are solved by Cramer's rule to obtain a_j, $j = 0(1)3$. Continuous two step method with trigonometric basis is constructed by substituting the values of a_j into (4). After some algebraic manipulation, our new two step method is of the form:

$$U(x) = y_n + h\{\alpha_0(w,x)f_n + \alpha_1(w,x)f_{n+1} + \alpha_2(w,x)f_{n+2}\}, \tag{7}$$

where w is the frequency, $\alpha_j(w,x)$, $j = 0(1)2$ are continuous coefficients. The continuous method (7) is used to generate the main method of the form (2) and an additional method of the form (3) by evaluating (7) at

$x = \{x_{n+1}, x_{n+2}\}$ and letting $u = wh$, we obtained the coefficients of (2) and (3) as follows:

$$\alpha_0 = \frac{1}{2}\frac{\left(-u\sin(u) - 2\,(\cos(u))^2 + 2\,\cos(u)\right)}{u\sin(u)(\cos(u)-1)}$$

$$\alpha_1 = \frac{1}{2}\frac{\left(2\,u\sin(u)\cos(u) + 2\,(\cos(u))^2 - 2\right)}{u\sin(u)(\cos(u)-1)} \tag{8}$$

$$\alpha_2 = \frac{1}{2}\frac{(-u\sin(u) - 2\,\cos(u) + 2)}{u\sin(u)(\cos(u)-1)}$$

$$\hat{\alpha_0} = \frac{(-u+\sin(u))}{u(\cos(u)-1)}$$

$$\hat{\alpha_1} = \frac{(2\,u\cos(u) - 2\,\sin(u))}{u(\cos(u)-1)} \tag{9}$$

$$\hat{\alpha_2} = \frac{(-u+\sin(u))}{u(\cos(u)-1)}$$

3 Error analysis and stability

3.1 Local truncation error

Taylor series is used for small values of u (see Simos[10]). The coefficient of (8) and (9) can be expressed as:

$$\alpha_0 = \frac{5}{12} + \frac{19\,u^2}{720} + \frac{23\,u^4}{10080} + \frac{263\,u^6}{1209600} + \frac{1033\,u^8}{47900160} + \frac{945979\,u^{10}}{435891456000} + \cdots$$

$$\alpha_1 = \frac{2}{3} - \frac{u^2}{90} - \frac{u^4}{2520} - \frac{u^6}{75600} - \frac{u^8}{2395008} - \frac{691\,u^{10}}{54486432000} + \cdots \tag{10}$$

$$\alpha_2 = -\frac{1}{12} - \frac{11\,u^2}{720} - \frac{19\,u^4}{10080} - \frac{247\,u^6}{1209600} - \frac{1013\,u^8}{47900160} - \frac{940451\,u^{10}}{435891456000} + \cdots$$

$$\hat{\alpha_0} = \frac{1}{3} + \frac{u^2}{90} + \frac{u^4}{2520} + \frac{u^6}{75600} + \frac{u^8}{2395008} + \frac{691\,u^{10}}{54486432000} + \cdots$$

$$\hat{\alpha_1} = \frac{4}{3} - \frac{u^2}{45} - \frac{u^4}{1260} - \frac{u^6}{37800} - \frac{u^8}{1197504} - \frac{691\,u^{10}}{27243216000} + \cdots \tag{11}$$

$$\hat{\alpha_2} = \frac{1}{3} + \frac{u^2}{90} + \frac{u^4}{2520} + \frac{u^6}{75600} + \frac{u^8}{2395008} + \frac{691\,u^{10}}{54486432000} + \cdots$$

The Local Truncation Error(LTE) for methods (2) and (3) are given by

$$\text{LTE}(2) = \frac{h^4}{24}\left(w^2 y^{(2)}(x_n) + y^{(4)}(x_n)\right)$$

$$\text{LTE}(3) = -\frac{h^5}{90}\left(w^2 y^{(2)}(x_n) + y^{(4)}(x_n)\right) \tag{12}$$

3.2 Stability

We define the block by block method for computing vectors $Y_0, Y_1, Y_2, \cdots$ in sequence (see Fatunla [11]).Let the η-vector($\eta = 2$ is the number of points within the block) Y_μ, $Y_{\mu-1}$, F_μ and $F_{\mu-1}$ be given as $Y_\mu = (y_{n+1}, y_{n+2})^T$,$Y_{\mu-1} = (y_{n-1}, y_n)^T$, $F_\mu = (f_{n+1}, f_{n+2})^T$, $F_{\mu-1} = (f_{n-1}, f_n)^T$, then the 1-block 2-point method for (1) is given as:

$$Y_\mu = \sum_{i=1}^{1} A^{(i)} Y_{\mu-1} + \sum_{i=0}^{1} B^{(i)} F_{\mu-1}, \tag{13}$$

where $A^{(i)}$, $B^{(i)}$, $i = 0, 1$ are 2×2 matrices whose entries are given by the coefficient of (2) and (3).

Zero stability

The block method in (13) is zero stable provided the roots R_j, $j = 1, 2$ of the first characteristic polynomial $\rho(R)$ is specified by

$$\rho(R) = det\left[\sum_{i=0}^{1} A^{(i)} R^{i-1}\right] = 0, \quad A^{(0)} = -I \tag{14}$$

satisfies $|R_j| \leq 1$, $j = 1, 2$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1 (see Fatunla [11]).

Consistency

Our block method in (13) is consistent(it has order $p > 1$).

The block method is convegent(Convergence = Zero stability+ consistency)

4 Implementation of the Scheme

Method (2) and (3) are implemented more efficiently as simultaneous intergrator for IVPs without requiring starting values and predictors. We proceed by explicitly obtaining initial conditions at x_{n+2}, $n = 0, 2, \cdots, N-2$ using the computed values $y(x_{n+2}){=}y_{n+2}$ over sub-intervals $[x_0,\, x_2]$, $\cdots$, $[x_{N-2},\, x_N]$. For instance, using equations $(12-13)$, and with $n = 0$. $(y_1,\, y_2)^T$, are simultaneously obtained over the sub-interval $[x_0,\, x_2]$, as y_0 is known from the IVP (1). Also for $n = 2$, $(y_3,\, y_4)^T$ are simultaneosly obtained over the sub-interval $[x_2,\, x_4]$, as y_2 is known from the previous block, where T is the transpose and so on. Hence, the sub-interval do not over-lap and the solution in this manner is more accurate than those obtain in the conventional fashion.

On the choice of frequency(w), we adopt the method in Vigo-Aguiar and Ramos [12]. On their paper titled " On the choice of the frequency in trigonometrically-fitted method." use the trigonometrically-fiited method to obtain an approximate solution to some nonlinear oscillators and also present a strategy for the choice of frequency in trigonometrically-fiited method.

5 Numerical examples

In order to study the accuracy and efficiency of the developed methods, we present some numerical experiments:

Example 1.1

$$y'' = -100y + 99\sin(x), \quad y(0) = 1,\ y'(0) = 11 \quad \text{(Adeniran et.al.[2])}$$

Exact solution: $y(x) = \cos(10x) + \sin(10x) + \sin(x)$

Example(1.2)

We consider the initial value problem

$$y''(x) = \frac{(y')^2}{2y} - 2y,\ y(\frac{\pi}{6}) = \frac{1}{4} \text{ and } y'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \quad \text{(Alabi et.al[13])}$$

$$\text{exact solution: } y = \sin^2 x$$

Table 1: Showing the exact solutions, computed results and error from the proposed methods for example 1.1 with $h = \frac{1}{320}$, $w = 1$

x	yExact	yComputed	error	Error in Adeniran et.al.[2]
$\frac{1}{320}$	1.03388166738420	1.03388166738410	9.99×10^{-14}	9.170×10^{-11}
$\frac{2}{320}$	1.06675678785246	1.06675678785234	1.20×10^{-13}	-
$\frac{3}{320}$	1.09859628036501	1.09859628036580	7.90×10^{-13}	3.0905×10^{-10}
$\frac{4}{320}$	1.12937207509627	1.12937207509610	1.69×10^{-13}	-
$\frac{5}{320}$	1.15905714081491	1.15905714081441	5.00×10^{-13}	-
$\frac{6}{320}$	1.18762550988244	1.18762550988224	2.00×10^{-13}	4.8987×10^{-10}
$\frac{7}{320}$	1.21505230844501	1.21505230844510	8.99×10^{-14}	-
$\frac{8}{320}$	1.24131377434580	1.24131377434560	2.00×10^{-13}	-
$\frac{9}{320}$	1.26638728387076	1.1.26638728387046	3.00×10^{-13}	-
$\frac{10}{320}$	1.29025137290459	1.29025137290430	2.99×10^{-13}	-

The numerical result for Example 1.1 were presented in Tables 1. The problem was compared to other existing method. The new two step trigonometric method displayed better accuracy within the range of integration.

Table 2: Showing the exact solutions, computed results and error from the proposed methods for Problem 1.2, $h = 0.1$, $w = 1$.

x	yExact	yComputed	Error	Error in Alabi et.al[13]
0.1	0.0099667110793792	0.00996671107827122	1.1×10^{-12}	
0.2	0.0394695029985574	0.0394695029786944	2.0×10^{-11}	
0.3	0.0873321925451611	0.0873321924526160	9.3×10^{-11}	
0.4	0.1516466453264171	0.151646645281091	4.5×10^{-11}	
0.5	0.229848847065930	0.229848847017083	4.9×10^{-11}	
0.6	0.318821122761663	0.318821122673452	8.8×10^{-11}	1.013×10^{-08}
0.7	0.415016428549879	0.415016428485594	6.4×10^{-11}	4.782×10^{-08}
0.8	0.514599761150645	0.514599761134647	1.6×10^{-11}	1.109×10^{-07}
0.9	0.613601047346543	0.613601047330533	1.6×10^{-10}	1.892×10^{-07}
1.0	0.708073418273572	0.708073418262838	1.1×10^{-10}	1.196×10^{-07}
1.1	0.794250558627672	0.794250558333421	2.9×10^{-10}	3.019×10^{-07}
1.2	0.868696857770622	0.868696848083653	9.7×10^{-09}	2.561×10^{-07}
1.3	0.928444376684474	0.928444372840036	3.8×10^{-09}	1.435×10^{-07}
1.4	0.971111170334329	0.971111163223212	7.1×10^{-09}	1.019×10^{-07}
1.5	0.994996248300222	0.994996243303974	5.0×10^{-09}	2.319×10^{-07}
1.6	0.999147387897376	0.999147386423497	1.5×10^{-09}	5.892×10^{-07}
1.7	0.983399096289731	0.983399092298768	4.0×10^{-09}	1.013×10^{-06}
1.8	0.948379208167073	0.948379206787865	1.4×10^{-09}	1.211×10^{-06}
1.9	0.895483855957207	0.895483801118647	5.5×10^{-08}	1.287×10^{-06}
2.0	0.826821810431807	0.826821783609997	2.7×10^{-08}	1.435×10^{-06}

The numerical result for Example 1.2 were presented in Tables 2. The problem was compared to other existing method. The new two step trigonometric method displayed better accuracy within the range of integration.

6 Conclusion

The two step trigonometrically fitted method for solving oscillatory IVPs generated in this paper is accurate, efficient, consistent and zero stable . This method is self-starting and require no predictor and requires only two functions evaluation at each integration step. The method complete favorably with other existing methods.

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