

GENERALIZED (α, β) -HIGHER DERIVATIONS ON LIE IDEALS OF RINGS WITH INVOLUTION

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ABSTRACT. In this manuscript, we investigate the behaviour of additive mappings which satisfy a functional identity associated with generalized (α, β) -higher derivations on Lie ideals of a prime rings with involution. As the consequences of our main theorem, many well known results can be deduced.

Keywords and Phrases: Involution, $*$ -closed Lie ideal, (α, β) -higher derivation, generalized (α, β) -higher derivation.

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1. INTRODUCTION

Hasse and Schmidt [11] were who extended the concept of derivations to higher derivations. More precisely, they supplied that $D = \{d_n\}_{n \in \mathbb{N}}$, a family of additive mappings on R , is said to be a higher derivation (resp. Jordan higher derivation) on R if $d_0 = I_R$ and $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ (resp. $d_n(x^2) = \sum_{i+j=n} d_i(x)d_j(x)$) for all $x, y \in R$ and for each $n \in \mathbb{N}$. It is easy to see that the first member of each higher derivation is itself a derivation. More related result can be find in [10]. Later on, Cortes and Haetinger [6] defined generalized higher derivations. A family $F = (f_i)_{i \in \mathbb{N}}$ of additive mapping of a ring R such that $f_0 = I_R$ is said to be a generalized higher derivation (resp. generalized Jordan higher derivation) of R if there exists a higher derivation (resp. Jordan higher derivation) $D = \{d_n\}_{n \in \mathbb{N}}$ and for each $n \in \mathbb{N}$, $f_n(xy) = \sum_{i+j=n} f_i(x)d_j(y)$ (resp. $f_n(x^2) = \sum_{i+j=n} f_i(x)d_j(x)$) hold for all $x, y \in R$. Obviously, every generalized higher derivation is a generalized Jordan higher derivation but the converse need not to be true. The converse have been proved in [6] for square closed Lie ideals of a prime ring. Later, Wei and Xiao [14] established this result for a 2-torsion free semiprime ring. For an account on higher derivations, we refer the reader to [2, 4, 9]. In 2010, Ashraf et al [4] introduced the concept of (θ, ϕ) -higher derivations as follows: A family D of additive mappings d_n on R is said to be (θ, ϕ) -higher derivation of R if $d_0 = I_R$ and $d_n(xy) = \sum_{i+j=n} d_i(\theta^{n-i}(x))d_j(\phi^{n-j}(y))$ for all $x, y \in R$ and for each $n \in \mathbb{N}$. Further, in [2], Ashraf and Khan have acquainted the concept of generalized (θ, ϕ) -higher derivations and proved that every generalized Jordan (θ, ϕ) -higher derivations is a generalized (θ, ϕ) -higher derivation on Lie ideals of a prime ring R .

In [13], Herstein proved that if a simple ring with $\text{char}(R) \neq 2$ and $\dim \geq 4$ admitting additive map $d : R \rightarrow R$ such that $d(xx^*) = d(x)x^* + xd(x)$ for all $x \in R$, then d must be a derivation. Later Daif and El-Sayiad [7] obtained the following result: Let R be a 2-torsion free semiprime $*$ -ring and $F : R \rightarrow R$ be an additive mapping associated with a derivation d on R such that $F(xx^*) =$

$F(x)x^* + xd(x^*)$ holds for all $x \in R$. Then F is a generalized Jordan derivation. In the same manner in [5], Ashraf et al. established Daif and El-Sayiad result in more general form and proved the following: Let R be a 2-torsion free semiprime $*$ -ring. Suppose that α, β are endomorphisms of R such that α is an automorphism of R . If there exists an additive mapping $F : R \rightarrow R$ associated with a (α, β) -derivation d of R such that $F(xx^*) = F(x)\alpha(x^*) + \beta(x)d(x^*)$ holds for all $x \in R$, then $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$. In 2015, Ezzat [8] have studied aforementioned results on generalized higher derivations. Very recently, Husain et al. [1] extended Ezzat result for generalized (θ, ϕ) -higher derivations on a semiprime ring with involution. In the present paper, we study above mentioned theorem in the setting of generalized (θ, ϕ) -higher derivations on Lie ideal of a prime ring with involution.

2. PRELIMINARIES AND MAIN RESULT

Throughout this article, unless otherwise mentioned, R will denote an associative ring. A ring endowed with involution $*$ is called a ring with involution or $*$ -ring. For basic definitions and notations we refer the reader to [12] and [13]. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$ and $r \in R$. U is also called $*$ -closed if $U^* = U$.

We begin our discussion with following key lemma which will be extensively used to prove the main result.

Lemma 2.1. [1, Lemma 1.2] Let R be a prime ring with involution with $\text{char}(R) \neq 2$ and U be a non central $*$ -closed Lie ideal of R such that $u^2 \in U$, for all $u \in U$. Suppose that α is an automorphism of U . If there exists an element $a \in U$ such that $a\alpha(u^*) = a\alpha(u)$ holds for all $u \in U$, then $a \in Z(R)$.

The main result of the present paper is the following theorem.

Theorem 2.2. Let R be a prime ring of characteristic different from two with involution and U be a square close Lie ideal on R . Suppose there exists a family of additive mappings $F = (f_i)_{i \in N_0}$ of U into R associated with some (α, β) -higher derivation $D = (d_i)_{i \in N_0}$ of U into R , where α, β are commuting automorphisms on R , such that $f_n(uu^*) = \sum_{i+j=n} f_i(\alpha^{n-i}(u))d_j(\beta^{n-j}(u^*))$ is fulfilled for each $n \in N$ and for all $u \in U$. Then, F is a generalized (α, β) -higher derivation on U into R .

Proof. Our hypothesis is

$$(2.1) \quad f_n(uu^*) = \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(u^*))$$

for all $u \in U$. Linearization of above expression yields

$$(2.2) \quad \begin{aligned} f_n(uv^* + vu^*) &= \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(v^*)) \\ &\quad + \sum_{i+j=n} f_i(\beta^{n-i}(v))d_j(\alpha^{n-j}(u^*)) \end{aligned}$$

for all $u, v \in U$. Taking $v = u^*$, we get

$$(2.3) \quad \eta_n(u) + \eta_n(u^*) = 0$$

for all $u \in U$, where $\eta_n(u)$ stands for $f_n(u^2) - \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(u))$. Our aim is to show $\eta_n(u) = 0$ for all $u \in U$. To show $\eta_n(u) = 0$, we prosecute by induction. If $n = 0$, then it is easy to obtain $\eta_0(u) = 0$ for all $u \in U$. If $n = 1$, then one can easily prove the result by following [3, Theorem 2.3]. Now suppose $\eta_m(u) = 0$ for all $u \in U$, and for $m < n$. Set $A = 2(f_n(u(uv^* + vu^*) + (uv^* + vu^*)u^*))$. In view of (2.2), we have

$$\begin{aligned}
A &= 2\left(\sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(uv^* + vu^*)) + f_i(\beta^{n-i}(uv^* + vu^*))d_j(\alpha^{n-j}(u^*))\right) \\
&= 2\left(\sum_{i+j=n} f_i(\beta^{n-i}(u))\left(\sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(u))d_q(\alpha^{j-q}\alpha^{n-j}(v^*))\right.\right. \\
&\quad \left.\left. + \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(v))d_q(\alpha^{j-q}\alpha^{n-j}(u^*))\right)\right) \\
&\quad + \sum_{i+j=n} \left(\sum_{s+t=i} f_s(\beta^{i-s}\beta^{n-i}(u))d_t(\alpha^{i-t}\beta^{n-i}(v^*))\right) \\
&\quad + \sum_{s+t=i} f_s(\beta^{i-s}\beta^{n-i}(v))d_t(\alpha^{i-t}\beta^{n-i}(u^*))\left)d_j(\alpha^{n-j}(u^*))\right) \\
&= 2\left(\sum_{i+j=n} f_i(\beta^{n-i}(u)) \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(u))d_q(\alpha^{j-q}\alpha^{n-j}(v^*))\right) \\
&\quad + \sum_{i+j=n} f_i(\beta^{n-i}(u)) \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(v))d_q(\alpha^{j-q}\alpha^{n-j}(u^*)) \\
&\quad + \sum_{i+j=n} \sum_{s+t=i} f_s(\beta^{i-s}\beta^{n-i}(u))d_t(\alpha^{i-t}\beta^{n-i}(v^*))d_j(\alpha^{n-j}(u^*)) \\
&\quad + \sum_{i+j=n} \sum_{s+t=i} f_s(\beta^{i-s}\beta^{n-i}(v))d_t(\alpha^{i-t}\beta^{n-i}(u^*))d_j(\alpha^{n-j}(u^*))\right)
\end{aligned}$$

This can be written as

$$\begin{aligned}
A &= 2\left(\sum_{i+j=n} f_i(\beta^{n-i}(u)) \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(u))d_q(\alpha^{n-q}(v^*))\right) \\
&\quad + \sum_{i+j=n} f_i(\beta^{n-i}(u)) \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(v))d_q(\alpha^{n-q}(u^*)) \\
&\quad + \sum_{i+j=n} \sum_{s+t=i} f_s(\beta^{n-s}(u))d_t(\alpha^{i-t}\beta^{n-i}(v^*))d_j(\alpha^{n-j}(u^*)) \\
&\quad + \sum_{i+j=n} \sum_{s+t=i} f_s(\beta^{n-s}(v))d_t(\alpha^{i-t}\beta^{n-i}(u^*))d_j(\alpha^{n-j}(u^*))\right)
\end{aligned}$$

In particular, we have

$$\begin{aligned}
(2.4) \quad A &= 2\left(\sum_{i+p=n} f_i(\beta^{n-i}(u))d_p(\alpha^{n-p}(u))\alpha^n(v^*)\right) \\
&+ \sum_{\substack{i+p+q=n \\ i+p \neq n}} f_i(\beta^{n-i}(u))d_p(\beta^q \alpha^i(u))d_q(\alpha^{n-q}(v^*)) \\
&+ \sum_{i+j+k=n} f_i(\beta^{n-i}(u))d_j(\beta^k \alpha^i(v+v^*))d_k(\alpha^{n-k}(u^*)) \\
&+ \sum_{i+j=n} \sum_{s+t=i} f_s(\beta^{n-s}(v))d_t(\alpha^{i-t} \beta^{n-i}(u^*))d_j(\alpha^{n-j}(u^*))
\end{aligned}$$

On the other hand, A can be written as

$$\begin{aligned}
A &= 2(f_n(u(v+v^*)u^*) + f_n(u^2v^* + v(u^*)^2)) \\
&= 2(f_n(u(v+v^*)u^*) + \sum_{i+j=n} f_i(\beta^{n-i}(u^2))d_j(\alpha^{n-j}(v^*))) \\
&+ \sum_{i+j=n} f_i(\beta^{n-i}(v))d_j(\alpha^{n-j}(u^*)^2) \\
(2.5) \quad A &= 2(f_n(u(v+v^*)u^*) + f_n(u^2)\alpha^n(v^*)) \\
&+ \sum_{\substack{i+p+q=n \\ i+p \neq n}} f_i(\beta^{n-i}(u))d_p(\beta^q \alpha^i(u))d_q(\alpha^{n-q}(v^*)) \\
&+ \sum_{i+j=n} f_i(\beta^{n-i}(v)) \sum_{k+l=j} d_k(\alpha^{j-k} \beta^{n-j}(u^*))d_l \alpha^{n-l}(u^*)
\end{aligned}$$

Compare (2.4) and (2.5) and use the fact that the characteristic of R is different from two, so we have

$$f_n(u(v+v^*)u^*) = -\eta_n(u)\alpha^n(v^*) + \sum_{i+j+k=n} f_i(\beta^{n-i}(u))d_j(\beta^k \alpha^i(v+v^*))d_k(\alpha^{n-k}(u^*))$$

Taking v as $v - v^*$ gives

$$(2.6) \quad \eta_n(u)\alpha^n(v^*) = \eta_n(u)\alpha^n(v)$$

In view of Lemma 2.1, we get $\eta_n(u) \in Z(R)$ for all $u \in U$.

Next, putting v as v^* in (2.2), we obtain

$$\begin{aligned}
(2.7) \quad f_n(uv + v^*u^*) &= \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(v)) \\
&+ \sum_{i+j=n} f_i(\beta^{n-i}(v^*))d_j(\alpha^{n-j}(u^*))
\end{aligned}$$

Replacing v by $2uv$ in above expression and using the fact that the characteristic of R is different from two, we get

$$\begin{aligned}
f_n(u^2v + v^*u^{*2}) &= \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(uv)) \\
&\quad + \sum_{i+j=n} f_i(\beta^{n-i}(v^*u^*))d_j(\alpha^{n-j}(u^*)) \\
&= \sum_{i+j=n} f_i(\beta^{n-i}(u)) \sum_{p+q=j} d_p(\beta^{j-p}\alpha^{n-j}(u))d_q(\alpha^{j-q}\alpha^{n-j}(v)) \\
&\quad + \sum_{i+j=n} f_i(\beta^{n-i}(v^*u^*))d_j(\alpha^{n-j}(u^*)) \\
&= \sum_{i+p=n} f_i(\beta^{n-i}(u))d_p(\alpha^{n-p}(u))\alpha^n(v) \\
&\quad + \sum_{\substack{i+p+q=n \\ i+p \neq n}} f_i(\beta^{n-i}(u))d_p(\beta^q\alpha^i(u))d_q(\alpha^{n-q}(v)) \\
&\quad + f_n(v^*u^*)\alpha^n(u^*) + \sum_{\substack{i+j=n \\ i \neq n}} f_i(\beta^{n-i}(v^*u^*))d_j(\alpha^{n-j}(u^*))
\end{aligned}$$

On the other hand, replacement of u by u^2 in (2.7) gives

$$\begin{aligned}
f_n(u^2v + v^*u^{*2}) &= \sum_{i+j=n} f_i(\beta^{n-i}(u^2))d_j(\alpha^{n-j}(v)) + \sum_{i+j=n} f_i(\beta^{n-i}(v^*))d_j(\alpha^{n-j}(u^{*2})) \\
&= f_n(u^2)\alpha^n(v) + \sum_{\substack{i+p+q=n \\ i \neq n}} f_i(\beta^{n-i}(u))d_p(\beta^q\alpha^i(u))d_q(\alpha^{n-q}(v)) \\
&\quad + \sum_{\substack{s+t+j=n \\ s+t \neq n}} f_s(\beta^{n-s}(v^*))d_t(\alpha^j\beta^s(u^*))d_j\alpha^{n-j}(u^*) \\
&\quad + \sum_{s+t=n} f_s(\beta^{n-s}(v^*))d_t(\alpha^{n-t}(u^*))\alpha^n(u^*)
\end{aligned}$$

In view of the last two expressions, we have

$$\begin{aligned}
0 &= \eta_n(u)\alpha^n(v) + \left(-f_n(v^*u^*) + \sum_{s+t=n} f_s(\beta^{n-s}(v^*))d_t(\alpha^{n-t}(u^*)) \right) \alpha^n(u^*) \\
&\quad - \sum_{\substack{s+t+j=n \\ s+t \neq n}} f_s(\beta^{n-s}(v^*))d_t(\alpha^j\beta^s(u^*))d_j\alpha^{n-j}(u^*) - \sum_{\substack{i+j=n \\ i \neq n}} f_i(\beta^{n-i}(v^*u^*))d_j(\alpha^{n-j}(u^*))
\end{aligned}$$

In particular, $v = u$ gives

$$(2.8) \quad \eta_n(u)\alpha^n(u) - \eta_n(u^*)\alpha^n(u^*) = 0$$

for all $u \in U$. From (2.3), we have

$$(2.9) \quad \eta_n(u)\alpha^n(u) + \eta_n(u)\alpha^n(u^*) = 0$$

for all $u \in U$. Replacing v by u in (2.6), we obtain

$$(2.10) \quad \eta_n(u)\alpha^n(u) - \eta_n(u)\alpha^n(u^*) = 0$$

for all $u \in U$. Combining above expression with (2.9), we get

$$(2.11) \quad \eta_n(u)\alpha^n(u) = 0$$

for all $u \in U$. Linearization of (2.11) gives

$$\eta_n(u)\alpha^n(v) + \mu(u, v)\alpha^n(u) + \eta_n(v)\alpha^n(u) + \mu(u, v)\alpha^n(v) = 0$$

for all $u \in U$, where $\mu(u, v) = f_n(uv + vu) - \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(v)) + \sum_{i+j=n} f_i(\beta^{n-i}(v))d_j(\alpha^{n-j}(u))$ for all $u, v \in U$. Putting $u = -u$ in above expression and combining with it, we get

$$\eta_n(u)\alpha^n(v) + \mu(u, v)\alpha^n(u) = 0$$

for all $u, v \in U$. On right multiplying by $\eta(u)$ and using (2.11) and Lemma 2.1, we get

$$\eta_n(u)\alpha^n(v)\eta_n(u) = 0$$

for all $u, v \in U$. Since α^n is an automorphism and R is semiprime, so we conclude that $\eta(u) = 0$ for all $u \in U$ i.e., $f_n(u^2) = \sum_{i+j=n} f_i(\beta^{n-i}(u))d_j(\alpha^{n-j}(u))$ for all $u \in U$. Therefore in view of [2] Theorem 2.1, we get the required result. This completes the proof of the theorem. \square

As special case of the above theorem, and which are of independent interests, are the following corollaries:

Corollary 2.1. [8, Theorem 2.3] *Let R be a 2-torsion free semiprime $*$ -ring. Suppose there exists a family of additive mappings $F = (f_i)_{i \in \mathbb{N}_0}$ of R associated with some higher derivation $D = (d_i)_{i \in \mathbb{N}_0}$ of R such that $f_0 = id_R$, and the relation $f_n(xx^*) = \sum_{i+j=n} f_i(x)d_j(x^*)$ is fulfilled for each $n \in \mathbb{N}_0$ and for all $x \in R$. Then, F is a generalized higher derivation.*

Corollary 2.2. *Let R be a 2-torsion free semiprime ring with involution. Suppose there exists a family of additive mappings $D = (d_i)_{i \in \mathbb{N}_0}$ of R , where α, β are commuting automorphisms on R , such that $d_n(xx^*) = \sum_{i+j=n} d_i(\alpha^{n-i}(x))d_j(\beta^{n-j}(x^*))$ is fulfilled for each $n \in \mathbb{N}$ and for all $x \in R$. Then, d is a (α, β) -higher derivation on R .*

Conclusion. This paper deals with the following $*$ -differential identity $f_n(xx^*) = \sum_{i+j=n} f_i(\alpha^{n-i}(x))d_j(\beta^{n-j}(x^*))$ on a square close Lie ideal U of a prime ring R , where $F = (f_i)_{i \in \mathbb{N}_0}$ is a family of additive maps associated with (α, β) -higher derivations of U into R . We inferred that F is a generalized (α, β) -higher derivation of U into R .

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