Method Article

ANew Type of Exponential Curve and Its Application in the Tertiary Industry

Abstract：This paper proposes an improved new exponential curve model based on the classical exponential curve and the modified exponential curve. System parameters of the new exponential curve are estimated by the nonlinear least squares method and the trust region algorithm. Furthermore, numerical examples are provided to validate and verify the accuracy of the new exponential curve model. Finally, the new model is applied to model and predict the tertiary industry in China. The computational results show that the proposed new exponential curve has higher precision than the others.

Key words: Exponential curve; Trust region algorithm; Nonlinear Least Square method; Parameter estimation; Tertiary industry

0 Introduction

Social and economic phenomena are always changing and developing. Sometimes we not only conduct static analysis of the social economy, but also conduct dynamic analysis of various economic phenomena. When the long-term trend of socio-economic shows a stable linear trend, it can be predicted by various methods, such as moving average method, exponential smoothing method and linear trend equation fitting method. However, when the long-term trend is nonlinear with certain regularity, it is necessary to match the appropriate nonlinear trend curve. Such as the parabolic curve, the exponential curve, the modified exponential curve, the Gompertz curve and the Logistic curve type. For long-term trends that increase or decrease at roughly the same rate of growth, an exponential curve or a modified exponential curve can be used for the study. For the exponential curve model, the commonly used parameter estimation methods are least squares or three-sum methods。

In the previous research, Mao and Wang [1]proposed a new idea of parameter estimation for growth curve models. Baiet al[2]. studied the problems, applicable conditions and improved methods of parameter estimation of traditional exponential curve models. Zhanget al[3]. proposed a new method for correcting the parameter estimation of exponential curve models. Tang et al[4].studied the two-step least squares method for parameter estimation of exponential curve models. Yang et al[5].discussed the parameter estimation and error analysis methods of the new model in the direct area. Wang[6]discussed the theory and application of nonlinear model parameter estimation. Xiuet al[7].gave a trust region method for solving general constrained optimization problems. Xue et al[8].explored the trust region method for solving nonlinear equations. Tian [9]used the quasi-Newton method and the trust region method to jointly invert the epicenter distribution and the one-dimensional velocity structure. Li et al[10].studied the reliability assessment of the power grid capacity of urban power grid based on the trust region method. Xu et al[11]. studied the convergence of the unconstrained optimization cone model quasi-Newton trust region method, and proved the global convergence condition when solving.

Based on the above research, this paper proposes a new type of exponential curve and its parameter estimation method. Compared with the previous exponential curve and the modified exponential curve, the new exponential curve has better fitting and predictive effects, and is applied to the China's tertiary industry. The data relating to the tertiary industry from 1996 to 2007 was obtained from the China Statistical Yearbook[12]. From 1996 to 2005, the actual data of the tertiary industry in China was used as modeling data, and the data from 2006 to 2007 are the test data for the new exponential curve model. The comparative analysis of the exponential curve, the modified exponential curve and the new exponential curve proposed in this paper show that the new prediction accuracy of the exponential curve model is higher than that of the other two models.

1 Exponential curve and Modified exponential curve

1.1 Exponential curve

The exponential growth is characterized by the same growth rate of each period, or the trend value of the time series is increasing or decreasing according to a certain percentage, and the exponential curve equation is

$$\hat{Y}_t = ab^t . \tag{1}$$

Take the logarithm on both sides

$$\ln \hat{Y}_t = \ln a + t \ln b . \tag{2}$$

By the least squares method and equation (2), we get

$$\begin{cases} \sum \ln Y = n \ln a + \left(\sum t\right) \ln b \\ \sum t \ln Y = \left(\sum t\right) \ln a + \left(\sum t^2\right) \ln b \end{cases} . \tag{3}$$

Estimated parameter $\ln a$ and $\ln b$, then invert the logarithm, you can get the parameters a and b estimated value.

1.2 The modified exponential curve

Adding a constant K to the classical exponential curve yields a modified exponential curve equation

$$\hat{Y}_t = ab^t + K . \tag{4}$$

Among them, K , a , b are unknown parameters, $K \in (0, \infty)$, $a \neq 0$, $b \in (0,1) \cup (1, \infty)$.

The phenomenon described by the revised exponential curve is that the initial growth is rapid, and then the growth rate is reduced, and finally K is the growth limit. The parameters K , a , b are estimated based on the three-sum method: divide the entire time series into three equal arrays, each group has m items, three parameters are determined based on three local sums of the trend values $\hat{Y}_t$ equal to the three local sums of the original number column observations $\hat{Y}_t$, respectively. Specifically, the three local sums of the observed values are S_1, S_2, S_3, respectively:

$$S_1 = \sum_{t=0}^{m-1} Y_t , \quad S_2 = \sum_{t=m}^{2m-1} Y_t , \quad S_2 = \sum_{t=2m}^{3m-1} Y_t . \tag{5}$$

By the three-sum method, the equation is as follows

$$\begin{cases} S_1 = mK + a + ab + ab^2 + \cdots + ab^{m-1} \\ S_2 = mK + ab^m + ab^{m+1} + \cdots + ab^{2m-1} \\ S_3 = mK + ab^{2m} + ab^{2m+1} + \cdots + ab^{3m-1} \end{cases} . \tag{6}$$

From equation (6), we get equation (7)

$$\begin{cases} b = \left(\dfrac{S_3 - S_2}{S_2 - S_1} \right)^{\frac{1}{m}} \\ a = \left(S_2 - S_1 \right) \dfrac{b-1}{\left(b^m - 1 \right)^2} \\ K = \dfrac{1}{m} \left(S_1 - \dfrac{a\left(b^m - 1 \right)}{b-1} \right) \end{cases} . \tag{7}$$

2 New exponential curve model and its parameter estimation

2.1 New exponential curve model

The traditional exponential curve and the modified exponential curve model are suitable for rapid growth in the initial stage, and then the growth rate is gradually reduced, and finally the data is predicted by K as the growth limit. For example, a new product just listed has started to grow rapidly. When the market share tends to be saturated, the sales volume of this product gradually tends to a stable value K, so there are certain conditions for the system when using the modified index curve. limits. To this end, this paper proposes a new type of exponential curve model.

$$Y_t = ab^t + ct + d \quad , t = 1, 2, 3, \cdots, n . \tag{8}$$

The new exponential curve model adds a term related time t, which better reflects the trend of the system and provides a more perfect interpretation of system changes. The new exponential curve model has four unknowns a, b, c and d, and equation (8) is a nonlinear function, so the estimation of its parameters requires use the nonlinear least squares to obtain the estimators a, b, c and d.

2.2 Parameter estimation

Let $X = \left(a, b, c, d \right)$, $y = \left(y_1, y_2, y_3, \cdots, y_{n-1}, y_n \right)^T$ is the observed values.Then the following nonlinear equations are constructed by equation (8)

$$\hat{y} = f\left(X \right) = \begin{bmatrix} ab^1 + \ \ c + d \\ ab^2 + 2c + d \\ \vdots \\ ab^n + nc + d \end{bmatrix} . \tag{9}$$

Solving the least squares estimator $\hat{X} = \left(\hat{a}, \hat{b}, \hat{c}, \hat{d} \right)$ of the nonlinear equations (9) with the minimum residual sum of squares as the objective function

$$\begin{aligned} \min \quad I &= \sum_{t=1}^{n} \Delta E^2{}_t \\ &= \left\| f\left(\hat{X} \right) - y \right\|^2 \\ &= \left(f\left(\hat{X} \right) - y \right)^T \left(f\left(\hat{X} \right) - y \right) \end{aligned} , \tag{10}$$

where $\Delta E_t = \hat{a}\hat{b}^t + t\hat{c} + \hat{d} - y_t$ .This is a nonlinear unconstrained optimization problem.

For the above problems, the commonly used iterative methods are the Newton method, the quasi-Newton method. This paper uses the trust region algorithm to solve the equation (10). The trust region algorithm can be traced back to the Levenberg-Marquardt method. Instead of directly solving the objective function, it searches for a model similar to it, called the trust region subproblem, and then obtains the minimum value of the model to obtain the objective function's optimal solution. Compared with the quasi-Newton algorithm, the trust region algorithm not only converges to the stable point, but also satisfies the second-order necessary condition. The key lies in the construction and solution of the sub-problem. The basic idea are: first give an initial estimated solution of the objective function, and then solve the approximate model of the objective function. The obtained solution is called the trial step, through the trial step, the next iteration point can be adjusted and the radius of the trust region can be adjusted; The approximation model can continuously update the iteration point and adjust the radius of the trust region until the optimal solution of the objective function is obtained. In the iterative process of each step, the trust region algorithm solves the trust region subproblem, and the solution of the subproblem is always limited to a reliable generalized sphere. The most prominent advantage of the trust region method is its global convergence. Moreover, the trust region method ensures that the problem is globally convergent and requires the problem to have fast convergence locally, which not only improves the speed of problem convergence, but also ensures the global stability of the problem. The construction methods of the objective function in the trust region subproblem are: quadratic model, linear model and cone model method. This paper chooses quadratic model.

For the nonlinear unconstrained optimization problem of equation (10), using the quadratic approximation, the general form of the objective function of the trust region model is

$$Q_k(d_k) = I(\hat{X}_k) g_k^T d_k + \frac{1}{2} d_k^T H_k d_k . \tag{11}$$

Then the trust domain subproblem is constructed as follows

$$\begin{gathered} q_k(d_k) = I(\hat{X}_k) g_k^T d_k + \frac{1}{2} d_k^T H_k d_k , \\ s.t. \|d_k\| \le \Delta_k \end{gathered} \tag{12}$$

where $g_k = \nabla I(x_k)$ is the gradient of the objective function

$$\begin{aligned} g_k &= \left(g_1^{(k)},\ g_2^{(k)}, g_3^{(k)}, g_4^{(k)}\right) \\ &= \left.\left(\frac{\partial I}{\partial a} \quad \frac{\partial I}{\partial b} \quad \frac{\partial I}{\partial c} \quad \frac{\partial I}{\partial d}\right)\right|_{X=X^{(k)}} . \\ & \qquad k = 0,1,2,\cdots \end{aligned} \tag{13}$$

x_k is the Kth iteration point, $d_k = x_{k+1} - x_k$; $H_k = \nabla^2 I(x_k)$ is the Hessian matrix

$$H_k = \begin{bmatrix} \frac{\partial^2 I}{\partial a^2} & \frac{\partial^2 I}{\partial ab} & \frac{\partial^2 I}{\partial ac} & \frac{\partial^2 I}{\partial ad} \\ \frac{\partial^2 I}{\partial ba} & \frac{\partial^2 I}{\partial b^2} & \frac{\partial^2 I}{\partial bc} & \frac{\partial^2 I}{\partial bd} \\ \frac{\partial^2 I}{\partial ca} & \frac{\partial^2 I}{\partial cb} & \frac{\partial^2 I}{\partial c^2} & \frac{\partial^2 I}{\partial cd} \\ \frac{\partial^2 I}{\partial da} & \frac{\partial^2 I}{\partial db} & \frac{\partial^2 I}{\partial dc} & \frac{\partial^2 I}{\partial d^2} \end{bmatrix}_{X=X^{(k)}} \quad k = 0,1,2,\cdots \tag{14}$$

$\Delta_k > 0$ is the confidence domain radius, d_k is the correction amount obtained after optimization that is the trial step, which is within the trust domain, and $\|.\|$ is the Euclid norm.

Solving equation (10) to obtain correction amount d_k

$$Ared_k = I\left(\hat{X}_k\right) - I\left(\hat{X}_k + d_k\right) \tag{15}$$

$$\begin{aligned} Pred_k &= Q_k\left(0\right) - Q_k\left(d_k\right) \\ &= -\left(g_k{}^T d_k + \frac{1}{2} d_k{}^T H_k d_k\right) \end{aligned} \tag{16}$$

$$r_k = \frac{Ared_k}{Pred_k}, \tag{17}$$

where $Ared_k$ is the actual amount of decline, $Pred_k$ is the predicted amount of decline, and the rate of decline r_k is the radius of the next iteration. The degree of approximation is judged according to the magnitude of the ratio r_k. When r_k is closer to 1, the convergence is better, and r_k can be increased. When r_k is less than 0, it does not converge, and the radius needs to be reduced.

3 Numerical examples

In this paper, four random parameters that conform to the standard normal distribution are randomly generated as the estimated parameters of the new exponential curve model, and random factors are added to simulate the real data. Then,we use the exponential curve model, the modified curve model and the new exponential curve model to fit the data with random factors and compare their fitting accuracy. Figure 1 specifically shows the case of simulated data.

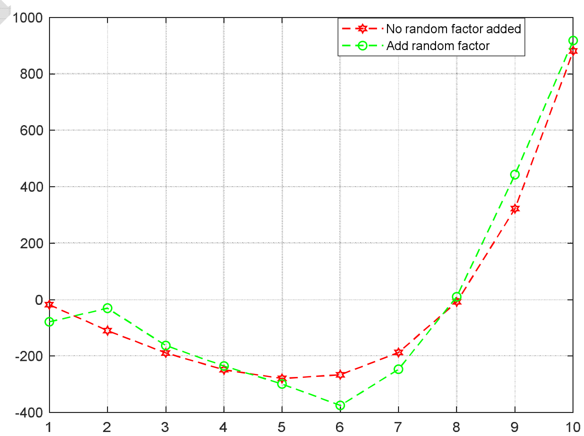

Figure 1 Comparison of random raw data

For the randomly generated analog data, it is realized by Matlab programming.

The calculation results are shown in Table 1 and Figure 2.

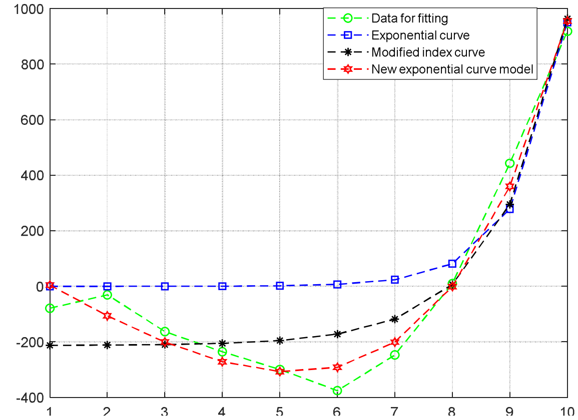

Figure 2 Fitting effects of three exponential curve models

Simulation data	Exponential curve model	Error	Corrected curve model	Error	New exponential curve model	Error
-79.2181	0.015	100.02%	-212.283	-167.97%	4.6272	105.84%
-30.518	0.0514	100.17%	-211.475	-592.95%	-106.168	-247.89%
-162.677	0.1754	100.11%	-209.603	-28.85%	-200.652	-23.34%
-235.526	0.5989	100.25%	-205.268	12.85%	-271.258	-15.17%
-299.233	2.0452	100.68%	-195.229	34.76%	-306.908	-2.57%
-375.356	6.9844	101.86%	-171.981	54.18%	-291.386	22.37%
-247.119	23.8516	109.65%	-118.144	52.19%	-200.952	18.68%
10.16933	81.4533	-700.97%	6.5329	35.76%	-0.8513	108.37%
442.9445	278.1628	37.20%	295.2583	33.34%	359.7912	18.77%
918.2308	949.9253	-3.45%	963.888	-4.97%	955.4549	-4.05%
Average relative error	4.56%		-57.17%		-1.90%	

Table 1: Fitting effects of three exponential curve models

It can be seen from Table 1 that the relative error of the new model is significantly smaller than the exponential curve model and the modified exponential curve model. Figure 2 shows that the new exponential curve model has the best fitting effect, the modified curve model has the second best fitting effect, and the exponential curve model has the worst fitting effect. The calculation results show that the new exponential curve model can fit this kind of data well and can predict the changes of such systems more accurately.

4 Applications

4.1Data source

In order to test the prediction accuracy and fitting effect of the new exponential curve, the actual data is used to verify the analysis, and the calculation results are compared with the exponential curve model and the modified exponential curve model. This article obtains the statistical data of China's tertiary industry from 1996 to 2007 from the China Statistical Yearbook. See Table 2 below for details:

Years	Observations	Years	Observations
1996	24107.2	2002	51421.7
1997	27903.8	2003	57754.4
1998	31558.3	2004	66648.9
1999	34934.5	2005	77427.8
2000	39897.9	2006	91759.7
2001	45700	2007	115810.7

Table 2 Statistics of China's tertiary industry

4.2 Model test accuracy test standard

In order to test the prediction accuracy of the model, the root mean square error ($RMSE$), the mean square error (MSE), and themean absolute percent error ($MAPE$) of the fitted data can be calculated according to the predicted value and the actual value.Their expressions are as follows

$$RMSE=\sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(Y(t)-\hat{Y}(t)\right)^2}. \tag{18}$$

$$MSE=\frac{1}{n}\sum_{t=1}^{n}\left(Y(t)-\hat{Y}(t)\right)^2. \tag{19}$$

$$MAPE=\sum_{t=1}^{n}\left|\frac{Y(t)-\hat{Y}(t)}{Y(t)}\right|\times\frac{100}{n}. \tag{20}$$

4.3 New exponential curve model and its parameter estimation

To test the prediction accuracy and fitting accuracy of the new exponential curve. First, the parameters a,b,c and d of the new model must be estimated. Since the nonlinear least squares method is involved in the parameter estimation of parameters a,b,c and d, in this example, the parameters of a,b,c and d are estimated using the trust region algorithm described above. The specific steps of the trust region method are as follows:

Step 1: Select initial point $\hat{X}_0\in R^n$, initial trust radius $\Delta_0=1,\varepsilon>0$. Let $k=0$;

Step 2: If $\|g_k\|\le\varepsilon$, the algorithm terminates, otherwise, proceeds to step 2;

Step 3: By calculating the solution of the trust region subproblem, the test step d_k is obtained;

Step 4: Calculate the values of $Ared_k$, $Pred_k$, and r_k using (15), (16), and (17), and update the confidence region radius.

$$\Delta_{k+1}=\begin{cases}0.25\|d_k\|, & if \quad r_k<0.1\\ 3\Delta_k, & if \quad r_k\ge 0.9\\ \Delta_k, & else\end{cases}. \tag{21}$$

Step 5: If $r_k>0.1$, let $\hat{X}_{k+1}=\hat{X}_k+d_k$, update H_k, and return to step 1; otherwise, $\hat{X}_{k+1}=\hat{X}_k$, return to step 2. Let $k=k+1$.

The parameter estimation of the new exponential curve model (retaining 4 decimal places) is calculated by the trust region algorithm. The specific expression is as follows

$$\hat{Y}(t)=1307.8593\times1.3636^t+2878.6676t+19492.7079. \quad t=1,2,3,\cdots n \tag{22}$$

4.4Numerical results

This paper combines MATLAB programming software, using the statistical data of China's tertiary industry for a total of 10 years from 1996 to 2005, and using it to fit the model. The data from 2 years of 2006-2007 were used as model extrapolation prediction test data. The fitting results of the above three models are shown in Table 3.

Years	Observations value	Exponential curve		Modified exponential curve		New exponential curve	
		Predictive value %	Average relative error	Predictive value%	Average relative error %	Predictive value	Average relative error %
1996	24107.2	23948.82	-0.66%	24716.37	2.53%	24154.75	0.20%
1997	27903.8	27237.58	-2.39%	27672.43	-0.83%	27681.81	-0.80%
1998	31558.3	30977.96	-1.84%	31106.30	-1.43%	31444.63	-0.36%
1999	34934.5	35231.98	0.85%	35095.22	0.46%	35528.90	1.70%
2000	39897.9	40070.19	0.43%	39728.91	-0.42%	40051.50	0.38%
2001	45700	45572.80	-0.28%	45111.58	-1.29%	45171.81	-1.16%
2002	51421.7	51831.05	0.80%	51364.31	-0.11%	51107.13	-0.61%
2003	57754.4	58948.71	2.07%	58627.72	1.51%	58153.80	0.69%
2004	66648.9	67043.80	0.59%	67065.18	0.62%	66715.87	0.10%
2005	77427.8	76250.54	-1.52%	76866.47	-0.72%	77344.31	-0.11%
MAPE		3.1747		2.6132		1.3840	
MSE		407592.5		228391		99027.39	
RMSE		638.4297		477.9027		314.6862	

Table 3:The fitting results of the above three models

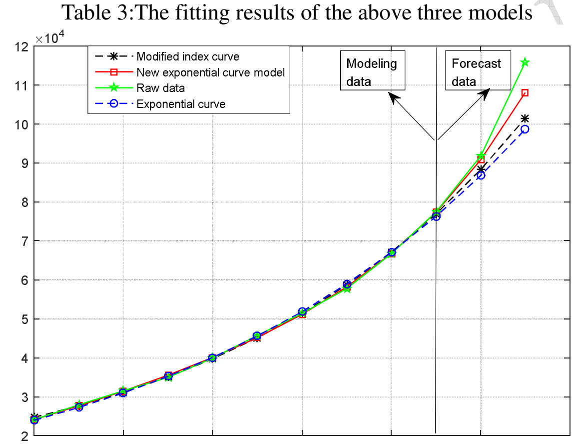

Figure 3 :Results among the above three models

It can be seen from Table 3 and Figure 3 that the exponential curve and the modified exponential curve have certain effects on the fitting and prediction of the tertiary industry statistical data in China, which can reflect the trend of the tertiary industry in China from 1996 to 2007. However, the *MAPE MSE RMSE* of the new modified exponential curve model are all smaller than the exponential curve and the modified exponential curve, and the relative error of the fitting is smaller than the other two models.

In order to more intuitively compare the accuracy of the fitting and extrapolation predictions of the three models of the exponential curve, the modified exponential curve and the new exponential curve, Table 4 and Figure 4 show the comparison of the fitting errors of the three models.

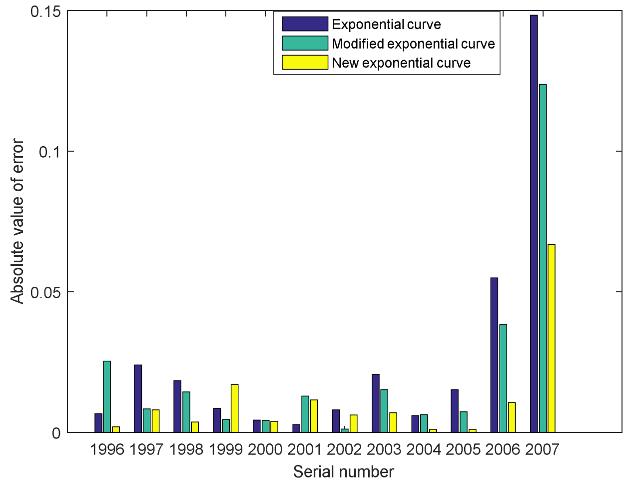

Figure 4 Fit errors graph for each model

Years	Observations value	Exponential curve Predictive value	Average relative error%	Modified exponential curve prediction value	Average relative error%	New exponential curve prediction value	Average relative error %
2006	91759.7	86721.59	-5.49%	88252.04	-3.82%	90790.42	-1.06%
2007	115810.7	98630.57	-14.83%	101477.95	-12.38%	108078.65	-6.68%

Table 4 Extrapolation prediction results and relative error size tables for each model

It can be seen from Fig. 4 that the fitting error of the new exponential curve model is only slightly larger than the other two models in 1999, 2001 and 2002, but significantly smaller than the other two models in other years. Moreover, as can be seen from Table 4, in the extrapolation prediction, the prediction accuracy of the new exponential curve model is also higher than the other two models, and the prediction result is closer to the true value.

5Conclusion

First of all, the new exponential curve model is effective for the prediction of China's tertiary industry data, which verifies the effectiveness and practicability of the improved model. Secondly, the prediction accuracy of the new exponential curve model is higher than that of the exponential curve and the modified exponential curve model.Finally, the new exponential curve model established in this paper is simple in calculation and small in calculation, and can be further applied to energy and economy.

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