Global existence and blow-up of classical solution for an attraction-repulsion chemotaxis system with logistic source

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Abstract

We consider the following quasilinear attraction-repulsion chemotaxis system of parabolic-elliptic-elliptic type with logistic source

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\chi S(u)\nabla v) + \nabla\cdot(\xi F(u)\nabla w) + f(u), & x\in\Omega, t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega, t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega, t>0, \end{cases}$$

under homegeneous Neumann boundary conditions in a bounded domain $\Omega \subset R^n (n \geq 2)$ with smooth boundary, where $D(u) \geq c_D(u+1)^{m-1}$ with $m \geq 1$ and $c_D > 0$, $f(u) \leq a - bu^\eta$ with $\eta > 1$. We show two cases that the system admits a unique global bounded classical solution depending on $0 \leq S(u) \leq C_s(u+1)^q, 0 \leq F(u) \leq C_F(u+1)^g$ by Gagliardo-Nirenberg inequality. For specific $D(u), S(u), F(u)$ with logistic source for $\eta > 1$ and $n = 2$, we establish the finite time blow-up conditions for solutions that the finite time blow-up occurs at $x_0 \in \Omega$ whenever $\int_\Omega u_0(x)dx > \frac{8\pi}{\chi\alpha - \xi\gamma}$ with $\chi\alpha - \xi\gamma > 0$, under $\int_\Omega u_0(x)|x-x_0|^2 dx$ sufficiently small.

Keywords: Chemotaxis, Attraction-repulsion, Boundedness, Blow-up, Logistic source

1. Introduction

We consider the quasilinear attraction-repulsion chemotaxis system of parabolic-elliptic-elliptic type with logistic source

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\chi S(u)\nabla v) + \nabla\cdot(\xi F(u)\nabla w) + f(u), & x\in\Omega, t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega, t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega, t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n (n \geq 2)$ is a bounded domain with smooth boundary and $\frac{\partial}{\partial\nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$, $\chi \geq 0$ and $\xi \geq 0$ are parameters referred to as chemosensitivity, α, β, γ and δ are positive parameters , $u(x,t), v(x,t)$ and $w(x,t)$ denote the cell density, the concentration of the chemoattractant and the concentration of

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the chemorepellent, respectively. We assume that $D(u), S(u), F(u)$ satisfy

$$D(u), S(u), F(u) \in C^2([0,\infty)), \tag{1.2}$$

and there exist some constants $c_D > 0$ and $m \geq 1$ such that

$$D(u) \geq c_D(u+1)^{m-1}. \tag{1.3}$$

The function $f : [0,\infty) \to R$ is smooth and it satisfies $f(0) \geq 0,\ a \geq 0,\ b > 0$ and $\eta > 1$,

$$f(u) \leq a - bu^\eta. \tag{1.4}$$

Chemotaxis describes the oriented movement of cells along the concentration gradient of a chemical signal produced by cells. The prototype of the chemotaxis model known as the Keller-Segel model was first proposed by Keller and Segel [12] in 1970. In its general form, the (attractive) Keller-Segel model is given by

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\chi S(u)\nabla v) + f(u), & x\in\Omega, t>0,\\ \tau v_t = \Delta v + u - v, & x\in\Omega, t>0. \end{cases} \tag{1.5}$$

In general, the chemicals diffuse much faster than cells because the chemical molecules are much smaller than the cells. Hence, the chemotaxis system (1.5) can be approximated by setting $\tau = 0$. The global solution with $\tau = 0$ and $\tau = 1$ have been investigated in the past four decades (1.5) by using some important estimates. When $D(u) = 1, S(u) = u$ and $f(u) = 0$, if $n = 1$, (1.5) admits a unique global solution; if $n = 2$, there is a critical mass phenomenon; if $n \geq 3$, finite-time blow-up occurs in [8, 9] by usinig Lyapunov Function. For general cases of $D(u), S(u)$ and $f(u) = 0$, many studies have considered the boundedness of the global solutions [5, 7, 23, 22]and many others have also addressed the finite time blow-up [1, 4] by using some important estimates. When $\tau = 0, S(u) = u$, $f(u)$ satisfies (1.4) and $D(u)$ fulfills (1.3), Wang et al.[18] showed that (1.5) admits a unique bounded global classical solution by Gagliardo-Nirenberg inequality. For (1.5) with more general cases of $D(u), S(u)$ and $f(u)$, we can refer to [2, 15].

In many biological processes, the interaction between cells and combinations of attractive and repulsive signal chemicals can produce various interesting biological patterns, the following attraction-repulsion chemotaxis model is produced in [3, 6].

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\chi S(u)\nabla v) + \nabla\cdot(\xi F(u)\nabla w) + f(u), & x\in\Omega, t>0,\\ \tau v_t = \Delta v + \alpha u - \beta v, & x\in\Omega, t>0,\\ \tau w_t = \Delta w + \gamma u - \delta w, & x\in\Omega, t>0. \end{cases} \tag{1.6}$$

Fewer results are available for system (1.6) than (1.5), because there exists a useful Lyapunov function for (1.5) and (1.6) does not admit such a function.When $D(u) = 1, S(u) = F(u) = u$ and $f(u) = 0$, (1.6) with $\tau = 1$ admits a unique global bounded solution [13, 10] by Gagliardo-Nirenberg inequality and some important estimates. When $D(u) = 1$ and $S(u) = F(u) = u$, $f(u)$ satisfies (1.4), (1.6) with $\tau = 0$, Jin and Wang [11] studied the boundness and blow-up in a bounded domain $\Omega \subset R^2$ with $\delta = \beta$,

$$\chi\alpha - \xi\gamma > 0 \ \textit{and} \ \int_\Omega u_0(x)dx > \frac{8\pi}{(\chi\alpha - \xi\gamma)}. \tag{1.7}$$

Tao and Wang [18] showed that the finite time blow-up for nonradial solutions, Zhang and Li [25] showed that (1.6) admits a unique global bounded solution and they proved the large-time behavior of a solution. For (1.6) with $\tau = 0$ for more general cases of $D(u), S(u)$ and $F(u) = u$, $f(u)$ satisfies (1.4), Wang [13] admits a unique global bounded solution and they proved the large-time behavior of a solution for a specific logistic source.

To the best of our knowledge, no rigorous result is available for more general case of (1.6) with $\tau = 0$. Thus the main aim of the present study is to explore on the global and blow-up solvability of system (1.1).

The remainder of this paper is organized as follows. In Section 2, we show the local existence and uniqueness of the solutions to system (1.1) and we give the mass estimates. In Section 3, two different cases of a priori estimation are applied to establish the desired estimates for (1.1). It need to be pointed out the distinction of these two cases lie in the mechanism which we take sufficient advantage of in the process of establishing the estimates of (1.1). In Case 1 (see Theorem 1), we mainly rely on the logistic dampening, while in Case 2 (see Theorem 2), the nonlinear diffusion plays the critical role. Finally, Theorem 1 and 2 are proved based on the estimates of (1.1). In Section 4, we obtain a sharp result on the blow-up for solutions to (1.1).

Theorem 1.1. Suppose that (1.2) and (1.3) are valid, f(u) fulfills (1.4) with $b > 0$ and $\eta \geq 2$. Assume that

$$q \leq \eta - 1, \;\; g \leq 1$$

and

$$0 \leq S(u) \leq C_s(u+1)^q, \;\; 0 \leq F(u) \leq C_F(u+1)^g, \tag{1.8}$$

then there exists a unique triple (u, v, w) of nonnegative functions which are bounded and solve (1.1) in the classical sense.

Theorem 1.2. Suppose that (1.2) and (1.3) are valid, f(u) fulfills (1.4) with $b > 0$ and $\eta > 1$,

$$0 \leq S(u) \leq C_s(u+1)^q, \;\; 0 \leq F(u) \leq C_F(u+1)^g$$

for $\max\{q, g\} < \frac{1}{n} + m - 1$, then there exists a unique triple (u, v, w) of nonnegative functions, which are bounded and solve (1.1) in the classical sense.

Theorem 1.3. Let

$$D(u) = 1, S(u) = F(u) = u, \;\; f(u) \leq a - bu^{\eta} \;\; for \; \eta > 1 \tag{1.9}$$

in (1.1), (1.7) hold with $\int_\Omega u_0(x)|x - x_0|^2 dx$ small enough for an $x_0 \in \Omega$. Then the solution of (1.1) blows up in finite time.

2. Preliminaries

The local existence and uniqueness of the system (1.1) can be derived from the reasoning of Lemma 2.1 in [14], so we only state the result and omit its proof here.

Lemma 2.1. Suppose that (1.2)-(1.4) are valid. Then there exists a maximal existence time $T_{max} \in (0, +\infty)$ and a unique triple (u,v,w) of functions which solve (1.1) in the classical sense. Also these functions have the following regularity properties:

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{max})) \bigcap C^{2,1}(\overline{\Omega} \times [0, T_{max})), \\ v \in C^0(\overline{\Omega} \times [0, T_{max})) \bigcap C^{2,1}(\overline{\Omega} \times [0, T_{max})) \bigcap L^\infty((0, T_{max}); W^{1,l}(\Omega)), \\ w \in C^0(\overline{\Omega} \times [0, T_{max})) \bigcap C^{2,1}(\overline{\Omega} \times [0, T_{max})) \bigcap L^\infty((0, T_{max}); W^{1,l}(\Omega)) \end{cases} \tag{2.1}$$

with $l > n$ and

$$u \geq 0, v \geq 0, w \geq 0 \ in \ \Omega \times (0, T_{max}).$$

In addition, if $T_{max} < +\infty$, then

$$\lim_{t \to T_{max}} sup(\|u(\cdot, t)\|)_{L^\infty}(\Omega) + \|v(\cdot, t)\|_{w^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{w^{1,\infty}(\Omega)} = \infty. \tag{2.2}$$

Lemma 2.2. Let the assumption in Lemma 2.1 hold. Then there exists a constant $C > 0$ such that

$$\int_\Omega u(x, t)dx \leq C, \qquad t \in (0, T_{max}), \tag{2.3}$$

$$\int_\Omega v(x, t)dx \leq C, \qquad t \in (0, T_{max}), \tag{2.4}$$

$$\int_\Omega w(x, t)dx \leq C, \qquad t \in (0, T_{max}). \tag{2.5}$$

Lemma 2.3. (Gagliardo-Nirenberg inequality). Let $r \in (0, \alpha)$ and $\psi \in W^{1,2}(\Omega) \bigcap L^r(\Omega)$. Then there exists a constant $C_{GN} > 0$ such that

$$\|\psi\|_{L^\alpha(\Omega)} \leq C_{GN} \left(\|\nabla \psi\|^{\lambda^*}_{L^2(\Omega)} \|\psi\|^{1-\lambda^*}_{L^r(\Omega)} + \|\psi\|_{L^r(\Omega)} \right), \tag{2.6}$$

where $\lambda^* \in (0, 1)$ satisfies

$$\lambda^* = \frac{\frac{n}{r} - \frac{n}{\alpha}}{1 - \frac{n}{2} + \frac{n}{r}}.$$

3. A priori estimates

In order to prove Theorem 1, we have a priori estimates for $\int_\Omega (u + 1)^p dx$ in the following lemma.

Lemma 3.1. Suppose that (1.2) and (1.3) are valid, f(u) fulfills (1.4) with $b > 0$ and $\eta \geq 2$. $S(u)$ and $F(u)$ satisfy (1.8) with $q \leq \eta - 1, g \leq 1$. Then for any $p > \frac{n}{2}$, there exists a constant $C > 0$ independent of t such that the solution (u, v, w) of system (1.1) satisfies

$$\int_\Omega (u + 1)^p)dx \leq C, \ t \in (0, T_{max}). \tag{3.1}$$

Proof. Let

$$\widetilde{S(u)} = \int_0^u (\zeta + 1)^{p-2} S(\zeta) d\zeta, \ \widetilde{F(u)} = \int_0^u (\zeta + 1)^{g-2} F(\zeta) d\zeta. \tag{3.2}$$

Substituting (1.8) into (3.2), we obtain that

$$\widetilde{S(u)} \le C_s \int_0^u (\zeta+1)^{p+q-2} d\zeta \le C_s \frac{1}{p+q-1}(u+1)^{p+q-1}, \tag{3.3}$$

and

$$\widetilde{F(u)} \le C_F \int_0^u (\zeta+1)^{p+g-2} d\zeta \le C_F \frac{1}{p+g-1}(u+1)^{p+g-1}. \tag{3.4}$$

After multiplying both sides of the first equation in (1.1) by $(u+1)^{p-1}$ and integrating by parts over Ω, we obtain that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx + \int_\Omega (p-1)D(u)(u+1)^{p-2}|\nabla u|^2 dx \\
&= \int_\Omega \chi(p-1)(u+1)^{p-2}S(u)\nabla u\cdot\nabla v dx - \int_\Omega \xi(p-1)(u+1)^{p-2}F(u)\nabla u\cdot\nabla w dx \\
&+ \int_\Omega (u+1)^{p-1} f(u) dx.
\end{aligned} \tag{3.5}$$

Adding (3.2) and (3.5) together,

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx + \int_\Omega (p-1)D(u)(u+1)^{p-2}|\nabla u|^2 dx \\
&= \chi(p-1)\int_\Omega \nabla\widetilde{S(u)}\cdot\nabla v dx - \xi(p-1)\int_\Omega \nabla\widetilde{F(u)}\cdot\nabla w dx + \int_\Omega (u+1)^{p-1} f(u) dx, \\
&= -\chi(p-1)\int_\Omega \widetilde{S(u)}\Delta v dx + \xi(p-1)\int_\Omega \widetilde{F(u)}\Delta w dx + \int_\Omega (u+1)^{p-1} f(u) dx.
\end{aligned} \tag{3.6}$$

From the second and the third equations of (1.1), we obtain that

$$-\Delta v \le \alpha u, \ \ \Delta w \le \delta w. \tag{3.7}$$

Inserting (3.7) into (3.6) yields

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx + \int_\Omega (p-1)D(u)(u+1)^{p-2}|\nabla u|^2 dx \\
&\le \alpha\chi(p-1)\int_\Omega \widetilde{S(u)} u dx + \xi\delta(p-1)\int_\Omega \widetilde{F(u)} w dx + \int_\Omega (u+1)^{p-1} f(u) dx.
\end{aligned} \tag{3.8}$$

By (3.3) and (3.4), we obtain that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx + \int_\Omega (p-1)D(u)(u+1)^{p-2}|\nabla u|^2 dx \\
&\le C_1\int_\Omega (u+1)^{p+q-1} u dx + C_2\int_\Omega (u+1)^{p+g-1} w dx + \int_\Omega (u+1)^{p-1} f(u) dx,
\end{aligned} \tag{3.9}$$

where $C_1 = \frac{C_s\alpha\chi(p-1)}{p+q-1}$ and $C_2 = \frac{C_F\xi\delta(p-1)}{p+g-1}$ are positive constants.

By $q \le \eta - 1$ and Young's inequality, there exists a positive constant C_3 such that

$$C_1\int_\Omega (u+1)^{p+q-1} u dx \le \frac{2^{1-\eta}b}{6}\int_\Omega (u+1)^{p+\eta-1} dx + C_3. \tag{3.10}$$

By $\eta \geq 2, g \leq 1$ and Young's inequality, there exist positive constants C_i $(i = 4, 5, 6)$ such that

$$\begin{aligned}
&C_2 \int_\Omega (u+1)^{p+g-1} w dx \\
&\leq C_4 \int_\Omega (u+1)^p w dx + C_5, \\
&\leq \frac{2^{1-\eta} b}{6} \int_\Omega (u+1)^{p+1} dx + C_6 \int_\Omega w^{p+1} dx + C_5, \\
&\leq \frac{2^{1-\eta} b}{6} \int_\Omega (u+1)^{p+\eta-1} dx + C_6 \int_\Omega w^{p+1} dx + C_5.
\end{aligned} \tag{3.11}$$

Substituting (3.10), (3.11) into (3.9), we obtain that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx + \int_\Omega (p-1) D(u)(u+1)^{p-2}|\nabla u|^2 dx \\
&\leq \frac{2^{1-\eta} b}{3} \int_\Omega (u+1)^{p+\eta-1} dx + C_6 \int_\Omega w^{p+1} dx + \int_\Omega (u+1)^{p-1} f(u) dx + C_7
\end{aligned} \tag{3.12}$$

with positive constants C_6 and C_7.

Now, we estimate the integral $\int_\Omega w^{p+1} dx$ according to a procedure similar to that employed by [19]. In the following, we provide a brief outline for the sake of completeness. Since $\delta > 0$ and w solves

$$\begin{cases} -\Delta w + \delta w = \gamma u, \ x \in \Omega, \\ \frac{\partial w}{\partial \nu} = 0, \ x \in \partial\Omega, \end{cases}$$

we can apply the L^p estimates to deduce that

$$\|w(\cdot,t)\|_{W^{2,p}(\Omega)} \leq C_8 \|u(\cdot,t)\|_{L^p(\Omega)}, \qquad t \in (0, T_{max}) \tag{3.13}$$

with some appropriate positive constant C_8. By (2.5) and the Gagliardo-Nirenberg inequality, there exist two constants $C_9 > 0$ and $C_{10} > 0$ such that

$$\begin{aligned}
\int_\Omega w^{p+1} dx &\leq C_9 \|D^2 w\|_{L^p(\Omega)}^{(p+1)\varepsilon} \|w\|_{L^1(\Omega)}^{(p+1)(1-\varepsilon)} + C_9 \|w\|_{L^1(\Omega)}^{(p+1)} \\
&\leq C_{10} \|u\|_{L^p(\Omega)}^{(p+1)\varepsilon} + C_{10}, \qquad t \in (0, T_{max}),
\end{aligned} \tag{3.14}$$

where $\varepsilon = \frac{1-\frac{1}{p+1}}{1+\frac{2}{n}-\frac{1}{p}}$. Since $p > \frac{n}{2}$, it is easy to check that $\varepsilon \in (0,1)$ and $(p+1)\varepsilon < p$. Hence, we use Young's inequality for $\eta \geq 2$ to obtain that

$$\begin{aligned}
&C_6 \int_\Omega w^{p+1} dx \\
&\leq C_{10} \|u\|_{L^p(\Omega)}^p + C_{10} \\
&\leq \frac{2^{1-\eta} b}{6} \int_\Omega u^{p+1} dx + C_{11} \\
&\leq \frac{2^{1-\eta} b}{6} \int_\Omega (u+1)^{p+\eta-1} dx + C_{11}.
\end{aligned} \tag{3.15}$$

Combining (3.12) with (3.15) yields

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx + \int_\Omega (p-1)D(u)(u+1)^{p-2}|\nabla u|^2 dx$$
$$\leq \frac{b2^{1-\eta}}{2}\int_\Omega (u+1)^{p+\eta-1}dx + \int_\Omega (u+1)^{p-1}f(u)dx + C_{12}. \tag{3.16}$$

Since $f(u)$ satisfies (1.4) with $\eta \geq 2$ and Young's inequality and $(u+1)^\eta \leq 2^{\eta-1}(u^\eta+1)$ for $\eta \geq 2$ implies $u^\eta \geq \frac{1}{2^{\eta-1}}(u+1)^\eta - 1$, (3.16) can be further written as

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx + \int_\Omega (p-1)D(u)(u+1)^{p-2}|\nabla u|^2 dx\\
&\leq \frac{2^{1-\eta}b}{2}\int_\Omega (u+1)^{p+\eta-1}dx + a\int_\Omega (u+1)^{p-1}dx - b\int_\Omega (u+1)^{p-1}u^\eta dx + C_{12}\\
&\leq \frac{2^{1-\eta}b}{2}\int_\Omega (u+1)^{p+\eta-1}dx + \frac{2^{1-\eta}b}{6}\int_\Omega (u+1)^{p+\eta-1}dx\\
&+ b\int_\Omega (u+1)^{p-1}dx - b2^{1-\eta}\int_\Omega (u+1)^{p+\eta-1}dx + C_{13}\\
&\leq \frac{2^{1-\eta}2b}{3}\int_\Omega (u+1)^{p+\eta-1}dx + \frac{2^{1-\eta}b}{6}\int_\Omega (u+1)^{p+\eta-1}dx - b2^{1-\eta}\int_\Omega (u+1)^{p+\eta-1}dx + C_{13}\\
&\leq -\frac{2^{1-\eta}b}{6}\int_\Omega (u+1)^{p+\eta-1}dx + C_{14}.
\end{aligned} \tag{3.17}$$

Using Young's inequality again, we obtain that

$$\frac{1}{p}\int_\Omega (u+1)^p dx \leq \frac{2^{1-\eta}b}{6}\int_\Omega (u+1)^{p+\eta-1}dx + C_{15}. \tag{3.18}$$

Thus, combining (3.17) and (3.18), we conclude that

$$\frac{d}{dt}\int_\Omega (u+1)^p dx + \int_\Omega (u+1)^p dx \leq C_{16},\ t\in(0,T_{max}), \tag{3.19}$$

from Gronwall's inequality, we obtain that

$$\int_\Omega (u+1)^p dx \leq 2\max\left\{\int_\Omega (u_0+1)^p dx, C_{16}\right\},\ t\in(0,T_{max}),$$

which implies the desired uniform estimates. □

In order to obtain Theorem 2, we have the following lemma.

Lemma 3.2. Suppose that (1.2) and (1.3) are valid, f(u) fulfills (1.4) with $b>0$ and $\eta>1$. Let $n\geq 2$, $m>1, \theta>1$ and

$$\alpha_1 = \frac{\frac{n(p+m-1)}{2}\left[1-\frac{1}{\theta(p+2q-m-1)}\right]}{1-\frac{n}{2}+\frac{n(p+m-1)}{2}},\ \alpha_2 = \frac{\frac{n(p+m-1)}{2}\left(1-\frac{\theta-1}{2\theta}\right)}{1-\frac{n}{2}+\frac{n(p+m-1)}{2}}, \tag{3.20}$$

$$\beta_1 = \frac{(p+2q-m-1)\alpha_1}{p+m-1} = \frac{\frac{n(p+2q-m-1)}{2}\left[1-\frac{1}{\theta(p+2q-m-1)}\right]}{1-\frac{n}{2}+\frac{n(p+m-1)}{2}},$$

$$\beta_2 = \frac{2\alpha_2}{p+m-1} = \frac{n\left(1-\frac{\theta-1}{2\theta}\right)}{1-\frac{n}{2}+\frac{n(p+m-1)}{2}}$$

satisfy $q < \frac{1}{n} + m - 1$.
Then there exist p sufficiently large such that

$$(i)\ \alpha_1 \in (0,1),\ (ii)\ \alpha_2 \in (0,1),\ (iii)\ \beta_1+\beta_2 \in (0,1). \tag{3.21}$$

Similarly,

$$\beta_3 = \frac{\frac{n(p+2g-m-1)}{2}\left[1-\frac{1}{\mu(p+2g-m-1)}\right]}{1-\frac{n}{2}+\frac{n(p+m-1)}{2}},$$

$$\beta_4 = \frac{n\left(1-\frac{\mu-1}{2\mu}\right)}{1-\frac{n}{2}+\frac{n(p+m-1)}{2}},$$

if $g < \frac{1}{n} + m - 1$, then

$$\beta_3+\beta_4 \in (0,1).$$

Proof. For

$$p > \max\left\{\frac{n}{2}, 2+m-2q\right\}, \tag{3.22}$$

we have that $1 - \frac{1}{\theta(p+2q-m-1)} > 0$. If $n = 2$, it is easy to derive (i) and (ii). If $n > 2$, let

$$\theta \in \left(\frac{-\frac{n(p+m-1)}{n-2}}{2-\frac{n(p+m-1)}{n-2}},\ \frac{n(p+m-1)}{(n-2)(p+2q-m-1)}\right). \tag{3.23}$$

Then it is easy to verify $\theta \neq \emptyset$ for choosing p sufficiently large. Since (3.23) also implies

$$-\frac{n(p+m-1)}{2\theta(p+2q-m-1)} < 1 - \frac{n}{2},$$

which together with the definition of α_1 in (3.20) implies (i) in (3.21). By a computation, we deduce that (3.23) is equivalent to

$$(p+m-1) > \frac{n-2}{n}\cdot\frac{2\theta}{\theta-1},$$

which implies

$$\frac{n}{2}(p+m-1)(1-\frac{\theta-1}{2\theta}) < 1-\frac{n}{2}+\frac{n}{2}(p+m-1), \tag{3.24}$$

we can infer (ii) in (3.21) by (3.24). In addition, if $q < \frac{1}{n} + m - 1$, we can verify that

$$p+2q-m-1-\frac{1}{\theta}+2-2(\frac{1}{2}-\frac{1}{2\theta}) < \frac{2}{n}+p+m-2. \tag{3.25}$$

Then we have that (3.25) is equivalent to

$$\frac{n(p+2q-m-1)}{2}\left[1-\frac{1}{\theta(p+2q-m-1)}\right]+n\left(1-\frac{\theta-1}{2\theta}\right) < 1-\frac{n}{2}+\frac{n}{2}(p+m-1) \tag{3.26}$$

which implies (iii) in (3.21). □

Lemma 3.3. Suppose that (1.2) and (1.3) are valid, f(u) fulfills (1.4) with $b > 0$ and $\eta > 1$, $\max\{q, g\} < \frac{1}{n} + m - 1$,

$$0 \le S(u) \le C_s(u+1)^q,\ 0 \le F(u) \le C_F(u+1)^g. \tag{3.27}$$

Then for any $p > \max\left\{\frac{n}{2}, 1+m-2q, 1+m-2g\right\}$ as well as sufficiently large, there exists a constant $C > 0$ independent of t such that the solution (u, v, w) of system (1.1) satisfies (3.1).

Proof. We test the first equation in (1.1) by $(u+1)^{p-1}$ and have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx + c_D(p-1)\int_\Omega (u+1)^{p+m-3}|\nabla u|^2 dx\\ &\le \chi(p-1)\int_\Omega S(u)(u+1)^{p-2}\nabla u\cdot\nabla v dx - \xi(p-1)\int_\Omega F(u)(u+1)^{p-2}\nabla u\cdot\nabla w dx\\ &+ a\int_\Omega (u+1)^{p-1}dx - b\int_\Omega (u+1)^{p-1}u^\eta dx\end{aligned} \tag{3.28}$$

for all $t \in (0, T_{max})$. Since $(u+1)^\eta \le 2^{\eta-1}(u^\eta+1)$ for $\eta > 1$ implies $u^\eta \ge \frac{1}{2^{\eta-1}}(u+1)^\eta - 1$, (3.28) can be further written as

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx + c_D(p-1)\int_\Omega (u+1)^{p+m-3}|\nabla u|^2 dx + \frac{b}{2^{\eta-1}}\int_\Omega (u+1)^{p+\eta-1}dx\\ &\le \chi(p-1)\int_\Omega S(u)(u+1)^{p-2}\nabla u\cdot\nabla v dx - \xi(p-1)\int_\Omega F(u)(u+1)^{p-2}\nabla u\cdot\nabla w dx\\ &+ (a+b)\int_\Omega (u+1)^{p-1}dx\end{aligned} \tag{3.29}$$

for all $t \in (0, T_{max})$. By virtue of the Young's inequality and (1.8), we obtain that

$$\begin{aligned}\chi(p-1)\int_\Omega S(u)(u+1)^{p-2}\nabla u\cdot\nabla v dx &\le \chi C_s(p-1)\int_\Omega (u+1)^{p+q-2}|\nabla u|\cdot|\nabla v| dx\\ &\le \frac{c_D(p-1)}{4}\int_\Omega (u+1)^{p+m-3}|\nabla u|^2 dx\\ &+ \frac{\chi^2 C_s^2(p-1)}{c_D}\int_\Omega (u+1)^{p+2q-m-1}|\nabla v|^2 dx.\end{aligned} \tag{3.30}$$

Similarly, we have that

$$\begin{aligned}&-\xi(p-1)\int_\Omega F(u)(u+1)^{p-2}\nabla u\cdot\nabla w dx\\ &\le \xi C_F(p-1)\int_\Omega (u+1)^{p+g-2}|\nabla u|\cdot|\nabla w| dx\\ &\le \frac{c_D(p-1)}{4}\int_\Omega (u+1)^{p+m-3}|\nabla u|^2 dx\\ &+ \frac{\xi^2 C_F^2(p-1)}{c_D}\int_\Omega (u+1)^{p+2g-m-1}|\nabla w|^2 dx.\end{aligned} \tag{3.31}$$

Since

$$(a+b)\int_\Omega (u+1)^{p-1}dx \le \frac{b}{2^\eta}\int_\Omega (u+1)^{p+\eta-1}dx + C_6,\ t\in(0,T_{max}),\ C_6>0,$$

we have the following result by (3.29), (3.30) and (3.31)

$$\frac{d}{dt}\int_\Omega (u+1)^p dx + \frac{2c_D p(p-1)}{(p+m-1)^2}\int_\Omega |\nabla (u+1)^{\frac{p+m-1}{2}}|^2 dx + \frac{bp}{2^\eta}\int_\Omega (u+1)^{p+\eta-1}dx$$
$$\le \frac{\chi^2 C_s^2 p(p-1)}{c_D}\int_\Omega (u+1)^{p+2q-m-1}|\nabla v|^2 dx + \frac{\xi^2 C_F^2 p(p-1)}{c_D}\int_\Omega (u+1)^{p+2g-m-1}|\nabla w|^2 dx + C_6.$$

By using the Holder inequality, we can find $C_7, C_8 > 0$ such that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (u+1)^p dx + \frac{2c_D p(p-1)}{(p+m-1)^2}\int_\Omega |\nabla (u+1)^{\frac{p+m-1}{2}}|^2 dx + \frac{bp}{2^\eta}\int_\Omega (u+1)^{p+\eta-1}dx\\ &\le C_7\left(\int_\Omega (u+1)^{\theta(p+2q-m-1)}dx\right)^{\frac{1}{\theta}}\left(\int_\Omega |\nabla v|^{\frac{2\theta}{\theta-1}}dx\right)^{\frac{\theta-1}{\theta}}\\ &+ C_8\left(\int_\Omega (u+1)^{\mu(p+2g-m-1)}dx\right)^{\frac{1}{\mu}}\left(\int_\Omega |\nabla w|^{\frac{2\mu}{\mu-1}}dx\right)^{\frac{\mu-1}{\mu}} + C_6 \end{aligned}\qquad (3.32)$$

with $\theta > 1$ and $\mu > 1$ for all $t\in(0,T_{max})$. More precisely, (3.21) in Lemma 3.2 enable us to apply (2.6) to derive

$$\begin{aligned}&\left(\int_\Omega (u+1)^{\theta(p+2q-m-1)}dx\right)^{\frac{1}{\theta}} = \|(u+1)^{\frac{p+m-1}{2}}\|_{L^{\frac{2\theta(p+2q-m-1)}{p+m-1}}(\Omega)}^{\frac{2(p+2q-m-1)}{p+m-1}}\\ &\le C_9\left(\|\nabla (u+1)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\alpha_1}\|(u+1)^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{1-\alpha_1} + \|(u+1)^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}\right)^{\frac{2(p+2q-m-1)}{p+m-1}}\\ &\le C_{10}\left\{\int_\Omega |\nabla (u+1)^{\frac{p+m-1}{2}}|^2 + 1\right\}^{\beta_1}\end{aligned}\qquad (3.33)$$

with α_1, β_1 defined in Lemma 3.2 and positive constants C_9, C_{10} for all $t\in(0,T_{max})$.

Since $\beta>0$, v solves

$$\begin{cases} -\Delta v + \beta v = \alpha u,\ x\in\Omega,\\ \frac{\partial v}{\partial \nu} = 0,\ x\in\partial\Omega,\end{cases}$$

we can apply the L^p estimates to deduce that

$$\|v(\cdot,t)\|_{W^{2,p}(\Omega)} \le C_{11}\|u(\cdot,t)\|_{L^p(\Omega)}\ \textit{for all}\ t\in(0,T_{max}). \qquad (3.34)$$

By (3.34), we obtain that

$$\left(\int_\Omega |\nabla v|^{\frac{2\theta}{\theta-1}}dx\right)^{\frac{\theta-1}{\theta}} = \|\nabla v\|^2_{L^{\frac{2\theta}{\theta-1}}(\Omega)} \le C\|u+1\|^2_{L^{\frac{2\theta}{\theta-1}}(\Omega)}. \qquad (3.35)$$

Moreover,

$$\begin{aligned}
&\|u+1\|^2_{L^{\frac{2\theta}{\theta-1}}(\Omega)}\\
&=\|(u+1)^{\frac{p+m-1}{2}}\|^{\frac{4}{p+m-1}}_{L^{\frac{4\theta}{(p+m-1)(\theta-1)}}(\Omega)}\\
&\le C_{12}\left(\|\nabla(u+1)^{\frac{p+m-1}{2}}\|^{\alpha_2}_{L^2(\Omega)}\|(u+1)^{\frac{p+m-1}{2}}\|^{1-\alpha_2}_{L^{\frac{2}{p+m-1}}(\Omega)}+\|(u+1)^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}\right)^{\frac{4}{p+m-1}}\\
&\le C_{13}\left\{\int_\Omega|\nabla(u+1)^{\frac{p+m-1}{2}}|^2+1\right\}^{\beta_2}
\end{aligned}$$

with $\alpha_2,\ \beta_2$ defined in Lemma 3.2 and positive constants C_{12}, C_{13} for all $t\in(0,T_{max})$, which along with (3.35) gives

$$\left(\int_\Omega|\nabla v|^{\frac{2\theta}{\theta-1}}dx\right)^{\frac{\theta-1}{\theta}}\le C_{13}\left\{\int_\Omega|\nabla(u+1)^{\frac{p+m-1}{2}}|^2dx+1\right\}^{\beta_2}. \tag{3.36}$$

We substitute (3.33) and (3.36) into (3.32),

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega(u+1)^pdx+\frac{2c_Dp(p-1)}{(p+m-1)^2}\int_\Omega|\nabla(u+1)^{\frac{p+m-1}{2}}|^2dx+\frac{bp}{2^{\eta-1}}\int_\Omega(u+1)^{p+\eta-1}dx\\
&\le C_{14}\left\{\int_\Omega|\nabla(u+1)^{\frac{p+m-1}{2}}|^2+1\right\}^{\beta_1+\beta_2}\\
&+C_8\left(\int_\Omega(u+1)^{\mu(p+2g-m-1)}dx\right)^{\frac{1}{\mu}}\left(\int_\Omega|\nabla w|^{\frac{2\mu}{\mu-1}}dx\right)^{\frac{\mu-1}{\mu}}+C_6,
\end{aligned}\tag{3.37}$$

similarly, if $g<\frac{1}{n}+m-1$, by using the same argument as in Lemma 3.2, we can also find $\beta_3<1,\beta_4<1$ and $\beta_3+\beta_4<1$, where

$$\beta_3=\frac{\frac{n(p+2g-m-1)}{2}\left[1-\frac{1}{\mu(p+2g-m-1)}\right]}{1-\frac{n}{2}+\frac{n(p+m-1)}{2}},\ \ \beta_4=\frac{n\left(1-\frac{\mu-1}{2\mu}\right)}{1-\frac{n}{2}+\frac{n(p+m-1)}{2}},$$

then by Young's inequality, (3.37) can be written as

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega(u+1)^pdx+\frac{2c_Dp(p-1)}{(p+m-1)^2}\int_\Omega|\nabla(u+1)^{\frac{p+m-1}{2}}|^2dx+\frac{bp}{2^{\eta-1}}\int_\Omega(u+1)^{p+\eta-1}dx\\
&\le C_{15}\left\{\int_\Omega|\nabla(u+1)^{\frac{p+m-1}{2}}|^2+1\right\}^{\beta_1+\beta_2}+C_{16}\left\{\int_\Omega|\nabla(u+1)^{\frac{p+m-1}{2}}|^2+1\right\}^{\beta_3+\beta_4}+C_{17}.
\end{aligned}\tag{3.38}$$

Due to $\beta_1+\beta_2<1$, $\beta_3+\beta_4<1$, applying the Young's inequality to (3.38) and by an ODE comparison argument, we obtain that

$$\int_\Omega(u+1)^pdx\le C,\qquad t\in(0,T_{max}).$$

□

Proof of Theorems 1.1 and 1.2. By recalling the L^p estimates of w in (3.13), we can use Lemma 3.1 and Lemma 3.3 to find a positive constant C_1 such that

$$\sup_{0<t<T_{max}}\|w(\cdot,t)\|_{W^{2,p}(\Omega)}\le C_1.$$

Then, by selecting a sufficiently large p from the Sobolev embedding theorem, we can find a positive constant C_2 such that

$$\sup_{0<t<T_{max}} \|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \le C_2. \tag{3.39}$$

Similarly, from (3.34), there exists a positive constant C_3 such

$$\sup_{0<t<T_{max}} \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le C_3. \tag{3.40}$$

By using Lemma A.1 in [20], we can conclude that u is uniformly bounded in $\Omega \times (0, T_{max})$. Thus, we can find a positive constant C_4 such that

$$\|u(\cdot,t)\|_{L^\infty} \le C_4 \ for\ all\ t \in (0, T_{max}), \tag{3.41}$$

which together with Lemma 2.1, we obtain (u, v, w) is a global bounded classical solution to system (1.1). □

4. Blow-up of solutions

Denote $B := \{x \in R^2 |\ |x| < R\}$ and $B_i := \{x \in R^2 |\ |x| < r_i\}$ with $R_i, r_i > 0, i = 1, 2, 3, 4$. The next two lemmas are the well-known conclusions on elliptic equations and the Green function (refer to [16]).

Lemma 4.1. Let u solves

$$\begin{cases} -\Delta u = f,\ x \in B, \\ u = 0,\ x \in \partial B, \end{cases}$$

with $f \in L^p(B), 1 \le p \le \infty$. Then

$$u(x) = \int_B G(x,y) f(y) dy \ for\ x \in B,$$

where $G(x, y)$ is the Green function with the following properties:

(i) $G(x,y) = N(x-y) + K(x,y), where\ N(x-y) = -\frac{1}{2\pi}\log|x-y|\ and\ K \ \in C^2(B\times B),$

(ii) $G(x,y) = G(y,x)\ for\ x,y \in \overline{B},$

(iii) $|\nabla_x G(x,y)| \le C/|x-y|\ in\ B\times B\ for\ some\ C > 0.$

Lemma 4.2. Let $u \in C^2(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u + \zeta u = f,\ x \in B, \\ \frac{\partial u}{\partial \nu} = 0,\ x \in \partial B, \end{cases}$$

with $f \in C^0(\overline{\Omega})$, $\zeta > 0$. Then there exist $C_p, C_q > 0$ such that

$$\|u\|_{L^p(\Omega)} \le C_p \|f\|_{L^1(\Omega)},\ 1 \le p < \infty,$$
$$\|\nabla u\|_{L^q(\Omega)} \le C_q \|f\|_{L^1(\Omega)},\ 1 \le q < 2.$$

We construct $\phi \in C^1([0,\infty)) \cap W^{2,\infty}((0,\infty))$ with $r_2 > r_1 > 0$ by

$$\phi(r)=\begin{cases} r^2, & if\ 0\le r\le r_1,\\ a_1r^2+a_2r+a_3, & if\ r_1\le r\le r_2,\\ r_1r_2, & if\ r\ge r_2\end{cases} \tag{4.1}$$

with $a_1=-\frac{r_1}{r_2-r_1}$, $a_2=\frac{2r_1r_2}{r_2-r_1}$ and $a_3=-\frac{r_1^2r_2}{r_2-r_1}$.

$\Phi\in[0,\infty)$ is defined as $\Phi(x)=\phi(|x|)\in C^1(R^2)\cap W^{2,\infty}(R^2)$, which will play a key role in the proof for the finite time blow-up of nonradial solutions.

Lemma 4.3. The function $\Phi(x)$ satisfies

$$\nabla\Phi(x)=\begin{cases} 2x, & if\ |x|\le r_1,\\ \frac{2r_1}{r_2-r_1}(r_2-|x|)\frac{x}{|x|}, & if\ r_1\le |x|\le r_2,\\ 0, & if\ |x|\ge r_2\end{cases}$$

and

$$|\nabla\Phi(x)|\le 2(\Phi(x))^{\frac{1}{2}}, \tag{4.2}$$
$$\Delta\Phi(x)=4\ for\ |x|\le r_1, \tag{4.3}$$
$$\Delta\Phi(x)\le 2\ for\ |x|>r_1. \tag{4.4}$$

Proof. We can see [[16], lemma 2.1] for details.

Moreover,

$$\nabla\Phi(x)-\nabla\Phi(y)\cdot\nabla N(x-y)=-\frac{1}{\pi},\ (x,y)\in B_1\times B_1, \tag{4.5}$$
$$|\nabla\Phi(x)-\nabla\Phi(y)|\cdot\nabla N(x-y)\le\frac{2r_2+r_1}{\pi(r_2-r_1)},\ (x,y)\in R^2\times R^2, \tag{4.6}$$

Let (u,v,w) be the solution of (1.1) ensured by Lemma 2.1. We should show $T_{max}<\infty$. It suffices to find a $T>0$ such that the Φ-weighted integral of $u(x,t)$ tends to zero as $t\to T$. Inspired by [16, 24] this will be realized via the following Lemma.

Lemma 4.4. Let

$$D(u)=1, S(u)=F(u)=u,\ f(u)\le a-bu^\eta\ for\ \eta>1 \tag{4.7}$$

for $n=2$ in (1.1) , $x_0\in\Omega$ and $0<r_1<r_2<dist(x_0,\partial\Omega)$, where $\mathrm{dist}(x_0,\partial\Omega)$ denotes the distance between x_0 and $\partial\Omega$. Then there exist $C_1,C_2>0$ only depending on r_1, r_2 and $\mathrm{dist}(x_0,\partial\Omega)$ such that for $t\in(0,T_{max})$,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u(x,t)\Phi(x-x_0)dx,\\ &\le 4\int_\Omega u_0(x)dx-\frac{\chi\alpha-\xi\gamma}{2\pi}\left(\int_\Omega u_0(x)dx\right)^2+C_1\left(\int_\Omega u_0(x)dx\right)\left(\int_\Omega u(x,t)\Phi(x-x_0)dx\right)\\ &+C_2\left(\int_\Omega u_0(x)dx\right)^{\frac{3}{2}}\left(\int_\Omega u(x,t)\Phi(x-x_0)dx\right)^{\frac{1}{2}},\end{aligned} \tag{4.8}$$

where $\Phi(x)=\phi(|x|)$ with ϕ defined by (4.1).

Proof. Without loss of generality, assume that x_0 is the origin. Multiply the first equation of (1.1) by $\Phi(x)$ and integrate over Ω. Due to the Neumann boundary condition of (1.1), $\frac{\partial\Phi}{\partial\nu} = 0$ with $r_2 < dist(x_0, \partial\Omega)$ by Lemma 4.3, $\int_\Omega u_0(x)dx = \int_\Omega u(x,t)dx$ and the estimates (4.3) and (4.4), we obtain that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u(x,t)\Phi(x)dx &= \int_\Omega \Phi(x)(\Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) + f(u))dx,\\ &\le 4\int_\Omega u_0(x)dx + \chi\int_\Omega u(x,t)\nabla\Phi(x)\cdot\nabla v(x,t)dx\\ &- \xi\int_\Omega u(x,t)\nabla\Phi(x)\cdot\nabla w(x,t)dx + \int_\Omega \Phi(x)f(u)dx. \end{aligned} \tag{4.9}$$

By the procedure in the proof of Lemma3.1 [16] and Proposition 3.1 [24], we have that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u(x,t)\Phi(x)dx\\ &\le 4\int_\Omega u_0(x)dx - \frac{\chi\alpha}{2\pi} + \frac{\xi\gamma}{2\pi}\left(\int_\Omega u_0(x)dx\right)^2 + C_3\left(\int_\Omega u_0(x)dx\right)\left(\int_\Omega u(x,t)\Phi(x)dx\right)\\ &+ C_2\left(\int_\Omega u_0(x)dx\right)^{\frac{3}{2}}\left(\int_\Omega u(x,t)\Phi(x)dx\right)^{\frac{1}{2}} + J_3 \end{aligned} \tag{4.10}$$

From (4.7) and Jensen's inequality , we obtain that

$$\begin{aligned} J_3 := \int_\Omega \Phi(x)f(u)dx &\le a\int_\Omega \Phi(x)dx - b\int_\Omega \Phi(x)u^\eta dx\\ &\le a\int_\Omega \Phi(x)dx - b\left(\int_\Omega \Phi(x)udx\right)^\eta dx \end{aligned} \tag{4.11}$$

Since $\eta > 1$, by using Young's inequality, we obtain that

$$-b\left(\int_\Omega \Phi(x)udx\right)^\eta dx \le C_4 - b\eta\int_\Omega \Phi(x)udx, \tag{4.12}$$

by the definition of $\Phi(x)$, we obtain $a\int_\Omega \Phi(x)dx$ is bounded. Thus, by (4.11) (4.12), we conclude that

$$J_3 \le C_5\int_\Omega u_0(x)dx\int_\Omega \Phi(x)udx. \tag{4.13}$$

Combining (4.9)-(4.13) yields (4.8). □

Proof of Theorem 1.3. Denote

$$M_\Phi(t) := \int_\Omega u(x,t)\Phi(x - x_0)dx \tag{4.14}$$

and

$$E(s) = 4\int_\Omega u_0(x)dx - \frac{\chi\alpha - \xi\gamma}{2\pi}\left(\int_\Omega u_0(x)dx\right)^2 + C_1\left(\int_\Omega u_0(x)dx\right)s + C_2\left(\int_\Omega u_0(x)dx\right)^{\frac{3}{2}} s^{\frac{1}{2}}.$$

By Lemma 4.4, we obtain that

$$\frac{d}{dt}M_\Phi(t) \le E(M_\Phi(t)),\ t\in(0,T_{max}). \tag{4.15}$$

By the definition of $\Phi(x)$ in (4.1) and (4.14), we have that

$$M_\Phi(0) = \int_\Omega u_0 \Phi(x - x_0)dx \le \int_\Omega u_0 |x - x_0|^2 dx.$$

Together with the condition (1.7), it is easy to check that for $s > 0$, $E(0) < 0$ and $E^{'}(s) > 0$. If $\int_\Omega u_0(x)|x - x_0|$ is small enough, then

$$E(M_\Phi(0)) < 0. \tag{4.16}$$

If the solution (u, v, w) exists for all $t > 0$, then $E\left(M_\Phi(s)\right)$ is bounded.
From (4.15), it is obtained that

$$\begin{aligned} M_\Phi(t) &< M_\Phi(0) + \int_0^t E(M_\Phi(s))ds \\ &< M_\Phi(0) + E(M_\Phi(s'))t \end{aligned}$$

for $s' \in (0, t)$. This concludes that there exists $T \in (0, \infty)$ such that $M_\Phi(0)+E(M_\Phi(s'))t \le 0$. This is a contradiction to the nonnegativity of $M_\Phi(t)$. The proof is complete. □

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