Solving Delay Differential Equations Using Reformulated Block Backward Differentiation Formulae Methods

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ABSTRACT

In this paper, the conventional backward differentiation formulae methods for step numbers $k = 3$ and 4 were reformulated by shifting them one-step backward to produce two and three approximate solutions respectively, in a step when implemented in block form. The derivation of the continuous formulations of the reformulated methods were carried out through multistep collocation method by matrix inversion technique. The discrete schemes were deduced from their respective continuous formulations. The convergence analysis of the discrete schemes were discussed. The stability analysis of these schemes were ascertained and the P- and Q-stability were also investigated. When the discrete schemes were implemented in block form to solve some first order delay differential equations together with an accurate and efficient formula for the solution of the delay argument, it was observed that the results obtained from the schemes for step number $k = 4$ performed slightly better than the schemes for step number $k = 3$ when compared with the exact solutions. More so, on comparing these methods with some existing ones, it was observed that the methods derived performed better in terms of accuracy

Keywords: *Delay Differential Equations, Reformulated Block Method, Backward Differentiation Formulae, Continuous Formulations.*

1. INTRODUCTION

Many real life problems encountered in the various branches of science, medicine and engineering give rise to ordinary differential equations (ODEs) of the form,

$$y'(t) = f(\mathrm{t}, \mathrm{y}), \qquad y(a) = y_0, \; a \le t \le b \tag{1}$$

This has been used to model the above physical phenomena since the concept of differentiation was first developed and nowadays complicated ODE models can be solved numerically with a high degree of confidence. However it was observed that some phenomena may have a delayed effect and the models described by (1) would be more realistic if some of the past history of the system is included in them, leading to what is called delay differential equations (DDEs) of the form:

$$\begin{aligned} y'(t) &= f(\mathrm{t}, \mathrm{y}, y(t-\tau)), && t > t_0, \tau > 0 \\ y(t) &= \varphi(t) && t \le t_0 \end{aligned} \tag{2}$$

where $\varphi(t)$ is the initial function, $\tau(t, y(t))$ is the delay or constant lag, $t-\tau(t, y(t))$ is the delay term and $y(t-\tau(t, y(t)))$ is the solution of the delay term.

Delay differential equations are similar to ordinary differential equations, but their evolution involves past values of the state variable. The solution of DDEs requires the knowledge of not only the current state, but also of the state at a certain time previously. An obvious distinction between a DDE and an ODE is that specifying the initial value $y(t_0)$ is not enough to determine the solution for $t \geq t_0$ it is necessary to specify the history $y(t)$ for $-\tau < t \leq t_0$ in the differential equation even to be defined for $t_0 \leq t < \tau$. Most of the numerical methods that have been developed to solve ODEs namely, the Runge-Kutta type of methods and multistep methods have also been used to solve DDEs together with their interpolation techniques by some researchers such as in [1-4]. All of these methods produce only one approximate solution in an integration step. Another approach that has gained interest recently is block methods. Block methods produce more than one approximate solution in a step [5-6]. Also using block methods greater efficiency is obtained since total number of steps taken will be reduced.

In this research, the reformulated block backward differentiation formulae (BDF), presented as a simple form of linear multistep methods would be used to solve DDEs. The block methods will be implemented using fixed step size and the delay term will be approximated without using the well-known interpolation techniques such as Hermite, Nordsieck, Newton divided difference, Neville's interpolation etc. According to [7], the order of interpolating polynomials used should be at least the same as that of the numerical method to preserve the desired accuracy. In order to circumvent this drawback, an accurate and efficient formula shall be proposed for approximating the delay term.

1.1 Existence and Uniqueness of Solutions

We shall state the theorem for existence and uniqueness solutions of (2) as in [8]

Theorem:-

Consider (2) and assume that the function $f(t,u,v)$ satisfies the condition $\|f(t,u,v)\| \leq M(t)+N(t)(\|u\|+\|v\|)$ in $[t_0,t_n)\times \mathbb{R}^d \times \mathbb{R}^d$,where $M(t)$ and $N(t)$ are continuous positive functions on $[t_0,t_n)$, then the solution of (2) exist and is unique on the entire interval $[t_0,t_n)$.

Consider the sequence of points $\{t_n\}$ defined by $t_n = t_0 + nh,\ n = 1, 2,..$ where the parameter, h is called the step size, a vital property of the most numerical methods for the solution of (2) is that of discretization i.e. an approximate solution is sought not on the continuous interval $t_0 \leq t \leq t_n$ but on the discrete point set $\left\{t_n \mid n = 1, 2, \ldots, \frac{t_n - t_0}{h}\right\}$

2. THE REFORMULATED METHOD

In this section, the continuous formulations of the reformulated BDF methods for step numbers $k = 3$ and 4 will be derived using multistep collocation method of [9]

2.1 The Multistep Collocation Method

In [8], a *k*-step multistep collocation method with *m* collocation points was obtained as

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \beta_j(x) f(x_j, y(x_j)) \tag{3}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients of the method defined as

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^i \quad \text{for} \quad j = 0,1,\ldots,t-1 \tag{4}$$

$$h\beta_j(x) = \sum_{i=0}^{t+m-1} h\beta_{j,i+1} x^i \quad \text{for} \quad j = 0,1,\ldots,m-1 \tag{5}$$

where $x_0, \ldots, x_{m-1}$ are the m collocation points and x_{n+j} , j = 0, 1, 2, …, t – 1 are the t arbitrarily chosen interpolation points.

To get $\alpha_j(x)$ and $\beta_j(x)$, [9] arrived at a matrix equation of the form

$$DC = I \tag{6}$$

where I is the identity matrix of dimension $(t+m)\times(t+m)$ while D and C are matrices defined as

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_0 & \cdots & (t+m-1)x_0^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{m-1} & \cdots & (t+m-1)x_{m-1}^{t+m-2} \end{bmatrix} \tag{7}$$

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & \cdots & h\beta_{m-1,t+m} \end{bmatrix} \tag{8}$$

It follows from (6), that the columns of $C = D^{-1}$ give the continuous coefficients of the continuous scheme (3).

2.2 Derivation of Continuous Formulation of Reformulated Block Backward Differentiation Formulae Method for *k* = 3

Using the idea of [9], we choose $t = 3$ interpolation points at x_{n+j}, $j = -1, 0, 1$ and $m = 1$ collocation point at x_{n+2}. Then (3) takes the form

$$y(x) = \alpha_{-1}(x)y_{n-1} + \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + h\beta_2(x)f_{n+2} \tag{9}$$

and the matrix D in (7) becomes

$$D = \begin{bmatrix} 1 & x_n - h & (x_n - h)^2 & (x_n - h)^3 \\ 1 & x_n & x_n^2 & x_n^3 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 \\ 0 & 1 & 2x_n + 4h & 3(x_n + 2h)^2 \end{bmatrix} \quad (10)$$

The columns of the $C = D^{-1}$ obtained using Maple 18 are used in (4) and (5) to yield the continuous coefficients of the method. Substituting these coefficients into (9) gives

$$y(x) = \frac{(x - x_n)^3}{h^3}\left(-3y_{n-1} + 8y_n - 5y_{n+1}\right) - \frac{4(x - x_n)}{11h}\left(y_n + y_{n-1} - 2y_{n+1}\right)$$

$$+ \frac{(x - x_n)^2}{2h^2}\left(y_{n-1} - 2y_n + y_{n+1} + \frac{2}{11}(x - x_n)f_{n+2}\right) + \left(y_n - \frac{1}{11}(x - x_n)f_{n+2}\right)$$

(11)

Next evaluating (11) at $x = x_{n+2}$ and its derivative at $x = x_{n+1}$, the reformulated block BDF for $k = 3$ is obtained as

$$y_{n+1} = -\frac{5}{23}y_{n-1} + \frac{28}{23}y_n + \frac{22}{23}hf_{n+1} - \frac{4}{23}hf_{n+2}$$

$$y_{n+2} = \frac{2}{11}y_{n-1} - \frac{9}{11}y_n + \frac{18}{11}y_{n+1} + \frac{6}{11}hf_{n+2}$$

(12)

2.3 Derivation of Continuous Formulation of Reformulated Block Backward Differentiation Formulae Method for *k* = 4

With the interpolation points at x_{n+j}, $j = -1, 0, 1, 2$ and the collocation point at x_{n+3}, (3) and (6) become respectively

$$y(x) = \alpha_{-1}(x)y_{n-1} + \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + h\beta_3(x)f_{n+3} \quad (13)$$

and

$$D = \begin{bmatrix} 1 & x_n - h & (x_n - h)^2 & (x_n - h)^3 & (x_n - h)^4 \\ 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 \\ 1 & x_n + 2h & (x_n + 2h)^2 & (x_n + 2h)^3 & (x_n + 2h)^4 \\ 0 & 1 & 2x_n + 6h & 3(x_n + 3h)^2 & 4(x_n + 3h)^3 \end{bmatrix} \quad (14)$$

Similarly, the continuous formulation (13) becomes

$$y(x) = \frac{7(x - x_n)^4}{50h^4}\left(\frac{11}{42}y_{n-1} - y_n + \frac{19}{14}y_{n+1} - \frac{13}{21}y_{n+2}\right) + \left(y_n + \frac{(x - x_n)}{25}f_{n+3}\right)$$
$$- \frac{(x - x_n)}{50h}\left(39y_n + 13y_{n-1} - 69y_{n+1} + 17y_{n+2} + (x - x_n)f_{n+3}\right)$$
$$- \frac{(x - x_n)^2}{25h^2}\left(\frac{43}{2}y_n - \frac{139}{12}y_{n-1} - \frac{31}{4}y_{n+1} - \frac{13}{6}y_{n+2} + (x - x_n)f_{n+3}\right) \quad (15)$$

and evaluating (15) at $x = x_{n+3}$, and its derivative at $x = x_{n+1}, x_{n+2}$, the reformulated block BDF for $k = 4$ is obtained as

$$y_{n+1} = -\frac{7}{9}y_{n-1} + 6y_n - \frac{38}{9}y_{n+2} + \frac{25}{3}hf_{n+1} + \frac{1}{3}hf_{n+3}$$
$$y_{n+2} = \frac{17}{197}y_{n-1} - \frac{99}{197}y_n + \frac{279}{197}y_{n+1} + \frac{150}{197}hf_{n+2} - \frac{18}{197}hf_{n+3}$$
$$y_{n+3} = -\frac{3}{25}y_{n-1} + \frac{16}{25}y_n - \frac{36}{25}y_{n+1} + \frac{48}{25}y_{n+2} + \frac{12}{25}hf_{n+3} \quad (16)$$

3. CONVERGENCE ANALYSIS

In this section, the order, error constants, consistency and zero stability of the derived discrete schemes shall be examined.

3.1 Order and Error constants

The order and error constants of the discrete schemes in (12) are found in block form as follows:

$$C_0 = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_1 = -\alpha_{-1} + \alpha_1 + 2\alpha_2 - \beta_{-1} - \beta_0 - \beta_1 - \beta_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_2 = -\frac{1}{2}\alpha_{-1} + \frac{1}{2}\alpha_1 + 2\alpha_2 + \beta_{-1} - \beta_1 - 2\beta_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_3 = -\frac{1}{6}\alpha_{-1} + \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 - \frac{1}{2}\beta_{-1} - \frac{1}{2}\beta_1 - 2\beta_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_4 = -\frac{1}{24}\alpha_{-1} + \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{1}{6}\beta_{-1} - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 = \begin{bmatrix} \frac{17}{138} \\ \frac{-3}{22} \end{bmatrix}$$

Therefore, (12) has order, $p = 3$ and error constants $= \frac{17}{138}, -\frac{3}{22}$

Similarly, the order and error constants of the discrete schemes in (16) are found in block form as follows:

$$C_0 = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = -\alpha_{-1} + \alpha_1 + 2\alpha_2 + 3\alpha_3 - \beta_{-1} - \beta_0 - \beta_1 - \beta_2 - \beta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_2 = -\frac{1}{2}\alpha_{-1} + \frac{1}{2}\alpha_1 + 2\alpha_2 + \frac{9}{2}\alpha_3 + \beta_{-1} - \beta_1 - 2\beta_2 - 3\beta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_3 = -\frac{1}{6}\alpha_{-1} + \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 + \frac{9}{2}\alpha_3 - \frac{1}{2}\beta_{-1} - \frac{1}{2}\beta_1 - 2\beta_2 - \frac{9}{2}\beta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_4 = -\frac{1}{24}\alpha_{-1} + \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{27}{8}\alpha_3 - \frac{1}{6}\beta_{-1} - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 - \frac{9}{2}\beta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_5 = -\frac{1}{120}(\alpha_{-1} - \alpha_1) + \frac{4}{15}\alpha_2 + \frac{8}{40}\alpha_3 - \frac{1}{24}(\beta_{-1} + \beta_1) - \frac{2}{3}\beta_2 - \frac{27}{8}\beta_3 = \begin{bmatrix} \frac{-31}{90} \\ \frac{111}{1970} \\ \frac{-12}{125} \end{bmatrix}$$

Therefore, (16) has order, $p = 4$ and error constants $= -\frac{31}{90}, \frac{111}{1970}, -\frac{12}{125}$

3.2 Consistency

All the schemes in (12) and (16) have their orders greater than one, so as in [10], the schemes are consistent.

3.3 Zero Stability

The zero stability of the discrete schemes in (12) is determined in a block form as follows

$$\begin{pmatrix} 1 & 0 \\ -\frac{18}{11} & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} & \frac{28}{25} \\ \frac{2}{11} & -\frac{9}{11} \end{pmatrix}\begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix} + h\begin{pmatrix} \frac{22}{23} & -\frac{4}{23} \\ 0 & \frac{6}{11} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} + h\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}$$

where $A_2^{(1)} = \begin{pmatrix} 1 & 0 \\ -\frac{18}{11} & 1 \end{pmatrix}, A_1^{(1)} = \begin{pmatrix} -\frac{5}{3} & \frac{28}{25} \\ \frac{2}{11} & -\frac{9}{11} \end{pmatrix}$ and $B_2^{(1)} = \begin{pmatrix} \frac{22}{23} & -\frac{4}{23} \\ 0 & \frac{6}{11} \end{pmatrix}$

The first characteristics polynomial of the block method of the discrete schemes in (12) is given by

$$p(\xi)=\det(\xi A_2^{(1)}-A_1^{(1)})=0$$

$$=\left|\xi A_2^{(1)}-A_1^{(1)}\right|$$

$$=0$$

Now we have,

$$\rho(\xi)=\left|\xi\begin{pmatrix}1 & 0\\ -\frac{18}{11} & 1\end{pmatrix}-\begin{pmatrix}-\frac{5}{3} & \frac{28}{25}\\ \frac{2}{11} & -\frac{9}{11}\end{pmatrix}\right|=\left|\begin{pmatrix}\xi & 0\\ -\frac{18}{11}\xi & \xi\end{pmatrix}-\begin{pmatrix}-\frac{5}{3} & \frac{28}{25}\\ \frac{2}{11} & -\frac{9}{11}\end{pmatrix}\right|$$

$$\rho(\xi)=\begin{vmatrix}\xi+\frac{5}{3} & -\frac{28}{25}\\ -\frac{18}{11}\xi-\frac{2}{11} & \xi+\frac{9}{11}\end{vmatrix}=\xi^2+\frac{538}{825}\xi+\frac{29}{25}=0\ ,$$

$$\Rightarrow \xi_1=-\frac{269}{825}+\frac{14}{825}i\sqrt{3659} \text{ and } \xi_2=-\frac{269}{825}-\frac{14}{825}i\sqrt{3659}$$

$$\Rightarrow |\xi_1|=1 \text{ and } |\xi_2|=1, \text{ but } \xi_1\neq\xi_2$$

then we observe that the discrete schemes in (12) satisfies the root condition and hence zero stable as in [10]

Similarly, the zero stability of the discrete schemes in (16) is determined in block form as follows

$$\begin{pmatrix}1 & \frac{38}{9} & 0\\ -\frac{297}{197} & 1 & 0\\ \frac{36}{25} & -\frac{48}{25} & 1\end{pmatrix}\begin{pmatrix}y_{n+1}\\ y_{n+2}\\ y_{n+3}\end{pmatrix}=\begin{pmatrix}0 & -\frac{7}{9} & 6\\ 0 & \frac{17}{97} & \frac{99}{197}\\ 0 & -\frac{3}{25} & \frac{16}{25}\end{pmatrix}\begin{pmatrix}y_{n-2}\\ y_{n-1}\\ y_n\end{pmatrix}$$

$$+h\begin{pmatrix}\frac{25}{3} & 0 & \frac{1}{3}\\ 0 & \frac{150}{197} & -\frac{18}{197}\\ 0 & 0 & \frac{12}{25}\end{pmatrix}\begin{pmatrix}f_{n+1}\\ f_{n+2}\\ f_{n+3}\end{pmatrix}+h\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}f_{n-2}\\ f_{n-1}\\ f_n\end{pmatrix}$$

where $A_2^{(2)}=\begin{pmatrix}1 & \frac{38}{9} & 0\\ -\frac{297}{197} & 1 & 0\\ \frac{36}{25} & -\frac{48}{25} & 1\end{pmatrix}, A_1^{(2)}=\begin{pmatrix}0 & -\frac{7}{9} & 6\\ 0 & \frac{17}{97} & \frac{99}{197}\\ 0 & -\frac{3}{25} & \frac{16}{25}\end{pmatrix}$

and $B_2^{(2)}=\begin{pmatrix}\frac{25}{3} & 0 & \frac{1}{3}\\ 0 & \frac{150}{197} & -\frac{18}{197}\\ 0 & 0 & \frac{12}{25}\end{pmatrix}$

The first characteristics polynomial of the block method of the discrete schemes in (16) is given by

$$p(\xi)=\det(\xi A_2^{(2)}-A_1^{(2)})=0$$

$$= \left| \xi A_2^{(2)} - A_1^{(2)} \right|$$

$$= 0$$

Now we have,

$$\rho(\xi) = \left| \xi \begin{pmatrix} 1 & \frac{38}{9} & 0 \\ -\frac{297}{197} & 1 & 0 \\ \frac{36}{25} & -\frac{48}{25} & 1 \end{pmatrix} - \begin{pmatrix} 0 & -\frac{7}{9} & 6 \\ 0 & \frac{17}{97} & \frac{99}{197} \\ 0 & -\frac{3}{25} & \frac{16}{25} \end{pmatrix} \right| = \left| \begin{pmatrix} \xi & \frac{38}{9}\xi & 0 \\ -\frac{297}{197}\xi & \xi & 0 \\ \frac{36}{25}\xi & -\frac{48}{25}\xi & \xi \end{pmatrix} - \begin{pmatrix} 0 & -\frac{7}{9} & 6 \\ 0 & \frac{17}{97} & \frac{99}{197} \\ 0 & -\frac{3}{25} & \frac{16}{25} \end{pmatrix} \right|$$

$$\rho(\xi) = \begin{vmatrix} \xi & \frac{38}{9}\xi + \frac{7}{9} & -\frac{1}{3} \\ -\frac{297}{197}\xi & \xi - \frac{17}{197} & -\frac{99}{197} \\ \frac{36}{25}\xi & -\frac{48}{25} + \frac{3}{25}\xi & \xi - \frac{16}{25} \end{vmatrix} = \frac{1451}{197}\xi^3 + \frac{1642}{197}\xi^2 + \frac{29}{197}\xi = 0$$

$$\Rightarrow \xi_1 = \frac{821}{1451} - \frac{14}{1451}\sqrt{7802}\ , \xi_2 = \frac{821}{1451} + \frac{14}{1451}\sqrt{7802} \text{ and } \xi_3 = 0$$

$\Rightarrow |\xi_1| < 1, |\xi_2|$ □ 1 and $|\xi_3| < 1$. Since $|\xi_i| \le 1$, i = 1, 2, 3, then we observe that the discrete schemes in (16) satisfies the root condition and hence zero stable as in [10]

3.4 Convergence

The block discrete schemes methods in (12) and (16) are convergent as in [10], since they are both consistent and zero-stable.

4. STABILITY ANALYSIS

In this section, the stability analysis of derived methods as it regards to P- and Q-stability will be investigated by means of the following test equation.

$$\begin{aligned} y'(t) &= \lambda y(\mathrm{t}) + \mu y(t-\tau), \qquad & t > t_0 \\ y(t) &= \varphi(t), & t \le t_0 \end{aligned} \tag{17}$$

where $\varphi(t)$ is the initial function λ, μ are complex coefficients and h is the step size.

Then from the discrete schemes in (12),

let $Y_{N+2} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix}, Y_N = \begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix}, F_{N+2} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}$ and $F_N = \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}$

Since, $A_2^{(1)} = \begin{pmatrix} 1 & 0 \\ -\frac{18}{11} & 1 \end{pmatrix}, A_1^{(1)} = \begin{pmatrix} -\frac{5}{3} & \frac{28}{25} \\ \frac{2}{11} & -\frac{9}{11} \end{pmatrix}$ and $B_2^{(1)} = \begin{pmatrix} \frac{22}{23} & -\frac{4}{23} \\ 0 & \frac{6}{11} \end{pmatrix}$

we have, $A_2^{(1)}Y_{N+2} = A_1^{(1)}Y_{N+1} + h\sum_{i=1}^{2} B_i^{(1)}F_{N+i}$ (18)

Also from the discrete schemes in (16),

let $Y_{N+3} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix}, Y_N = \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}, F_{N+3} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix}$ and $F_N = \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$

Since, $A_2^{(2)} = \begin{pmatrix} 1 & \frac{38}{9} & 0 \\ -\frac{297}{197} & 1 & 0 \\ \frac{36}{25} & -\frac{48}{25} & 1 \end{pmatrix}, A_1^{(2)} = \begin{pmatrix} 0 & -\frac{7}{9} & 6 \\ 0 & \frac{17}{97} & \frac{99}{197} \\ 0 & -\frac{3}{25} & \frac{16}{25} \end{pmatrix}$ and $B_2^{(2)} = \begin{pmatrix} \frac{25}{3} & 0 & \frac{1}{3} \\ 0 & \frac{150}{197} & -\frac{18}{197} \\ 0 & 0 & \frac{12}{25} \end{pmatrix}$

we have, $A_2^{(2)}Y_{N+2} = A_1^{(2)}Y_{N+1} + h\sum_{i=1}^{2} B_i^{(2)}F_{N+i}$ (19)

According to [7], the P- and Q-stability polynomials are obtained by applying (18) and (19) to (17). Thus the P-stability polynomials for the discrete schemes in (12) and (16) are given respectively by

$$\psi^{(1)}(\xi) = \det\left[(A_2^{(1)} - H_1 B_2^{(1)})\xi^{2+r} - (A_1^{(1)} - H_1 B_1^{(1)})\xi^{1+r} - H_2 \sum_{i=1}^{2} B_i^{(1)}\xi^{i} \right]$$

and

$$\psi^{(2)}(\xi) = \det\left[(A_2^{(2)} - H_1 B_2^{(2)})\xi^{2+r} - (A_1^{(2)} - H_1 B_1^{(2)})\xi^{1+r} - H_2 \sum_{i=1}^{2} B_i^{(2)}\xi^{i} \right].$$

Also the Q-stability polynomials for the discrete schemes in (12) and (16) are given respectively by

$$\pi^{(1)}(\xi) = \det\left[A_2^{(1)}\xi^{2+r} - A_1^{(1)}\xi^{1+r} - H_2 \sum_{i=1}^{2} B_i^{(1)}\xi^{i} \right]$$

and

$$\pi^{(2)}(\xi) = \det\left[A_2^{(2)}\xi^{2+r} - A_1^{(2)}\xi^{1+r} - H_2 \sum_{i=1}^{2} B_i^{(2)}\xi^{i} \right],$$

where $r = \frac{\tau}{h} \in \mathbb{Z}$, $H_1 = h\lambda$ and $H_2 = h\mu$. Using Maple 18 and MATLAB the P- and Q-stability regions for r =1 for the schemes (12) and (16) are shown in Fig. 1 to 4

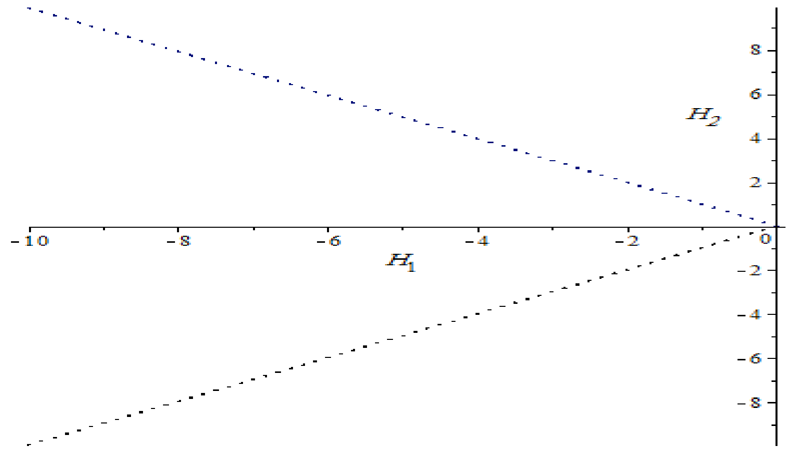

Fig.1 The P-stability region of the schemes in (12)

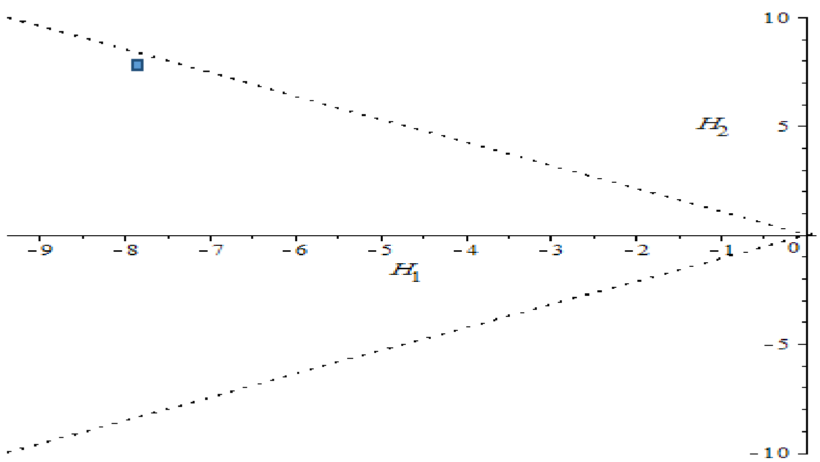

Fig.2 The P-stability region of the schemes in (16)

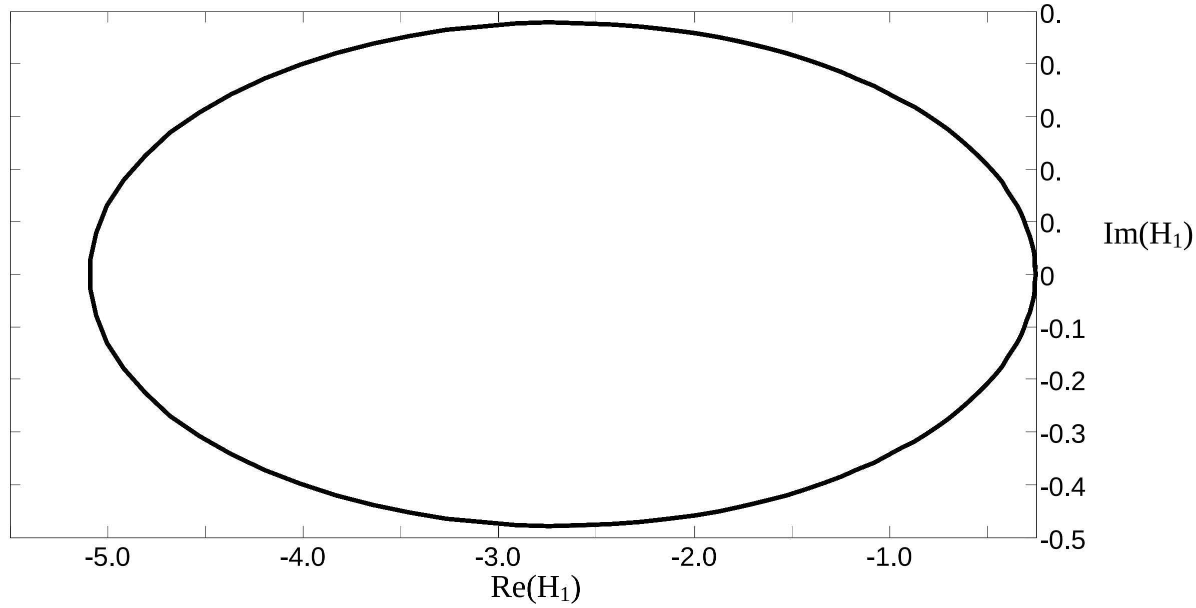

Fig.3 The Q-stability region of the schemes in (12)

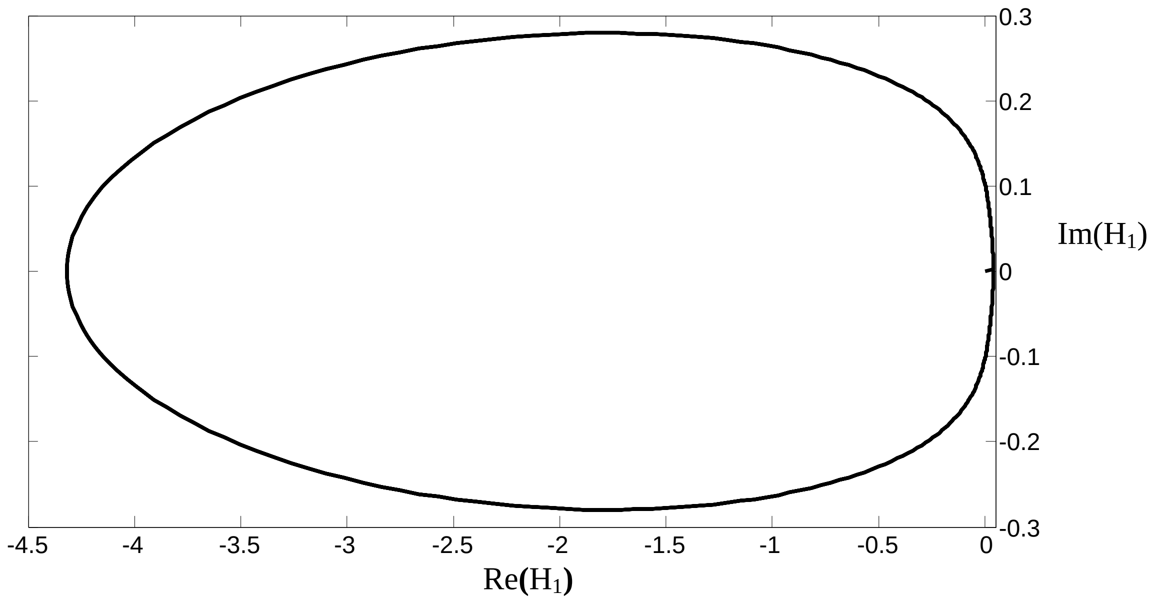

Fig.4 The Q-stability region of the schemes in (16)

From Figures 1 and 2, it is observed that the P-stability region of the schemes in (12) is about the same with that of the schemes in (16). Also from Figures 3 and 4, it is observed that the Q-stability region of the schemes in (12) is larger than that of the schemes in (16). Note that in figures 1 and 2, the P-stability regions lie inside the open ended region while in Figures 3 and 4, the Q-stability regions lie inside the enclosed region.

5. IMPLEMENTATION

The corresponding values of $f_n = f\ (t_n, y, y(t_n - \tau))$, where f is the function, were substituted using the discrete schemes in (12) and (16) with an accurate formula of the form.

$$p_{n+j}(t) = p((n+j-r)h) \tag{20}$$

where $j \in \{-k, k\}, k$ is a step number, $r = \frac{\tau}{h} \in \square$, $n = 0,1,2,\ldots,N-1$ and N is the number of solutions in the given interval, is implemented to approximate the delay term at the point $t = t_n - \tau$ using previous values of $p_{n+j} = \varphi(t)$ at $t_n - \tau \le t_0$ whenever $t_n - \tau > t_0$, where $p_{n+j}(t)$ is the approximation to $y(t_n - \tau)$ .The results of the above are obtained in block form using Maple 18 varying $n = 0,1,2,\ldots,N-1$ and evaluating the values of y_n

6. NUMERICAL RESULTS

In order to study the performance of the discrete schemes in (12) and (16) together with the formula (20), we present some numerical results for the following problems:

Problem 1

$$y'(t) = -24y(t) - e^{(-25)}y(t-1), \quad 0 \le t \le 3$$

$$y(t) = e^{(-25)t}, t \le 0$$

Exact Solution $y(t) = e^{(-25)t}$

Problem 2

$$y'(t) = -1000y(t) + 997e^{-3}y(t-1) + (1000 - 997e^{-3}), \quad 0 \le t \le 3$$

$$y(t) = 1 + e^{-3t}, t \le 0$$

Exact Solution $y(t) = 1 + e^{-3t}$

Problem 3

$$y'(t) = -1000y(t) + y(t - (\ln(1000-1))), \quad 0 \le t \le 3$$

$$y(t) = e^{-t}, t \le 0$$

Exact Solution $y(t) = e^{-t}$

Problem 4

$$y'(t) = -y(t - 1 + e^{-t}) + \sin(t - 1 + e^{-t}) + \cos(t), \quad 0 \le t \le 3$$

$$y(t) = \sin(t), t \le 0$$

Exact Solution $y(t) = \sin(t)$

Problem 5

$$y'(t) = \cos(t)(y(y(t) - 2)) \quad 0 \le t \le 3$$

$$y(t) = 1, t \le 0$$

Exact Solution $y(t) = 1 + \sin(t)$

The above problems were also solved using the schemes in [9], which are obtained by shifting Reformulated Block BDF methods one step forward, together with the formula (20). The results obtained are summarized in the tables 1 to 5 and the notations used in the tables are as follows

h	Step size
TS	Total steps taken
MAXE	Maximum Error
2BBDF	Implicit 2-point Block BDF method in [11]
CBBDF	Conventional Block BDF method for step number $k = 2$ in [9]
*CBBDF**	Conventional Block BDF method for step number $k = 3$ in [9]
RBBDF	Reformulated Block BDF method for step number $k = 3$
*RBBDF**	Reformulated Block BDF method for step number $k = 4$

The maximum error *MAXE* is a highest value of the absolute error for total number of steps taken.

Table 1. Comparison between 2BBDF, CBBDF, CBBDF*, RBBDF and RBBDF* using Problem 1

h	METHOD	TS	MAXE
10^{-2}	2BBDF	150	4.41E-02
	CBBDF	150	1.58E-03
	CBBDF*	150	3.47E-03
	RBBDF	150	3.36E-04
	RBBDF*	100	2.56E-04
10^{-3}	2BBDF	1500	9.28E-04
	CBBDF	1500	2.33E-06
	CBBDF*	1500	1,89E-06
	RBBDF	1500	1.73E-07
	RBBDF*	1000	1.12E-07
10^{-4}	2BBDF	15000	9.97E-06
	CBBDF	15000	8.62E-07
	CBBDF*	15000	6.73E-07
	RBBDF	15000	7.56E-08
	RBBDF*	10000	5.00E-08

Table 2. Comparison between 2BBDF, CBBDF, CBBDF*, RBBDF and RBBDF* using Problem 2

h	METHOD	TS	MAXE
10^{-2}	2BBDF	150	3.41E-03
	CBBDF	150	6.32E-06
	CBBDF*	150	5.10E-07
	RBBDF	150	1.54E-09
	RBBDF*	100	1.04E-09
10^{-3}	2BBDF	1500	2.34E-06
	CBBDF	1500	5.40E-07
	CBBDF*	1500	4.18E-08
	RBBDF	1500	3.02E-09
	RBBDF*	1000	2.56E-09
10^{-4}	2BBDF	15000	1.20E-07
	CBBDF	15000	4.78E-08
	CBBDF*	15000	1.22E-08
	RBBDF	15000	9.90E-09
	RBBDF*	10000	7.36E-09

Table 3. Comparison between 2BBDF, CBBDF, CBBDF*, RBBDF and RBBDF* using Problem 3

h	METHOD	TS	MAXE
10^{-2}	2BBDF	150	3.80E-04
	CBBDF	150	8.96E-05
	CBBDF*	150	9.39E-06
	RBBDF	150	4.88E-06
	RBBDF*	100	4.38E-06
10^{-3}	2BBDF	1500	2.61E-07
	CBBDF	1500	3.12E-08

	CBBDF*	1500	1.43E-08
	RBBDF	1500	7.52E-09
	RBBDF*	1000	7.02E-09
10^{-4}	2BBDF	15000	1.34E-08
	CBBDF	15000	1.27E-08
	CBBDF	15000	8.40E-09
	RBBDF	15000	4.26E-09
	RBBDF*	10000	3.70E-09

Table 4. Comparison between CBBDF, CBBDF*, RBBDF and RBBDF* using Problem 4

h	**METHOD**	**TS**	**MAXE**
10^{-2}	CBBDF	150	1.66E-05
	CBBDF*	150	2.22E-07
	RBBDF	150	1.61E-07
	RBBDF*	100	1.54E-08
10^{-3}	CBBDF	1500	2.71E-07
	CBBDF*	1500	3.21E-08
	RBBDF	1500	1.28E-08
	RBBDF*	1000	2.58E-09
10^{-4}	CBBDF	15000	7.23E-08
	CBBDF	15000	5.56E-09
	RBBDF	15000	2.67E-09
	RBBDF*	10000	3.31E-10

Table 5. Comparison between CBBDF, CBBDF*, RBBDF and RBBDF* using Problem 5

h	**METHOD**	**TS**	**MAXE**
10^{-2}	CBBDF	150	1.66E-05
	CBBDF*	150	2.65E-07
	RBBDF	150	2.16E-07
	RBBDF*	100	2.96E-08
10^{-3}	CBBDF	1500	7.45E-07
	CBBDF*	1500	5.04E-08
	RBBDF	1500	2.14E-08
	RBBDF*	1000	2.27E-09

	CBBDF	15000	1.51E-08
	CBBDF	15000	2.54E-09
10^{-4}	RBBDF	15000	1.33E-09
	RBBDF*	10000	4.30E-10

7. CONCLUSION

In this paper, it was observed that the results obtained from the schemes for step number k = 4 performed slightly better than the schemes for step number k = 3 when compared with the exact solutions. When comparing RBBDF with other existing methods, like CBBDF in [9] and 2BBDF in [11], it was observed that RBBDF achieved better results in terms of accuracy. Therefore it can be concluded that the Reformulated Block Backward Differentiation Formulae methods are suitable for solving Delay Differential Equations.

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