Original Research Article

Construction of Stable High Order One-Block Methods Using Multi-Block Triple

Abstract

This paper deals with the construction of *l*-stable implicit one-block methods for the solution of stiff initial value problems. The constructions are done using three different multi-block methods. The first multi-block method is composed using Generalized Backward Differentiation Formula (GBDF) and Backward Differentiation Formula (BDF), the second is composed using Reversed Generalized Adams Moulton (RGAM) and Generalized Adams Moulton (GAM) while the third is composed using Reversed Adams Moulton (RAM) and Adams Moulton (AM). Shift operator is then applied to the combination of the three multi-block methods in such a manner that the resultant block is a one-block method and self-starting. These one-block methods are *l*-stable at order six and $l(\alpha)$–*stable* with $\alpha = 79.75^o$ an order ten. Numerical experiments show that they are good for solving stiff initial problems.

Keywords: l-stable; multi-block; stiff initial value problem; one-block and self-starting.

Subject classification: 65L04, 65L05, 65L06.

Introduction

In [7], it was pointed out that unconventional means were adopted by many researchers in order to circumvent the Dahlquist order barrier [6]. In other to construct high order and stable methods, many researchers in recent time have followed this unconventional means (see [1, 4, 9, 11, 12, 13, 14, 15, 16]. In this paper, the same trend is followed in constructing methods for finding the numerical solution $y(t)$ to the initial value problems (ivp) in ode

$$\begin{aligned} &y'(t) = f(t, y(t))\,; \quad y(t_0) = y_0\,; \quad t \in [a, b]\,; \\ &f : \Re \times \Re^m \to \Re^m\,; \quad y : \Re \to \Re^m \end{aligned} \qquad (1)$$

The classical linear multistep formula which is given by

$$\sum_{r=0}^{k} \alpha_r y_{n+r} = h_n \sum_{r=0}^{k} \beta_r f(t_{n+r}, y_{n+r}) \qquad (2)$$

where the step number $k > 1$ and $h_n = t_{n+1} - t_n$ is a variable step length, $\{\alpha_r\}_{r=0}^{k}$ $\{\beta_r\}_{r=0}^{k}$ and are real constants and both not zero. Formula (1) can be represented by two polynomials

$$\rho(z) = \sum_{r=0}^{k} \alpha_r z^r, \quad \sigma(z) = \sum_{r=0}^{k} \beta_r z^r \tag{3}$$

such that (2) can be rewritten as

$$\rho(E) y_n = h\sigma(E) f_n \tag{4a}$$

where E is the shift operator defined by $E^j y_n = y_{n+j}$. When (2) is applied to the scalar test equation

$$y' = \lambda y, \ \mathrm{Re}(\lambda) < 0. \tag{4b}$$

yields the stability polynomial

$$\rho(z) - h\sigma(z) = 0 \tag{5}$$

Now let us redefine (3) as

$$\rho(z) = \sum_{r=0}^{k} A_r z^r, \quad \sigma(z) = \sum_{r=0}^{k} B_r z^r \tag{6}$$

where $\{A_i\}_{i=0}^{k}$ and ,$\{B_i\}_{i=0}^{k}$ are matrices (block coefficients), then (2) becomes a linear multi-block method (LMBM)

$$\sum_{r=0}^{k} A_r y_{n+r} = h_n \sum_{r=0}^{k} B_r f(t_{n+r}, y_{n+r}) \tag{7}$$

which can be rewritten as (4a)

Construction of the block methods

The methodology for the construction of the methods is explained in the following proposition:

Proposition

Let the family of Linear Multi-Block Methods (LMBM) $\left\{\rho_k^{[j]}(R), \sigma_k^{[j]}(R)\right\}_{j=1,k=1}^{m,T}$ *be given, that is,*

$$\rho_k^{[j]}(E)Y_n = h\,\sigma_k^{[j]}(E)F_n\,;\; j = 1(1)m\,,\;\; k = 1(1)T \qquad (8)$$

with $\{\rho_k^{[j]}, \sigma_k^{[j]}\}$ *for a fixed j forming a family of variable order* $p_{k,j}$ *of variable step number k. Then the resultant system of composite LMBM*

$$E^i \rho_k^{[j]}(E)Y_n = hE^i \sigma_k^{[j]}(E)F_n\,;\; i = 0(1)k-l\;\;;\; j = 1,2,\ldots,m \text{ (the number of LMBM)} \qquad (9)$$

arising from the E-operator transformation of (8) *can be composed as the one-block method*

$$C_1Y_{n+1} + C_0Y_n = h(D_1F_{n+1} + D_0F_n)\,;\; \det(C_1) \neq 0 \qquad (10)$$

if k is chosen such that l is an integer given as

$$l = \frac{k(ms-1-s)+ms}{s(m-1)}\;;\;\; k \geq 4; m,s \geq 2; (s \text{ is the number of rows in each LMBM})$$

and $k - l \geq 0$. (11)

where Y_{n+1}, Y_n; F_{n+1} *and* F_n $n = 0,1,2,\ldots$ *are as defined below and* C_1, C_0, D_1, D_0 *are square matrices also defined below for a fixed s and m.*

$$C_0 = \left(\begin{array}{c|c} & A_0^{[1]} \\ & A_0^{[2]} \\ & \vdots \\ & A_0^{[m]} \\ O & 0 \\ & \vdots \\ & 0 \end{array}\right)_{(k+s(k-l))\times(k+s(k-l))} ; \quad D_0 = \left(\begin{array}{c|c} & B_0^{[1]} \\ & B_0^{[2]} \\ & \vdots \\ & B_0^{[m]} \\ O & 0 \\ & \vdots \\ & 0 \end{array}\right)_{(k+s(k-l))\times(k+s(k-l))} \qquad (12)$$

.

$$
C_1 = \begin{pmatrix}
A_1^{[1]} & \cdot & \cdot & \cdot & A_k^{[1]} & 0 & 0 & 0 & \cdots & & & 0 \\
\vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & 0 \\
A_1^{[m]} & \cdot & \cdot & \cdot & A_k^{[m]} & 0 & \cdot & \cdot & & & & 0 \\
A_0^{[1]} & A_1^{[1]} & \cdot & \cdot & \cdot & A_k^{[1]} & \cdot & \cdot & & & & 0 \\
\cdot & \cdot & \cdot & \cdot & & \vdots & 0 & & & & & 0 \\
\vdots & \vdots & \cdot & \cdot & \cdot & \cdot & 0 & & & & & \cdot \\
A_0^{[m]} & A_1^{[m]} & & & & A_k^{[m]} & 0 & & & & & \cdot \\
0 & A_0^{[1]} & A_1^{[1]} & & & & A_k^{[1]} & & & & & \cdot \\
0 & \cdot & \cdot & & & \cdot & \cdot & & & & & \\
 & \vdots & \vdots & \cdot & \cdot & \cdot & \vdots & & & & & \\
 & A_0^{[m]} & A_1^{[m]} & \cdot & \cdot & \cdot & A_k^{[m]} & 0 & & & & \\
\vdots & \vdots & \vdots & & & & \vdots & & & & \cdots & \\
\vdots & \cdot & \cdot & & & & \cdot & & \cdot & \cdot & & 0 \\
\cdot & & & & & & & & \cdot & A_0^{[1]} & & A_k^{[1]} \\
 & & & & & & & & \cdot & \vdots & & \vdots \\
0 & \cdot & \cdot & \cdot & & & & & 0 & A_0^{[m]} & \cdots & A_k^{[m]}
\end{pmatrix}_{(k+s(k-l))\times(k+s(k-l))}
$$

.

$$D_1 = \begin{pmatrix}
B_1^{[1]} & \cdot & \cdot & \cdot & B_k^{[1]} & 0 & 0 & 0 & \cdot & \cdot & \cdot & & & & 0 \\
\cdot & & & & & & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & 0 \\
\cdot & & & & & & & & & & & & & & \\
B_1^{[m]} & \cdot & \cdot & \cdot & B_k^{[m]} & 0 & \cdot & \cdot & & & & & & & 0 \\
B_0^{[1]} & B_1^{[1]} & \cdot & \cdot & \cdot & B_k^{[1]} & \cdot & \cdot & & & & & & & 0 \\
\cdot & \cdot & \cdot & \cdot & & \cdot & 0 & & & & & & & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & & & & & & & & \cdot \\
B_0^{[m]} & B_1^{[m]} & & & & B_k^{[m]} & 0 & & & & & & & & \cdot \\
0 & B_0^{[1]} & B_1^{[1]} & & & B_{k-1}^{1} & B_k^{[1]} & & & & & & & & \cdot \\
0 & \cdot & \cdot & & & \cdot & \cdot & & & & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & \\
 & B_0^{[m]} & B_1^{[m]} & \cdot & \cdot & B_{k-1}^{[m]} & B_k^{[m]} & 0 & & & & & & & \\
\cdot & \cdot & \cdot & & & & \cdot & & & & & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & & & & \cdot & & & \cdot & \cdot & & & & 0 \\
\cdot & & & & & & & & & \cdot & B_0^{[1]} & & & & B_k^{[1]} \\
 & & & & & & & & & \cdot & \cdot & & & & \cdot \\
0 & \cdot & \cdot & \cdot & & & & & & 0 & B_0^{[m]} & \cdot & \cdot & \cdot & B_k^{[m]}
\end{pmatrix}_{(k+s(k-l))\times(k+s(k-l))}$$

$$Y_{n+1} = (y_{n+1}, y_{n+2}, \ldots, y_{n+k+s(k-l)})^T; \quad Y_n = (y_{n-(k+s(k-l))+1}, y_{n-2k+l+2}, \ldots, y_{n-1}, y_n)^T; \quad (13)$$

$$F_{n+1} = (f_{n+1}, f_{n+2}, \ldots, f_{n+k+s(k-l)})^T; \quad F_n = (f_{n-(k+s(k-l))+1}, f_{n-2k+l+2}, \ldots, f_{n-1}, f_n)^T$$

$$n = 0, 1, 2, \ldots$$

Proof:

Notice that the *E*-operator is effectively applied *k-l* times on the system of LMBM $\{\rho_k^{[j]}, \sigma_k^{[j]}\}_{k,j}$. Thus there are $(k + s(k-l)) \times (k + s(k-l))$ unknown solution points captured in the block of solution $Y_{n+1} = (y_{n+1}, y_{n+2}, \ldots, y_{n+k+s(k-l)})^T$. By this, the block definition in (10) is realized if the coefficient matrices C_1, C_0, D_1, D_0 are square matrices of dimension $(k + s(k-l)) \times (k + s(k-l))$.

This simply implies that $ms + ms(k-l) = k + s(k-l)$ so that l is as in (11) and for a fixed m and s, k is chosen such that $k - l \geq 0$. ∎

In particular:

$$(1.)\ m = 2\ ;\ s = 2;\ l = \frac{k+4}{2};\quad k = 4,6,8,10,\ldots$$

$$(2.)\ m = 3\ ;\ s = 2;\ l = \frac{3k+6}{4}\ ;\ k = 6,10,14,\ldots$$

When $k - l = 0$ the method requires zero shiftings. This is so if $ms = k$. However, the case of interest in this paper is when $m = 3$ and $s = 2$. Consider the family composed using GBDF/BDF [3], RGAM/GAM and RAM/AM [2] methods, the coefficients are respectively given below:

The method constructed using the pair of GBDF and BDF of order 6, that is k=6.

$$[\text{illegible}]\ \frac{1}{30}\ [\text{illegible}]\ \begin{matrix}0 & 1\end{matrix}\ F_{n+3} + [\text{illegible}]\ 4\ \frac{7}{12}\ [\text{illegible}]\ \begin{matrix}0 & 0\end{matrix}\ F_{n+2} + [\text{illegible}]\ 2\ \frac{1}{2}\ \frac{15}{4}\ [\text{illegible}]\ \begin{matrix}0 & 0\end{matrix}\ F_{n+1} + [\text{illegible}]\ \frac{1}{60}\ [\text{illegible}]\ \begin{matrix}0 & 0\end{matrix}\ F_n \quad (3)$$

The method constructed using the pair of RGAM and GAM of order 7, that is k=6.

$$[\text{illegible}]\ Y_{n+3}\ [\text{illegible}]\ Y_{n+2}\ [\text{illegible}]\ Y_{n+1}\ [\text{illegible}]\ Y_n = [\text{illegible}]\ \frac{67}{[\text{illegible}]}\ -\frac{191}{60480}\ [\text{illegible}]\ -\frac{271}{60480}\ [\text{illegible}] + [\text{illegible}]\ \frac{586}{[\text{illegible}]}\ -\frac{2257}{20160}\ [\text{illegible}]\ -\frac{10273}{20160}\ [\text{illegible}] + [\text{illegible}]\ \frac{23}{[\text{illegible}]}\ \frac{10273}{20160}\ [\text{illegible}]\ \frac{2257}{20160}\ [\text{illegible}] + [\text{illegible}]\ \frac{271}{60480}\ \frac{191}{[\text{illegible}]}\ [\text{illegible}] \quad (4)$$

The method constructed using the pair of RAM and AM of order 7, that is k=6.

$$[\text{illegible}]\ Y_{n+3}\ [\text{illegible}]\ Y_{n+2}\ [\text{illegible}]\ Y_{n+1}\ [\text{illegible}]\ Y_n = [\text{illegible}]\ \frac{263}{[\text{illegible}]}\ -\frac{863}{60480}\ [\text{illegible}]\ \frac{19087}{60480}\ [\text{illegible}] + [\text{illegible}]\ \frac{586}{[\text{illegible}]}\ -\frac{6737}{20160}\ [\text{illegible}]\ -\frac{15487}{20160}\ [\text{illegible}] + [\text{illegible}]\ \frac{2713}{[\text{illegible}]}\ -\frac{15487}{20160}\ [\text{illegible}]\ -\frac{6737}{20160}\ [\text{illegible}] + [\text{illegible}]\ \frac{19087}{60480}\ \frac{863}{60480}\ [\text{illegible}] \quad (5)$$

The three multi-block (Three-block) methods are then used to construct a one-block method given as in (10) where

$$C_1 = \begin{pmatrix} [\text{illegible}] & \frac{2}{15} & \frac{1}{2} & -\frac{4}{3} & \frac{7}{12} & \frac{2}{5} & -\frac{1}{30} \\ [\text{illegible}] & \frac{6}{5} & \frac{15}{4} & -\frac{20}{3} & \frac{15}{2} & -6 & \frac{49}{20} \\ [\text{illegible}] & 0 & -1 & 1 & 0 & 0 & 0 \\ [\text{illegible}] & 0 & 0 & 1 & -1 & 0 & 0 \\ [\text{illegible}] & [\text{illegible}] & 0 & 0 & 0 & 0 & 0 \\ [\text{illegible}] & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \quad [\text{illegible}]_0 = \begin{pmatrix} [\text{illegible}] & 0 & 0 & 0 & 0 & \frac{1}{60} \\ [\text{illegible}] & 0 & 0 & 0 & 0 & \frac{1}{6} \\ [\text{illegible}] & 0 & 0 & 0 & 0 & 0 \\ [\text{illegible}] & 0 & 0 & 0 & 0 & 0 \\ [\text{illegible}] & 0 & 0 & 0 & 0 & -1 \\ [\text{illegible}] & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{271}{60480} \\ 0 & 0 & 0 & 0 & 0 & \frac{191}{60480} \\ 0 & 0 & 0 & 0 & 0 & \frac{19087}{60480} \\ 0 & 0 & 0 & 0 & 0 & -\frac{863}{60480} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{23}{504} & \frac{10273}{20160} & \frac{586}{945} & -\frac{2257}{20160} & \frac{67}{2520} & -\frac{191}{60480} \\ \frac{67}{2520} & \frac{2257}{20160} & -\frac{586}{945} & -\frac{10273}{20160} & \frac{23}{504} & -\frac{271}{60480} \\ \frac{2713}{2520} & -\frac{15487}{20160} & \frac{586}{945} & -\frac{6737}{20160} & \frac{263}{2520} & -\frac{863}{60480} \\ \frac{263}{2520} & -\frac{6737}{20160} & \frac{586}{945} & -\frac{15487}{20160} & \frac{2713}{2520} & \frac{19087}{60480} \end{pmatrix}$$

Case of k=10, five-block methods constructed are

$$[illegible]\ \frac{1}{840},\ \frac{7381}{2520},\ \frac{1}{1260},\ \ldots\ F_{n+5} + \ldots \frac{4}{\ldots},\ -\frac{3}{28},\ \frac{45}{\ldots}\ F_{n+4} + \ldots \frac{6}{\ldots},\ \frac{11}{30},\ \frac{105}{2}\ F_{n+3} + \ldots \frac{4}{\ldots},\ \frac{1}{2}\ F_{n+2} + \ldots \frac{1}{\ldots},\ \frac{3}{56},\ \frac{45}{\ldots}\ F_{n+1} + \ldots F_n \tag{11}$$

$$[illegible]\ Y_{n+5} \ldots Y_{n+3} \ldots Y_{n+1} \ldots\ \frac{32309}{\ldots07200},\ -\frac{14797}{95800320},\ -\frac{90817}{479001600},\ \frac{1394959}{\ldots958400},\ \frac{14296081}{26611200},\ \frac{3216337}{26611200} + \ldots \frac{163459}{\ldots1680},\ -\frac{1746433}{159667200},\ -\frac{493837}{31933440},\ \frac{292531}{\ldots9750400},\ \frac{493837}{31933440},\ \frac{1746433}{159667200} + \ldots \frac{379571}{\ldots700},\ -\frac{3216337}{26611200},\ -\frac{14296081}{\ldots},\ \frac{90817}{479001600},\ \frac{14797}{\ldots5800320} \tag{12}$$

$$[illegible]\ Y_{n+5} \ldots Y_{n+3} \ldots Y_{n+1} \ldots\ \frac{9071219}{\ldots750400},\ -\frac{3250433}{479001600},\ \frac{26842253}{95800320},\ \frac{12051709}{\ldots1680},\ -\frac{33765029}{8870400},\ -\frac{21677723}{8870400} + \ldots \frac{23643791}{\ldots58400},\ -\frac{12318413}{31933440},\ -\frac{296725183}{159667200},\ \frac{164046413}{\ldots750400},\ -\frac{296725183}{159667200},\ -\frac{12318413}{31933440} + \ldots \frac{2227571}{\ldots700},\ -\frac{21677723}{8870400},\ -\frac{33765029}{\ldots},\ \frac{26842253}{95800320},\ \frac{3250433}{479001600} \tag{13}$$

Putting all the three multi-block methods together, we have (10) where

$$C_1 = \begin{bmatrix} \frac{1}{105} & \frac{3}{56} & -\frac{4}{21} & \frac{1}{2} & -\frac{6}{5} & \frac{11}{30} & \frac{4}{7} & -\frac{3}{28} & \frac{1}{63} & -\frac{1}{840} & 0 & 0 \\ -\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & \frac{1}{1260} & -\frac{1}{105} & \frac{3}{56} & -\frac{4}{21} & \frac{1}{2} & -\frac{6}{5} & \frac{11}{30} & \frac{4}{7} & -\frac{3}{28} & \frac{1}{63} & -\frac{1}{840} \\ 0 & \frac{1}{10} & -\frac{10}{9} & \frac{45}{8} & -\frac{120}{7} & 35 & -\frac{252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1260} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{292531}{119750400} & \frac{493837}{31933440} & -\frac{1394959}{19958400} & \frac{14296081}{26611200} & \frac{379571}{623700} & -\frac{3216337}{26611200} & \frac{163459}{3991680} & -\frac{1746433}{159667200} & \frac{32309}{17107200} & -\frac{14797}{95800320} & 0 & 0 \\ -\frac{32309}{17107200} & \frac{1746433}{159667200} & -\frac{163459}{3991680} & \frac{3216337}{26611200} & -\frac{379571}{623700} & -\frac{14296081}{26611200} & \frac{1394959}{19958400} & -\frac{493837}{31933440} & \frac{292531}{119750400} & -\frac{90817}{479001600} & 0 & 0 \\ \frac{164046413}{119750400} & -\frac{296725183}{159667200} & \frac{12051709}{3991680} & -\frac{33765029}{8870400} & \frac{2227571}{623700} & -\frac{21677723}{8870400} & \frac{23643791}{19958400} & -\frac{12318413}{31933440} & \frac{9071219}{119750400} & -\frac{3250433}{479001600} & 0 & 0 \\ \frac{9071219}{119750400} & -\frac{12318413}{31933440} & \frac{23643791}{19958400} & -\frac{21677723}{8870400} & \frac{2227571}{623700} & -\frac{33765029}{8870400} & \frac{12051709}{3991680} & -\frac{296725183}{159667200} & \frac{164046413}{119750400} & \frac{26842253}{95800320} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{90817}{479001600} & -\frac{292531}{119750400} & \frac{493837}{31933440} & -\frac{1394959}{19958400} & \frac{14296081}{26611200} & \frac{379571}{623700} & -\frac{3216337}{26611200} & \frac{163459}{3991680} & -\frac{1746433}{159667200} & \frac{32309}{17107200} & -\frac{14797}{95800320} \\ 0 & \frac{14797}{95800320} & -\frac{32309}{17107200} & \frac{1746433}{159667200} & -\frac{163459}{3991680} & \frac{3216337}{26611200} & -\frac{379571}{623700} & -\frac{14296081}{26611200} & \frac{1394959}{19958400} & -\frac{493837}{31933440} & \frac{292531}{119750400} & -\frac{90817}{479001600} \\ 0 & \frac{26842253}{95800320} & \frac{164046413}{119750400} & -\frac{296725183}{159667200} & \frac{12051709}{3991680} & -\frac{33765029}{8870400} & \frac{2227571}{623700} & -\frac{21677723}{8870400} & \frac{23643791}{19958400} & -\frac{12318413}{31933440} & \frac{9071219}{119750400} & -\frac{3250433}{479001600} \\ 0 & -\frac{3250433}{479001600} & \frac{9071219}{119750400} & -\frac{12318413}{31933440} & \frac{23643791}{19958400} & -\frac{21677723}{8870400} & \frac{2227571}{623700} & -\frac{33765029}{8870400} & \frac{12051709}{3991680} & -\frac{296725183}{159667200} & \frac{164046413}{119750400} & \frac{26842253}{95800320} \end{bmatrix}$$

$$D_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{90817}{479001600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14797}{95800320} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{26842253}{95800320} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3250433}{479001600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Stability of the Implicit Block Methods

When (10) is applied to the test equation

$$y' = \lambda y\,,\ \mathrm{Re}(\lambda) < 0 \qquad (14)$$

it yields the characteristics equation.

$$\pi(w,z) = \det(C_1 w + C_0 - z(D_1 w + D_0)),\quad z = \lambda h \qquad (15)$$

For order k = 6,

$\pi(w,z) =$

$$-\frac{571\,w^5}{720}+\frac{571\,w^6}{720}-\frac{9647\,w^5 z}{5040}-\frac{2867\,w^6 z}{1008}-\frac{61333\,w^5 z^2}{30240}+\frac{145717\,w^6 z^2}{30240}-\frac{26147\,w^5 z^3}{21600}-\frac{764341\,w^6 z^3}{151200}-\frac{4517\,w^5 z^4}{10800}+\frac{270707\,w^6 z^4}{75600}-\frac{6109\,w^5 z^5}{90720}-\frac{110011\,w^6 z^5}{64800}+\frac{3223\,w^6 z^6}{7560}$$

The region of absolute stability R_A associated with (10) is the set

$$R_A = \{z \in \mathbb{C} : |w_j(z)| \le 1,\ j = 1(1)(k + s(k-l))\} \qquad (16)$$

For order 6 above $w_j(z),\ j = 1(1)6$ are given below

[illegible] $\frac{359730+868230\,z+919995\,z^2+549087\,z^3+189714\,z^4+30545\,z^5}{359730-1290150\,z+2185755\,z^2-2293023\,z^3+1624242\,z^4-770077\,z^5+193380\,z^6}$ [illegible]

he only non-zero value of $w(z)$ for this family of methods are given as a rational function $T(z) = \frac{P(z)}{Q(z)}$. where $P(z)$ $Q(z)$ and are polynomials. From the above $k = 6$,

$T(z) =$

$$\frac{359730+868230\,z+919995\,z^2+549087\,z^3+189714\,z^4+30545\,z^5}{359730-1290150\,z+2185755\,z^2-2293023\,z^3+1624242\,z^4-770077\,z^5+193380\,z^6}$$

This value tends to zero as z tends to infinity.

Definition1: A block method is said to be pre-stable if the roots of $Q(z)$ are contained in C^+ (see [5]). The roots of $Q(z)$ are

{{z→0.2210288675951737 - 1.2587046977754033 ™},{z→0.2210288675951737 + 1.2587046977754033 ™}, {z→0.7560441897235561 - 0.701940199394596 ™}, {z→0.7560441897235561 + 0.701940199394596 ™}, {z→1.0140247811340575 - 0.2047642418277674 ™}, {z→1.0140247811340575 + 0.2047642418277674 ™}}

They are contained in C^+.

Definition2: A one block method is *A*-stable if and only if it is stable on the imaginary axis (*I*-stable) [8]:

That is $T(iy) \le 1$ for all $y \in \Re$, and $T(z)$ is analytic for $z < 0$ (i.e. $Q(z)$ does not have roots with negative or zero real parts), *I*-stability is equivalent to the fact that the Norsett polynomial defined by

$$G(y) = |Q(iy)|^2 - |P(iy)|^2 = Q(iy)Q(-iy) - P(iy)P(-iy) \qquad (19)$$

satisfies $G(y) > 0$ for all $y \in \Re$ [8].

Definition 3: A block method is said to be $L-Stable$ if it is $A-Stable$ and also $T(z) \to 0$ as $z \to \infty$ [10].

The none zero solution, $T(z)$ of order 6 has no pole on C^-, all the roots of $Q(z)$ are contained in C^+. The orders 6 satisfies condition (19) and definitions 1 and 2, therefore is $L-Stable$.

Definition 4: A LMF is said to be A(α)-Stable, with $\alpha \in (0, \frac{\pi}{2})$ if its region of absolute stability (RAS) contains the infinite wedge w_α, $w_\alpha = \{\lambda h : -\alpha \le |\pi - \arg(z)| \le \alpha\ \}$

Following the analysis as above, the order 10 of the constructed method is $L(\alpha)-Stable$ with $\alpha = 79.75^o$.

Numerical Experiments

In this section, we considered two problems to test the effectiveness of the method

Problem: Van der Pol problem (cf: [5])

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 + \mu\, y_2(1 - y_1^2);\ y_1(0) = 2,\ y_2(0) = 0, \mu = 200 \end{aligned}$$

The phase diagram of the problem of the computed solution and that of ode15s are plotted in figure 1.

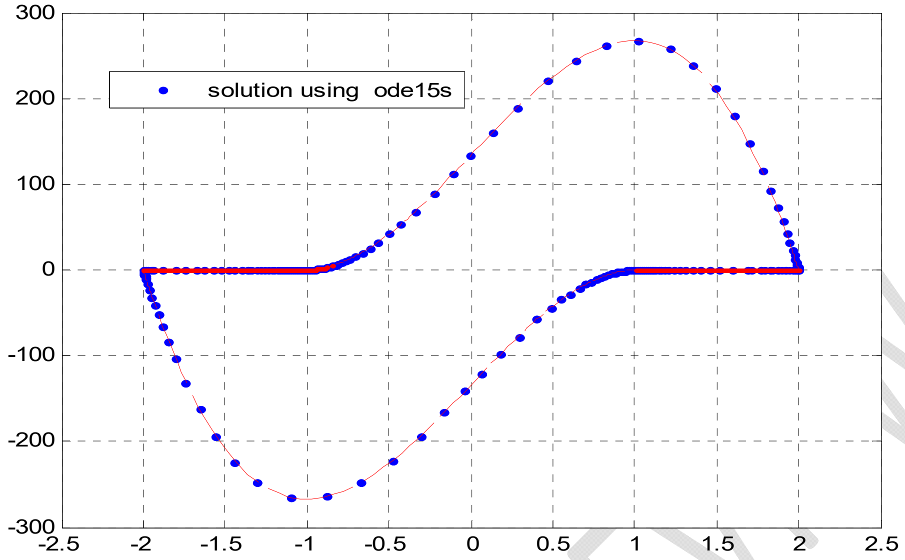

Figure 1: The phase diagram of problem computed with order 6 of the method

Conclusion

The work done in [2, 3, 9] using linear multistep methods has been extended to multi-block methods. The order 6 of the methods constructed is $L-Stable$, while the order 10 is $L(\alpha)-Stable$ with $\alpha = 79.75^o$. The result of the implementation of order 6 of the method on a stiff initial value problem shows that it is effective.

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