Perturbation approach in the dynamic buckling of a model structure with a cubic-quintic nonlinearity subjected to an explicitly time dependent slowly varying load

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Abstract:

This investigation is concerned with analytically determining the dynamic buckling load of an imperfect cubic-quintic nonlinear elastic model structure struck by an explicitly time-dependent but slowly varying load. Besides, the load is continuously decreasing in magnitude. A multi-timing regular perturbation technique in asymptotic procedures is utilized to analyze the problem. The result shows that the dynamic buckling load depends, among other things, on the first derivative of the load function evaluated at the initial time. In the long run, the dynamic buckling load is related to its static equivalent, and that relationship is independent of the imperfection parameter. Thus, once any of the two buckling loads is known, then the other can easily be evaluated using this relationship.

Keywords: Dynamic buckling, perturbation approach, multi-timing perturbation technique, cubic-quintic nonlinear elastic structure, slowly varying load, imperfection parameter.

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1. INTRODUCTION

In a series of investigations [1-3], Budiansky and Hutchison extended the original work of Koiter [4, 5] to the case of dynamic buckling. Using mass-spring model structure arrangement, they (Budiansky and Hutchison) derived a series of equations of motion characterizing certain nonlinear elastic model structures trapped by various types of loading histories. Such loading histories include the step load, rectangular load, periodic load as well as impulse load, among others. Except for periodic load which normally results to non-autonomous equations of motion, most other loading histories give rise to autonomous loading histories, where the resultant equations of motion are overtly implicit in the time variable.

It must however be mentioned that Budiansky and Hutchison [1-3], named such systems and the resultant equations of motion after the degree of nonlinearity of the model structures investigated. Thus, there is, for instance, the quadratic structure satisfying the equation

$$\frac{d^2\xi}{dt^2} + (1 - \lambda f(t))\xi - \alpha\xi^2 = \lambda\bar{\xi} f(t), \quad t > 0 \tag{1a}$$

$$\xi(0) = \frac{d\xi(0)}{dt} = 0 \tag{1b}$$

where $\xi(t)$ is the displacement, λ is a nondimensional load amplitude, $f(t)$ is the loading history, t is the time variable, $\bar{\xi}$ is the imperfection amplitude while α is the imperfection sensitivity parameter. In all quadratic structures, the nonlinearity is quadratic. Budiansky and Hutchison likewise investigated other structures such as cubic structures (having cubic nonlinearity), quadratic-cubic structures (with quadric-cubic nonlinearity) and so on. In the light of the foregoing, this investigation shall extend the study to the case of an imperfect cubic-quintic nonlinearity elastic model structure (i.e. having cubic-quintic nonlinear)

with the sole aim of obtaining the dynamic buckling load in the case of the structure trapped by an explicitly time dependent but slowly-varying dynamic load that has a continuously decreasing but nonzero load amplitude. Other pertinent investigations include the studies by Sadovsky et al. [6] and Bisagni [7].
It is to be remarked that investigations into slowly-varying systems have been the pre-occupation of many researchers. Such include Kevorkian [8], Kuzmark [9], Luke [10], Kroll et al. [11] as well Li and Kevorkian [12], among others. Such earlier studies were largely made in the context of dynamical, mechanical and electrical systems, but not particularly in the landscape of dynamic buckling. To our knowledge, such earlier investigations involving explicitly time dependent but slowly-varying loading systems in a dynamic buckling setting, appear to be rare. It is also to be remarked that Askogan and Sofiyev [13] investigated the dynamic buckling load of spherical shells with variable thickness subjected to a time-dependent external pressure varying as a power function of time, while Kubiak [14], Wooseok [15], Kolakowski [16] and Kowal-Michalska [17] made similar investigations. Mention is also made of studies by Bisagni and Vescovini [18] and Patel et al. [19], while Reda and Forbes [20] carried out an investigation into the dynamic effects of lateral buckling of high temperature/high pressure offshore pipelines.
In this analysis, we actually adopt a similar approach as in Ette et al. [21, 22] with minor modifications. In [21] as well as [22], it is assumed that the structure is trapped by a step load such that $f(1) = 1$. Furthermore, [22] is a case of viscously damped imperfect finite column and the perturbation was in the viscous damping and the imperfection amplitude.

2. FORMULATION OF THE PROBLEM

Taking a cue from (1a, b), the relevant equations of motion are

$$\frac{d^2\xi}{dt^2} + \left(1 - \lambda f(\bar{\xi}^2 t)\right)\xi + \alpha\xi^3 - \beta\xi^5 = \lambda\bar{\xi} f\left(\bar{\xi}^2 t\right), \quad t > 0 \tag{2a}$$

$$\xi(0) = \frac{d\xi(0)}{dt} = 0, \quad 0 < \bar{\xi} \ll 1, \; 0 < \lambda < 1 \tag{2b}$$

where α and β are the imperfection sensitivity parameters and all other variables are as defined earlier. Here the load $f(\bar{\xi}^2 t)$ is such that

$$f(0) = 1, \quad \left|f(\bar{\xi}^2 t)\right| < 1, \quad t > 0 \tag{3}$$

In addition, $f(\bar{\xi}^2 t)$ is a continuously decreasing but slowly-varying function of time and having right hand derivatives of all orders at $t = 0$, otherwise $f(\bar{\xi}^2 t)$ is arbitrary. Our aim then is to analytically determine the dynamic buckling load of the structure subjected to the stipulated load.
The dynamic buckling load λ_D is defined as the maximum load amplitude for which the solution of (2a, b) remains bounded for all time $t > 0$. As in Budiansky and Hutchison [1-3] and Amazigo [6,7], the condition for dynamic buckling is

$$\frac{d\lambda}{d\xi_a} = 0 \tag{4}$$

where ξ_a is the maximum displacement. It is then incumbent on us to first obtain an asymptotic expression of the displacement after which the maximum displacement ξ_a is determined.

3. STATIC BUCKLING LOAD λ_S

Because the intention is to eventually relate the dynamic buckling load to the static buckling load λ_S, it is necessary that λ_S be first obtained.
The relevant equation in this case is obtained by neglecting the inertia term in (2a) and setting $f\left(\bar{\xi}^2 t\right) = 1$, while neglecting the initial conditions. This gives

$$(1-\lambda)\xi + \alpha\xi^3 - \beta\xi^5 = \lambda\bar{\xi} \quad (5)$$

The condition for static buckling, as in [1-3] and [6,7], is

$$\frac{d\lambda}{d\xi} = 0 \quad (6a)$$

This gives

$$(1-\lambda) + 3\alpha\xi_S^2 - 5\beta\xi_S^4 = 0 \quad (6b)$$

Where λ_S and ξ_S are the static buckling load and the displacement at static buckling respectively. The solution of (6b) is

$$\xi_S^2 = \frac{-3\alpha + \sqrt{9\alpha^2 + 20\beta(1-\lambda_S)}}{10\beta} = \frac{3\alpha(R_1 - 1)}{10\beta} \quad (6c)$$

where the positive square root has been taken in (6c), and

$$R_1(\lambda_S) = 1 + \frac{20(1-\lambda_S)}{9}\left(\frac{\beta}{\alpha^2}\right) \quad (6d)$$

$$\therefore \quad \xi_S = \sqrt{\frac{3}{10}}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}(R_1 - 1)^{\frac{1}{2}} \quad (6e)$$

To obtain the static buckling load λ_S, the procedure is to first multiply equation (5) by 5 and get

$$\xi_S[5(1-\lambda_S) + 5\alpha\xi_S^2 - 5\beta\xi_S^4] = 5\lambda_S\bar{\xi} \quad (6f)$$

Making $5\beta\xi_S^4$ the subject in (6b) and substituting same in (6f), gives

$$5\lambda_S\bar{\xi} = 2\xi_S[2(1-\lambda_S) + \alpha\xi_S^2]$$

On simplification, this gives

$$5\lambda_S\bar{\xi} = 2\sqrt{\frac{3}{10}}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}(R_1 - 1)^{\frac{1}{2}}(1-\lambda_S)R_2 \quad (6g)$$

where

$$R_2 = 1 + \frac{3(R_1 - 1)}{(1-\lambda_S)}\left(\frac{\alpha^2}{\beta}\right) \quad (6h)$$

Equations (6g, h) evaluate λ_S implicitly.

4. SOLUTION OF (2a, b)

Let

$$\tau = \bar{\xi}^2 t \quad (7a)$$

and now assume a time scale $\bar{t}$, such that

$$\frac{d\bar{t}}{dt} = \left(1 - \lambda f(\bar{\xi}^2 t)\right)^{1/2} = \left(1 - \lambda f(\tau)\right)^{1/2} \quad (7b)$$

Further let

$$\xi(t) = \eta(\bar{t}, \tau) \quad (7c)$$

$$\therefore \quad \frac{d\xi}{dt} = \frac{\partial\eta}{\partial\bar{t}}\frac{d\bar{t}}{dt} + \frac{\partial\eta}{\partial\tau}\frac{d\tau}{dt} = (1-\lambda f)^{\frac{1}{2}}\eta_{,\bar{t}} + \bar{\xi}^2\eta_{,\tau} \quad (7d)$$

Here, a subscript following a comma denotes partial differentiation. It follows that

$$\frac{d^2\xi}{dt^2} = (1-\lambda f)\eta_{,\bar{t}\bar{t}} + 2\bar{\xi}^2(1-\lambda f)^{\frac{1}{2}}\eta_{,\bar{t}\tau} + \bar{\xi}^4\eta_{,\tau\tau} - \frac{\lambda f'\bar{\xi}^2\eta_{,\bar{t}}}{2(1-\lambda f)^{\frac{1}{2}}} \quad (8)$$

where $(\quad)' = \frac{d(\dots)}{d\tau}$. Substituting in (2a) results to

$$(1-\lambda f)\eta_{,\bar{t}\bar{t}} + 2\bar{\xi}^2(1-\lambda f)^{\frac{1}{2}}\eta_{,\bar{t}\tau} + \bar{\xi}^4\eta_{,\tau\tau} - \frac{\lambda f'\bar{\xi}^2\eta_{,\bar{t}\tau}}{2(1-\lambda f)^{\frac{1}{2}}} + (1-\lambda f)\eta + \alpha\eta^3 - \beta\eta^5 = \lambda\bar{\xi}f(\tau) \quad (9)$$

Let

$$\eta(\bar{t}\tau) = \sum_{i=1}^{\infty}\eta^i(t,\tau)\,\bar{\xi}^i \quad (10)$$

The following are obtained on equating coefficients of orders of $\bar{\xi}$

$$O(\bar{\xi}): \quad \eta^{(1)}_{,\bar{t}\bar{t}} + \eta^{(1)} = B(\tau) = \frac{\lambda f(\tau)}{1-\lambda f(\tau)} \tag{11}$$

$$O(\bar{\xi}^3): \quad \eta^{(3)}_{,\bar{t}\bar{t}} + \eta^{(3)} = -2(1-\lambda f)^{-\frac{1}{2}}\eta^{(1)}_{,\bar{t}\tau} - \alpha(1-\lambda f)^{-1}{\eta^{(1)}}^3 + \frac{\lambda f\prime \eta^{(1)}_{,\bar{t}}}{2(1-\lambda f)^{\frac{3}{2}}} \tag{12}$$

$$O(\bar{\xi}^5): \quad \eta^{(5)}_{,\bar{t}\bar{t}} + \eta^{(5)} = -2(1-\lambda f)^{-\frac{1}{2}}\eta^{(3)}_{,\bar{t}\tau} - (1-\lambda f)^{-1}\eta^{(1)}_{,\tau\tau} + \frac{\lambda f' \eta^{(3)}_{,\bar{t}}}{2(1-\lambda f)^{\frac{3}{2}}}$$
$$-3\alpha(1-\lambda f)^{-1}{\eta^{(1)}}^2\eta^{(3)} - \beta(1-\lambda f)^{-1}{\eta^{(1)}}^5 \tag{13}$$

etc.

The initial conditions are

$$O(\bar{\xi}): \quad \eta^{(i)}(0,0) = 0 \ \ \forall i = 1,3,5,\dots \tag{14a}$$

$$\eta^{(1)}_{,\bar{t}}(0,0) = 0 \tag{14b}$$

$$O(\bar{\xi}^3): \quad \eta^{(3)}(0,0) + (1-\lambda f)^{-\frac{1}{2}}\eta^{(1)}_{,\tau}(0,0) = 0 \tag{14c}$$

$$O(\bar{\xi}^5): \quad \eta^{(5)}_{,\bar{t}}(0,0) + (1-\lambda f)^{-\frac{1}{2}}\eta^{(3)}_{,\tau}(0,0) = 0 \tag{14d}$$

etc.

Solving (11) with (14a), for $i = 1$, results to

$$\eta^{(1)}(\,\bar{t}\tau) = \alpha_1(\tau)\cos\bar{t} + \beta_1(\tau)\sin\bar{t} + B(\tau) \tag{15a}$$

$$\alpha_1(0) = -B(0) = -\frac{\lambda}{1-\lambda}, \quad \beta_1(0) = 0 \tag{15b}$$

Before substituting into (12) it is necessary to first simplify ${\eta^{(1)}}^3$ such that

$${\eta^{(1)}}^3 = (\alpha_1\cos\bar{t} + \beta_1\sin\bar{t} + B)^3$$
$$= \left(B^3 + \frac{3B\beta_1^2}{2} + \frac{3B\alpha_1^2}{2}\right) + \left(\frac{3a_1b_1^2}{4} + \frac{3\alpha_1^3}{4} + 3a_1B^2\right)\cos\bar{t} + \left(\frac{\alpha_1^2\beta_1}{4} + \frac{\beta_1^3}{4} + \beta_1B^2\right)\sin\bar{t}$$
$$+3\alpha_1\beta_1B\sin 2\bar{t} + \frac{3B}{2}(\alpha_1^2 - \beta_1^2)\cos 2\bar{t} + \left(\frac{\alpha_1^3}{4} - \frac{3\alpha_1\beta_1^2}{2}\right)\cos 3\bar{t} - \frac{\beta_1^3}{4}\sin 3\bar{t} \tag{16}$$

The substitution of (16) into (12) and subsequent simplification yields

$$\eta^{(3)}_{,\bar{t}\bar{t}} + \eta^{(3)} = 2(1-\lambda f)^{-\frac{1}{2}}(\alpha_1{}'\sin\bar{t} - \beta_1{}'\cos\bar{t}) + \frac{\lambda f\prime}{2(1-\lambda f)^{\frac{3}{2}}}(-\alpha_1\sin\bar{t} + \beta_1\cos\bar{t})$$
$$-\alpha(1-\lambda f)^{-1}\left[\left(B^3 + \frac{3B\beta_1^2}{2} + \frac{3B\alpha_1^2}{2}\right) + \left(\frac{3a_1b_1^2}{4} + \frac{3\alpha_1^3}{4} + 3\alpha_1B^2\right)\cos\bar{t} + \left(\frac{\alpha_1^2\beta_1}{4} + \frac{\beta_1^3}{4} + \beta_1B^2\right)\sin\bar{t}\right.$$
$$\left.+3\alpha_1\beta_1B\sin 2\bar{t} + \frac{3B}{2}(\alpha_1^2 - \beta_1^2)\cos 2\bar{t} + \left(\frac{\alpha_1^3}{4} - \frac{3\alpha_1\beta_1^2}{2}\right)\cos 3\bar{t} - \frac{\beta_1^3}{4}\sin 3\bar{t}\right] \tag{17}$$

To ensure a uniformly valid solution in $\bar{t}$, there is the need to equate to zero in (17), the coefficients of $\cos\bar{t}$ and $\sin\bar{t}$ and get, for $\cos\bar{t}$:

$$\beta_1' = \frac{\lambda f\prime \beta_1}{4(1-\lambda f)} = -\left(\frac{3\alpha\alpha_1^3}{8} + \frac{3\alpha_1\beta_1^2}{4} + \frac{3\alpha_1B^2}{2}\right)(1-\lambda f)^{-\frac{1}{2}}. \tag{18a}$$

For $\sin\bar{t}$:

$$\alpha_1' = \frac{\lambda f\prime \alpha_1}{4(1-\lambda f)} = 3\alpha(1-\lambda f)^{-\frac{1}{2}}\left(\frac{\alpha_1^2\beta_1}{4} + \frac{\beta_1^3}{4} + \beta_1B^2\right). \tag{18b}$$

The coupled equations (18a, b) need not be solved explicitly because the only terms needed of them are $\alpha_1'(0)$, $\beta_1'(0)$, $\alpha_1''(0)$ and $\beta_1''(0)$, all which can be evaluated from (18a, b) as follows:

$$\alpha_1'(0) = \frac{f'(0)B^2(0)}{4}, \quad \beta_1'(0) = \frac{15\alpha B^3(0)}{8(1-\lambda)^{\frac{1}{2}}} \tag{19a}$$

Similarly, the following terms are obtained

$$\alpha_1''(0) = \frac{-B^2(0)\varphi_2}{4(1-\lambda)} + \frac{45B^5(0)\alpha^2(1-\lambda)}{64} \quad (19b)$$

and where

$$\varphi_2 = (1-\lambda)f''(0) + \frac{5\lambda f'(0)}{4} \quad (19c)$$

$$\beta_1''(0) = \frac{-B^4(0)f'(0)\varphi_3}{(1-\lambda)^{\frac{1}{2}}} \quad (20a)$$

where

$$\varphi_3 = \frac{9\alpha}{32} + 3\left(\frac{1}{2} + \frac{1}{\lambda}\right) \quad (20b)$$

In passing, the following terms are worthy of note

$$B'(0) = \frac{B(0)f'(0)}{1-\lambda} \quad (21)$$

$$B''(0) = \frac{B(0)}{(1-\lambda)^2}\{(1-\lambda)f''(0) + 2\lambda f'^2(0)\} = B(0)\varphi_4 \quad (22a)$$

$$\varphi_4 = \frac{(1-\lambda)f''(0) + 2\lambda f'^2(0)}{(1-\lambda)^2} \quad (22b)$$

The remaining equation in (17) is

$$\eta^{(3)}_{,\bar{t}\bar{t}} + \eta^{(3)} = r_0(\tau) + r_1(\tau)\cos 2\bar{t} + r_2(\tau)\cos 3\bar{t} + r_3(\tau)\sin 3\bar{t} \quad (23a)$$

with

$$\eta^{(3)}(0,0) = 0, \qquad \eta^{(3)}_{,\bar{t}}(0,0) + (1-\lambda)^{\frac{1}{2}}\eta^{(1)}_{,\tau}(0,0) = 0 \quad (23b)$$

where

$$r_0(\tau) = \frac{-\alpha}{(1-\lambda f)}\left(B^3 + \frac{3B\beta_1^2}{2} + \frac{3B\alpha_1^2}{2}\right), \qquad r_1(\tau) = \frac{-3\alpha B}{2(1-\lambda f)}(\alpha_1^2 - \beta_1^2) \quad (24a)$$

$$r_2(\tau) = \frac{\alpha}{(1-\lambda f)}\left(\frac{\alpha_1^3}{4} - \frac{3\alpha_1\beta_1^2}{2}\right), \qquad r_3(\tau) = \frac{-\alpha\beta_1^3}{4(1-\lambda f)} \quad (24b)$$

$$r_0(0) = \frac{-5\alpha B^3(0)}{2(1-\lambda)}, \qquad r_1(0) = \frac{-3\alpha B^3(0)}{2(1-\lambda)} \quad (24c)$$

$$r_2(0) = \frac{-\alpha B^3(0)}{4(1-\lambda)}, \qquad r_3(0) = 0 \quad (24d)$$

Similarly, the following simplifications equally hold

$$r_0'(0) = \frac{B^4(0)f'(0)\alpha}{(1-\lambda)}\left(\frac{5}{2} - \frac{6}{\lambda}\right), \qquad r_1'(0) = \frac{-3\alpha B^4(0)f'(0)}{4\lambda(1-\lambda)}(2-\lambda-\lambda^2) \quad (25a)$$

$$r_2'(0) = \frac{\alpha B^4(0)f'(0)}{4(1-\lambda)}, \qquad r_3'(0) = 0 \quad (25b)$$

The solution of (23a, b) is

$$\eta^{(3)} = \alpha_3\cos\bar{t} + \beta_3\sin\bar{t} + r_0 - \frac{r_1\cos 2\bar{t}}{3} - \frac{r_2\cos 3\bar{t}}{8} - \frac{r_3\sin 3\bar{t}}{8} \quad (26a)$$

$$\Rightarrow \qquad \alpha_3(0) = r_0(0) + \frac{r_1(0)}{3} + \frac{r_3(0)}{8} = \frac{B^3(0)\alpha}{(1-\lambda)} \quad (26b)$$

$$\beta_3(0) = -(1-\lambda)^{-\frac{1}{2}}\left(\alpha_1'(0) + \beta_1'(0)\right) = \frac{-B^2(0)f'(0)}{4\lambda(1-\lambda)^{\frac{3}{2}}}(4-\lambda+\lambda^2) \quad (26c)$$

The substitution into (13) requires the following simplifications:

$$\begin{aligned}{\eta^{(1)}}^5 &= (\alpha_1\cos\bar{t} + \beta_1\sin\bar{t} + B)^5 \\ &= r_4 + r_5\cos\bar{t} + r_6\sin\bar{t} + r_7\cos 2\bar{t} + r_8\sin 2\bar{t} + r_9\cos 3\bar{t} + r_{10}\sin 3\bar{t} + r_{11}\cos 4\bar{t} \\ &\quad + r_{12}\sin 4\bar{t} + r_{13}\cos 5\bar{t} + r_{14}\sin 5\bar{t} \end{aligned} \quad (27a)$$

where

$$r_4 = \frac{15B\alpha_1^4}{2} + 5\alpha_1^2\left(\frac{3\beta_1^3}{4} + B^2\right) + \left(B^5 + 5B^3\beta_1^2 + \frac{15B\beta_1^4}{2}\right) \tag{27b}$$

$$r_5 = \frac{5\alpha_1^5}{8} + \frac{5\alpha_1^3}{2}\left(\frac{\beta_1^2}{2} + 3B^2\right) + 5\alpha_1\left(\frac{3\beta_1^4}{16} + \frac{3B^2\beta_1^4}{2} + B^4\right) \tag{27c}$$

$$r_6 = \frac{5\alpha_1^4\beta_1}{2} + 5\alpha_1^2\left(\frac{\beta_1^3}{4} + \frac{3B^2\beta_1}{2}\right) - \frac{5\alpha_1\beta_1^3B}{2} + \left(\frac{7\beta_1^5}{16} + 15B^2\beta_1^3\right) \tag{27d}$$

$$r_7 = 10B\alpha_1^4 + 5\alpha_1^2B^3 - (5\beta_1^4B + 5\beta_1^2B^3) \tag{27e}$$

$$r_8 = 5B\beta_1\alpha_1^3 + \frac{5\alpha_1}{2}(3\beta_1^3B + 4\beta_1B^3) \tag{27f}$$

$$r_9 = \frac{3\alpha_1^5}{8} + \frac{5\alpha_1^3}{2}\left(B^2 + \frac{\beta_1^2}{2}\right) - \frac{5\alpha_1}{2}\left(3\beta_1^2B + \frac{\beta_1^2}{2}\right) \tag{27g}$$

$$r_{10} = \frac{3\alpha_1^4\beta_1}{2} + 5\alpha_1^2\left(\frac{3\beta_1B^2}{2} - \frac{\beta_1^3}{8}\right) - \frac{5\alpha_1\beta_1^3B}{2} - \left(\frac{\beta_1^5}{8} + 5\beta_1^2B^2\right) \tag{27h}$$

$$r_{11} = 5\alpha_1^4B - \frac{15\alpha_1^2\beta_1^2B}{4} + \frac{5\beta_1^4B}{2} \tag{27i}$$

$$r_{12} = \frac{5\alpha_1^3\beta_1B}{2} \tag{27j}$$

$$r_{13} = \frac{\alpha_1^5}{8} - \frac{5\alpha_1^3\beta_1^2}{8} + \frac{5\alpha_1\beta_1^4}{16} \tag{27k}$$

$$r_{14} = \frac{\beta_1^5}{16} - \frac{5\alpha_1^4\beta_1}{2} - \frac{5\alpha_1^2\beta_1^3}{8} \tag{27l}$$

where

$$r_4(0) = \frac{27B^5(0)}{2}, \qquad r_5(0) = \frac{-105B^5(0)}{8}, \qquad r_6(0) = 0 \tag{27m}$$

$$r_7(0) = 15B^5(0), \qquad r_8(0) = 0, \qquad r_9(0) = \frac{-23B^5(0)}{8} \tag{27n}$$

$$r_{10}(0) = 0, \qquad r_{11}(0) = 5B^5(0), \qquad r_{12}(0) = 0 \tag{27o}$$

$$r_{13}(0) = \frac{-B^5(0)}{8}, \qquad r_{14}(0) = 0. \tag{27p}$$

Similarly, the expansion below to be substituted into equation (13) follows

$$\begin{aligned} {\eta^{(1)}}^2\eta^{(3)} &= (\alpha_1\cos\bar{t} + \beta_1\sin\bar{t} + B)^2\left(\alpha_3\cos\bar{t} + \beta_3\sin\bar{t} + r_0 - \frac{r_1\cos 2\bar{t}}{3} - \frac{r_3\cos 3\bar{t}}{8}\right) \\ &= r_{15} + r_{16}\cos\bar{t} + r_{17}\sin\bar{t} + r_{18}\cos 2\bar{t} + r_{19}\sin 2\bar{t} + r_{20}\cos 3\bar{t} + r_{21}\sin 3\bar{t} \\ &\quad + r_{22}\cos 4\bar{t} + r_{23}\sin 4\bar{t} + r_{24}\cos 5\bar{t} + r_{25}\sin 5\bar{t} \end{aligned} \tag{28a}$$

where

$$r_{15} = \left[\frac{(\alpha_1^2+\beta_1^2)}{2} + B^2\right]r_0 + B\alpha_1\alpha_3 + B\beta_1\beta_3 + 2B\beta_1r_0 + \frac{r_0(\alpha_1^2-\beta_1^2)}{2} - \frac{r_1(\alpha_1^2-\beta_1^2)}{2} \tag{28b}$$

$$r_{16} = \left[\frac{(\alpha_1^2+\beta_1^2)}{2} + B^2\right]\alpha_3 + 2B\alpha_1r_0 + \frac{B\alpha_1r_1}{2} - \frac{B\beta_1r_3}{8} + \frac{\alpha_3(\alpha_1^2-\beta_1^2)}{4} - \frac{r_2(\alpha_1^2-\beta_1^2)}{32} + \frac{\alpha_1\beta_1\beta_3}{2} \tag{28c}$$

$$r_{17} = \left[\frac{(\alpha_1^2+\beta_1^2)}{2} + B^3\right]\beta_3 + \frac{B\beta_1r_1}{3} + \frac{B\beta_1r_2}{8} + \frac{\beta_3(\alpha_1^2-\beta_1^2)}{4} - \frac{r_3(\alpha_1^2-\beta_1^2)}{32} + \frac{\alpha_1\beta_1\alpha_3}{2} + \frac{\alpha_1\beta_1r_2}{16} - \frac{\alpha_1\beta_1r_3}{16} \tag{28d}$$

$$r_{18} = -\frac{r_1}{3}\left[\frac{(\alpha_1^2+\beta_1^2)}{2} + B^3\right] + B\alpha_1\alpha_3 - \frac{B\alpha_1r_2}{8} - B\beta_1\beta_3 \tag{28e}$$

$$r_{19} = B\alpha_1\beta_3 - \frac{B\alpha_1r_3}{8} + B\beta_1\alpha_3 + \alpha_1\beta_1r_0 \tag{28f}$$

$$r_{20} = -\frac{r_2}{8}\left[\frac{(\alpha_1^2+\beta_1^2)}{2} + B^2\right] + \frac{B\alpha_1r_1}{3} + \frac{B\beta_1r_3}{8} + \frac{\alpha_3(\alpha_1^2-\beta_1^2)}{4} - \frac{\alpha_1\beta_1\beta_3}{2} \tag{28g}$$

$$r_{21} = -\frac{r_3}{8}\left[\frac{(\alpha_1^2+\beta_1^2)}{2} + B^2\right] - \frac{B\beta_1r_1}{3} + \frac{\beta_3(\alpha_1^2-\beta_1^2)}{4} + \frac{\alpha_1\beta_1\alpha_3}{2} - \frac{\alpha_1\beta_1r_2}{16} \tag{28h}$$

$$r_{22} = -\frac{B\alpha_1r_2}{8} - \frac{r_1(\alpha_1^2-\beta_1^2)}{12} \tag{28i}$$

$$r_{23} = -\left[\frac{B\alpha_1r_3}{8} + \frac{B\beta_1r_2}{8} - \frac{\alpha_1\beta_1r_1}{6}\right] \tag{28j}$$

$$r_{24} = -\frac{r_2(\alpha_1^2-\beta_1^2)}{32} \quad (28k)$$

$$r_{25} = -\frac{r_3(\alpha_1^2-\beta_1^2)}{32} - \frac{\alpha_1\beta_1 r_3}{16} \quad (28l)$$

where

$$r_{15}(0) = \frac{27\alpha B^5(0)}{4(1-\lambda)}, \qquad r_{16}(0) = \frac{17\alpha B^5(0)}{4(1-\lambda)}, \qquad r_{17}(0) = \frac{-7B^4(0)f\prime(0)(4-\lambda+\lambda^2)}{16\lambda(1-\lambda)^{\frac{3}{2}}} \quad (29a)$$

$$r_{18}(0) = \frac{-\alpha B^5(0)}{4(1-\lambda)}, \qquad r_{19}(0) = \frac{B^5(0)f\prime(0)(4-\lambda+\lambda^2)}{4\lambda^2(1-\lambda)^{\frac{1}{2}}}, \qquad r_{20}(0) = \frac{-13\alpha B^5(0)}{64(1-\lambda)} \quad (29b)$$

$$r_{21}(0) = \frac{-B^4(0)f\prime(0)(4-\lambda+\lambda^2)}{32\lambda(1-\lambda)^{\frac{3}{2}}}, \quad r_{23}(0) = 0, \quad r_{24}(0) = \frac{\alpha B^5(0)}{128(1-\lambda)}, \quad r_{23}(0) = 0 \quad (29c)$$

Now, substituting into (13), results to

$$\begin{aligned}\eta^{(5)}_{,\bar{t}\bar{t}} + \eta^{(5)} = &-2(1-\lambda f)^{-\frac{1}{2}}\left[-\alpha_3{}'\sin\bar{t} + \beta_3{}'\cos\bar{t} - \frac{2r_1'\sin 2\bar{t}}{3} + \frac{3r_2'\sin 3\bar{t}}{8} - \frac{3r_3'\cos 3\bar{t}}{8}\right] \\ &+ \frac{\lambda f\prime}{2(1-\lambda f)^{\frac{3}{2}}}\left(-\alpha_3\sin\bar{t} + \beta_3\cos\bar{t} - \frac{2r_1\sin 2\bar{t}}{3} + \frac{3r_2\sin 3\bar{t}}{8} - \frac{3r_3\cos 3\bar{t}}{8}\right) \\ &-3\alpha(1-\lambda f)^{-1}[r_{15} + r_{16}\cos\bar{t} + r_{17}\sin\bar{t} + r_{18}\cos 2\bar{t} + r_{19}\sin 2\bar{t} + r_{20}\cos 3\bar{t} \\ &+r_{21}\sin 3\bar{t} + r_{22}\cos 4\bar{t} + r_{23}\sin 4\bar{t} + r_{24}\cos 5\bar{t} + r_{25}\sin 5\bar{t}] \\ &-\beta(1-\lambda f)^{-1}[r_4 + r_5\cos\bar{t} + r_6\sin\bar{t} + r_7\cos 2\bar{t} + r_8\sin 2\bar{t} + r_9\cos 3\bar{t} + r_{10}\sin 3\bar{t} \\ &+r_{11}\cos 4\bar{t} + r_{12}\sin 4\bar{t} + r_{13}\cos 5\bar{t} + r_{14}\sin 5\bar{t}]\end{aligned} \quad (30a)$$

$$\eta^{(5)}(0,0) = 0, \qquad \eta^{(5)}_{,\bar{t}}(0,0) + (1-\lambda)^{\frac{1}{2}}\eta^{(3)}_{,\tau}(0,0) = 0 \quad (30b)$$

Ensuring a uniformly valid solution in $\bar{t}$, needs equating to zero in (30a), the coefficients of $\cos\bar{t}$ and $\sin\bar{t}$ and get,

for $\cos\bar{t}$:

$$\beta_3' - \frac{\lambda f\prime\beta_3}{4(1-\lambda f)} = -\frac{(1-\lambda f)^{-\frac{1}{2}}}{2}(\alpha_1'' + 3\alpha r_{16} - \beta r_5). \quad (30c)$$

For $\sin\bar{t}$:

$$\alpha_3' - \frac{\lambda f\prime\alpha_3}{4(1-\lambda f)} = \frac{(1-\lambda f)^{-\frac{1}{2}}}{2}(\beta_1'' + 3\alpha r_7 - \beta r_6). \quad (30d)$$

Equations (30c, d) are coupled equations but, fortunately, may not be solved explicitly because only $\beta_3'(0)$ and $\alpha_3'(0)$ are needed, which can be obtained easily.

Thus, it follows from (30c) that

$$\beta_3'(0) = \frac{\lambda f\prime\beta_3(0)}{4(1-\lambda)} - \frac{(1-\lambda)^{-\frac{1}{2}}}{2}(\alpha_1'' + 3\alpha r_{16} - \beta r_5)|_{\tau=0}.$$

Without further simplification, it is seen that

$$\beta_3'(0) = \frac{-B^3(0)f\prime^2(0)(4-\lambda+\lambda^2)}{16\lambda(1-\lambda)^{\frac{3}{2}}} - \frac{(1-\lambda)^{-\frac{1}{2}}}{2}\left(\alpha_1''(0) + \frac{51\alpha B^5(0)}{4(1-\lambda)} + \frac{105\beta B^5(0)}{8}\right) \quad (31a)$$

Similarly, it is seen that

$$\alpha_3'(0) = \frac{\lambda f'(0)\alpha_3(0)}{4(1-\lambda)} + \frac{(1-\lambda)^{-\frac{1}{2}}}{2}(\beta_1'' + 3\alpha r_7 - \beta r_5)|_{\tau=0}$$

This gives

$$\alpha_3'(0) = \alpha B^3(0)\left[\frac{B(0)f'(0)}{4(1-\lambda)} + \frac{3(1-\lambda)^{-\frac{1}{2}}}{2}\left(f'(0)\varphi_3 + 15B^2(0)\right)\right] \quad (31b)$$

$$= \frac{\alpha B^3(0)\varphi_{13}}{4(1-\lambda)}, \quad \varphi_{13} = f'(0) + \frac{6(1-\lambda)^{-\frac{1}{2}}}{B(0)}\left(f'(0)\varphi_3 + 15B^2(0)\right) \tag{31c}$$

The remaining equation in (30a) is

$$\eta^{(5)}_{,\bar{t}\bar{t}} + \eta^{(5)} = r_{26} + r_{27}\cos 2\bar{t} + r_{28}\sin 2\bar{t} + r_{29}\cos 3\bar{t} + r_{30}\sin 3\bar{t} + r_{31}\cos 4\bar{t} + r_{32}\sin 4\bar{t}$$
$$+ r_{33}\cos 5\bar{t} + r_{34}\sin 5\bar{t} \tag{32a}$$

$$\eta^{(5)}(0,0) = 0, \qquad \eta^{(5)}_{,\bar{t}}(0,0) + (1-\lambda)^{\frac{1}{2}}\eta^{(3)}_{,\tau}(0,0) = 0 \tag{32b}$$

where

$$r_{26} = -B''(1-\lambda f)^{-1} - 3\alpha(1-\lambda f)^{-1}r_{15} + \beta(1-\lambda f)^{-1}r_4 \tag{32c}$$
$$r_{27} = -3\alpha(1-\lambda f)^{-1}r_{18} + \beta(1-\lambda f)^{-1}r_7 \tag{32d}$$
$$r_{28} = 2(1-\lambda)^{-\frac{1}{2}}\left(\frac{2r_1'}{3}\right) - \frac{\lambda f'}{2(1-\lambda f)^{\frac{3}{2}}}\left(\frac{2r_1}{3}\right) - 3\alpha(1-\lambda)^{-1}r_{19} + \beta(1-\lambda)^{-1}r_8 \tag{32e}$$
$$r_{29} = 2(1-\lambda)^{-\frac{1}{2}}\left(\frac{3r_3'}{8}\right) - \frac{\lambda f'}{2(1-\lambda f)^{\frac{3}{2}}}\left(\frac{3r_3}{8}\right) - 3\alpha(1-\lambda)^{-1}r_{20} + \beta(1-\lambda)^{-1}r_9 \tag{32f}$$
$$r_{30} = -2(1-\lambda)^{-\frac{1}{2}}\left(\frac{3r_2'}{8}\right) - \frac{\lambda f'}{2(1-\lambda f)^{\frac{3}{2}}}\left(\frac{3r_2}{8}\right) - 3\alpha(1-\lambda)^{-1}r_{21} + \beta(1-\lambda)^{-1}r_{10} \tag{32g}$$
$$r_{31} = -3\alpha(1-\lambda f)^{-1}r_{22} + \beta(1-\lambda f)^{-1}r_{11} \tag{32h}$$
$$r_{32} = (1-\lambda f)^{-1}(-3\alpha r_{23} + \beta r_{12}) \tag{32i}$$
$$r_{33} = (1-\lambda f)^{-1}(-3\alpha r_{24} + \beta r_{13}) \tag{32j}$$
$$r_{34} = (1-\lambda f)^{-1}(-3\alpha r_{25} + \beta r_{14}) \tag{32k}$$
$$r_{26}(0) = \frac{27\beta B^5(0)\varphi_5}{2(1-\lambda)}, \quad \varphi_5 = 1 - \frac{2}{27\beta}\left(\frac{81\alpha^2}{4} + \frac{\varphi_4}{B^4(0)}\right) \tag{32l}$$
$$r_{27}(0) = \frac{15\beta B^5(0)\varphi_6}{(1-\lambda)}, \quad \varphi_6 = \left(1 + \frac{\alpha^2}{20\beta(1-\lambda)}\right) \tag{32m}$$
$$r_{28}(0) = B^4(0)\alpha\varphi_7, \quad \varphi_7 = f'(0)\left(\frac{1}{2(1-\lambda f)^{\frac{3}{2}}} + \frac{\lambda^2+\lambda-2}{\lambda(1-\lambda f)^{\frac{3}{2}}} - \frac{3B(0)}{1-\lambda}\right) \tag{32n}$$
$$r_{29}(0) = B^5(0)\beta\varphi_8, \quad \varphi_8 = \frac{1}{1-\lambda}\left(\frac{39\alpha}{64\beta} - \frac{23}{8}\right) \tag{32o}$$
$$r_{30}(0) = \frac{-B^4(0)\alpha f'(0)\varphi_9}{(1-\lambda)^{\frac{3}{2}}}, \quad \varphi_9 = \frac{15}{64} + \left(\frac{4-\lambda+\lambda^2}{32\lambda(1-\lambda)}\right) \tag{32p}$$
$$r_{31}(0) = \frac{5\beta B^5(0)\varphi_{10}}{(1-\lambda)}, \quad \varphi_{10} = \left(1 - \frac{3\alpha^2}{40\beta}\right) \tag{32q}$$
$$r_{32}(0) = 0 \tag{32r}$$
$$r_{33}(0) = \frac{-5\beta B^5(0)\varphi_{11}}{(1-\lambda)}, \quad \varphi_{11} = \left(\frac{1}{8} + \frac{3\alpha^2}{128\beta(1-\lambda)}\right) \tag{32s}$$
$$r_{34}(0) = 0 \tag{32t}$$

The remaining equation (32a) together with (32b) is now solved to get

$$\eta^{(5)}(\bar{t},\tau) = r_{26} + \alpha_5\cos\bar{t} + \beta_5\sin\bar{t} - \frac{1}{3}(r_{27}\cos 2\bar{t} + r_{28}\sin 2\bar{t}) - \frac{1}{8}(r_{29}\cos 3\bar{t} + r_{30}\sin 3\bar{t})$$
$$- \frac{1}{15}(r_{31}\cos 4\bar{t} + r_{32}\sin 4\bar{t}) - \frac{1}{24}(r_{33}\cos 5\bar{t} + r_{34}\sin 5\bar{t}) \tag{33a}$$

where

$$\alpha_5(0) = \left[\frac{r_{27}}{3} + \frac{r_{29}}{8} + \frac{r_{31}}{15} + \frac{r_{33}}{24}\right]\Big|_{\tau=0} - r_{26}(0) \tag{33b}$$
$$\beta_5(0) = \left[\frac{2r_{28}}{3} + \frac{3r_{30}}{8} + \frac{4r_{32}}{15} + \frac{5r_{33}}{24}\right]\Big|_{\tau=0} - (1-\lambda)^{-\frac{1}{2}}\left[\alpha_3'(0) + r_0'(0) - \frac{r_1'(0)}{3} - \frac{r_2'(0)}{8}\right] \tag{33c}$$

So far, the displacement can be written as

$$\eta(\bar{t},\tau) = \eta^{(1)}\bar{\xi} + \eta^{(3)}\bar{\xi}^3 + \eta^{(5)}\bar{\xi}^5 + \cdots$$

where

$$\eta^{(i)} = \eta^{(i)}(\bar{t},\tau)$$

5. CRITICAL VALUES OF DEPENDENT VARIABLES AT MAXIMUM DISPLACEMENT

In order to determine the maximum displacement η_a, there is the need to first determine the values of $\bar{t}, t$ and τ at maximum displacement. Let $\bar{t}_a, t_a$ and τ_a be the respective values of $\bar{t}, t$ and τ at maximum displacement and let them be expanded asymptotically as follows:

$$\bar{t}_a = \bar{t}_0 + \bar{\xi}^2\bar{t}_2 + \bar{\xi}^4\bar{t}_4 + \cdots \quad (35a)$$

$$t_a = t_0 + \bar{\xi}^2 t_2 + \bar{\xi}^4 t_4 + \cdots \quad (35b)$$

$$\tau_a = \bar{\xi}^2 t_a = \bar{\xi}^2\left[t_0 + \bar{\xi}^2 t_2 + \bar{\xi}^4 t_4 + \cdots\right] \quad (35c)$$

Following (7c), the condition for maximum displacement is

$$\eta_{,\bar{t}} + \bar{\xi}^2(1-\lambda f)^{-\frac{1}{2}}\eta_{,\tau} = 0 \quad (36)$$

By substituting (34) into (36), it is easily seen that the expansion of each term in (36) will be as follows:

$$\bar{\xi}\eta^{(1)}_{,\bar{t}}(\bar{t}_0,0) = \bar{\xi}\Big[\eta^{(1)}_{,\bar{t}} + \left(\bar{\xi}^2 t_2 + \bar{\xi}^4 t_4 + \cdots\right)\eta^{(1)}_{,\bar{t}\,\bar{t}} + \bar{\xi}^2\left(t_0 + \bar{\xi}^2 t_2 + \cdots\right)\eta^{(1)}_{,\bar{t}\tau}$$
$$+\frac{1}{2}\Big\{\left(\bar{\xi}^2\,\bar{t}_2 + \bar{\xi}^4\,\bar{t}_4 + \cdots\right)^2\eta^{(1)}_{,\bar{t}\,\bar{t}\,\bar{t}} + 2\bar{\xi}^2\left(\bar{\xi}^2\,\bar{t}_2 + \cdots\right)\left(t_0 + \bar{\xi}^2 t_2 + \cdots\right)\eta^{(1)}_{,\bar{t}\,\bar{t}\,\tau}$$
$$+\bar{\xi}^4\left(t_0 + \bar{\xi}^2 t_2 + \cdots\right)\eta^{(1)}_{,\bar{t}\tau\tau} + \cdots\Big\} + \cdots\Big] \quad (37a)$$

$$\bar{\xi}^3\eta^{(3)}_{,\bar{t}}(\bar{t}_0,0) = \bar{\xi}^3\Big[\eta^{(3)}_{,\bar{t}} + \left(\bar{\xi}^2\bar{t}_2 + \bar{\xi}^4\bar{t}_4 + \cdots\right)\eta^{(3)}_{,\bar{t}\,\bar{t}} + \bar{\xi}^2\left(t_0 + \bar{\xi}^2 t_2 + \cdots\right)\eta^{(3)}_{,\bar{t}\tau}$$
$$+\frac{1}{2}\Big\{\left(\bar{\xi}^2\,\bar{t}_2 + \bar{\xi}^4\,\bar{t}_4 + \cdots\right)^2\eta^{(3)}_{,\bar{t}\,\bar{t}\,\bar{t}} + 2\bar{\xi}^2\left(\bar{\xi}^2\,\bar{t}_2 + \cdots\right)\left(t_0 + \bar{\xi}^2 t_2 + \cdots\right)\eta^{(3)}_{,\bar{t}\,\bar{t}\,\tau}$$
$$+\bar{\xi}^4\left(t_0 + \bar{\xi}^2 t_2 + \cdots\right)\eta^{(3)}_{,\bar{t}\tau\tau} + \cdots\Big\} + \cdots\Big] \quad (37b)$$

$$\bar{\xi}^5\eta^{(5)}_{,\bar{t}}(\bar{t}_0,0) = \bar{\xi}^5\left[\eta^{(5)}_{,\bar{t}} + \cdots\right] \quad (37c)$$

$$\bar{\xi}^3(1-\lambda f)^{-\frac{1}{2}}\eta^{(1)}_{,\tau} = \bar{\xi}^3\Big[(1-\lambda)^{-\frac{1}{2}}\eta^{(1)}_{,\tau} + (1-\lambda)^{-\frac{1}{2}}\eta^{(1)}_{,\tau\,\bar{t}}\left\{\bar{\xi}^2\bar{t}_2 + \cdots\right\}$$
$$+\bar{\xi}^2\left\{(1-\lambda f)^{-\frac{1}{2}}\eta^{(1)}_{,\tau}\right\}_{,\tau}\left\{t_0 + \bar{\xi}^2 t_2 + \bar{\xi}^4 t_4 + \cdots\right\} + \cdots\Big] \quad (37d)$$

$$\bar{\xi}^5(1-\lambda f)^{-\frac{1}{2}}\eta^{(3)}_{,\tau} = \bar{\xi}^5\Big[(1-\lambda)^{-\frac{1}{2}}\eta^{(3)}_{,\tau} + (1-\lambda)^{-\frac{1}{2}}\eta^{(3)}_{,\tau\,\bar{t}}\left\{\bar{\xi}^2\bar{t}_2 + \cdots\right\}$$
$$+\bar{\xi}^2\left\{(1-\lambda f)^{-\frac{1}{2}}\eta^{(3)}_{,\tau}\right\}_{,\tau}\left\{t_0 + \bar{\xi}^2 t_2 + \bar{\xi}^4 t_4 + \cdots\right\} + \cdots\Big] + \cdots \quad (37e)$$

The terms in (37a – e), which are equated at $(\bar{t}_0, 0)$, are next substituted into (36) and the coefficients of powers of $\bar{\xi}$ are equated to zero. The following equations, in orders of $\bar{\xi}$, are obtained

$$O(\bar{\xi}): \quad \eta^{(1)}_{,\bar{t}}(\bar{t}_0,0) = 0 \quad (38a)$$

$$O(\bar{\xi}^3): \quad \bar{t}_2\eta^{(1)}_{,\bar{t}\bar{t}} + t_0\eta^{(1)}_{,\bar{t}\tau} + \eta^{(3)}_{,\bar{t}} + (1-\lambda)^{-\frac{1}{2}}\eta^{(1)}_{,\tau} = 0 \quad (38b)$$

$$O(\bar{\xi}^5): \quad \bar{t}_4\eta^{(1)}_{,\bar{t}\bar{t}} + t_2\eta^{(1)}_{,\bar{t}\tau} + \frac{1}{2}\left\{\bar{t}_2^2\eta^{(1)}_{,\bar{t}\bar{t}\bar{t}} + 2\bar{t}_2 t_0\eta^{(1)}_{,\bar{t}\bar{t}\tau} + \bar{t}_0^2\eta^{(1)}_{,\bar{t}\tau\tau}\right\} + \bar{t}_2\eta^{(3)}_{,\bar{t}\bar{t}} + t_0\eta^{(3)}_{,\bar{t}\tau} + \eta^{(5)}_{,\bar{t}}$$
$$+(1-\lambda)^{-\frac{1}{2}}\,\bar{t}_2\eta^{(1)}_{,\tau t} + t_0\left\{(1-\lambda)^{-\frac{1}{2}}\eta^{(1)}_{,\tau}\right\}_{,\tau} = 0 \quad (38c)$$

etc.

From (38a), it is seen that

$$\sin\bar{t}_0 = 0, \quad \therefore \quad \bar{t}_0 = \pi$$

where the least nontrivial value of $\bar{t}_0$ has been taken.
It follow from (38b) that

$$\bar{t}_2 = \frac{-\left\{\eta_{,\bar{t}}^{(3)} + (1-\lambda)^{-\frac{1}{2}}\eta_{,\tau}^{(1)} + t_0\eta_{,\bar{t}\tau}^{(1)}\right\}}{\eta_{,\bar{t}\bar{t}}^{(1)}} \tag{38d}$$

where

$$\eta_{,\bar{t}\tau}^{(1)}(\bar{t}_0,0) = 0, \quad \eta_{,\tau}^{(1)}(\bar{t}_0,0) = \frac{B(0)f'(0)(\lambda+4)}{4(1-\lambda)}, \quad \eta_{,\bar{t}\bar{t}}^{(1)}(\bar{t}_0,0) = -B(0), \quad \eta_{,\bar{t}}^{(3)}(\bar{t}_0,0) = 0$$

$$\therefore \quad \bar{t}_2 = \frac{f'(0)(\lambda+4)}{4(1-\lambda)^{\frac{3}{2}}} \tag{38e}$$

Most terms in (38c) vanish on evaluating them but on substitution, the final simplification is

$$\bar{t}_4 = \left.\frac{-\left[2\bar{t}_2 t_0\eta_{,\bar{t}\bar{t}\tau}^{(1)} + \bar{t}_2\eta_{,\bar{t}\bar{t}}^{(3)} + t_0\left\{(1-\lambda)^{-\frac{1}{2}}\eta_{,\tau}^{(1)}\right\}_{,\tau}\right]}{\eta_{,\bar{t}\bar{t}}^{(1)}}\right|_{\tau=0} \tag{38f}$$

The following terms are easily evaluated

$$\eta_{,\bar{t}\bar{t}}^{(3)}(\bar{t}_0,0) = \frac{-87\alpha B^3(0)}{32(1-\lambda)}, \quad \eta_{,\bar{t}\bar{t}\tau}^{(1)}(\bar{t}_0,0) = \frac{-B^2(0)f'(0)}{4}, \quad \eta_{,\tau\tau}^{(1)}(\bar{t}_0,0) = \left(B''(0) - \alpha_1''(0)\right)$$

On substituting in (38f) and simplifying for $\bar{t}_4$, the following is obtained

$$\bar{t}_4 = \left[\frac{B(0)f'(0)(\lambda+4)}{128(1-\lambda)^{\frac{3}{2}}} + \left(t_0 - \frac{87\alpha}{1-\lambda}\right) + \frac{(1-\lambda)^{\frac{1}{2}}}{\lambda}\left(B''(0) - \alpha_1''(0)\right)\right] \tag{38g}$$

where t_0 is yet to be determined and $B''(0)$ is as in (22a, b), while $\alpha_1''(0)$ is as in (19b, c).

6. MAXIMUM DISPLACEMENT η_a

Using (35a – c), the maximum displacement η_a will now be obtained using the critical values of the independent variables already obtained and where $\eta_a = \eta(\bar{t}_a, \tau_a)$.
Expansion of each component of η_a in the following series, gives

$$\bar{\xi}\eta_a^{(1)} = \bar{\xi}\eta^{(1)}(\bar{t}_a,\tau_a) = \bar{\xi}\left[\eta^{(1)} + \left(\bar{\xi}^2\bar{t}_2 + \bar{\xi}^4\bar{t}_4 + \cdots\right)\eta_{,\bar{t}}^{(1)} + \bar{\xi}^2\left(t_0 + \bar{\xi}^2 t_2 + \bar{\xi}^4 t_4 + \cdots\right)\eta_{,\tau}^{(1)} + \cdots\right]\Big|_{(\bar{t}_0,0)} \tag{39a}$$

$$\bar{\xi}^3\eta_a^{(3)} = \bar{\xi}^3\eta^{(1)}(\bar{t}_a,\tau_a) = \bar{\xi}^3\left[\eta^{(3)} + \left(\bar{\xi}^2\bar{t}_2 + \bar{\xi}^4\bar{t}_4 + \cdots\right)\eta_{,\bar{t}}^{(3)} + \bar{\xi}^2\left(t_0 + \bar{\xi}^2 t_2 + \bar{\xi}^4 t_4 \ldots\right)\eta_{,\tau}^{(3)} + \cdots\right]\Big|_{(\bar{t}_0,0)} \tag{39b}$$

$$\bar{\xi}^5\eta_a^{(5)} = \bar{\xi}^5\eta^{(2)}(\bar{t}_0,0) + \cdots \tag{39c}$$

On substituting (39a – c) into (34), evaluated at maximum values of the independent variables, the non-vanishing values of η_a is obtained as follows

$$\eta_a = \bar{\xi}\eta^{(1)}(\bar{t}_0,0) + \bar{\xi}^3\left[t_0\eta_{,\tau}^{(1)} + \eta^{(3)}\right] + \bar{\xi}^5\left[t_2\eta_{,\tau}^{(1)} + \frac{\bar{t}_2^2}{2}\eta_{,\bar{t}\bar{t}}^{(1)} + \frac{t_0^2}{2}\eta_{,\tau\tau}^{(1)} + t_0\eta_{,\tau}^{(3)} + \eta^{(5)}\right] + \cdots \tag{40}$$

where (40) is to be evaluated at $(\bar{t}_0, 0)$.
To determine t_0, it is noted from (7b), that

$$\frac{d\bar{t}}{dt} = (1-\lambda f(\tau))^{\frac{1}{2}} = (1-\lambda f(\bar{\xi}^2 t))^{\frac{1}{2}} = \left[(1-\lambda) - \lambda\left(f'(0)\bar{\xi}^2 t + \frac{f''(0)\bar{\xi}^2 t^2}{2} + \cdots\right)\right]^{\frac{1}{2}}$$

$$= (1-\lambda)^{\frac{1}{2}}\left[1 - \frac{\lambda}{2(1-\lambda)}\left(f'(0)\bar{\xi}^2 t + \frac{f''(0)\bar{\xi}^4 t^2}{2} + \frac{f'''(0)\bar{\xi}^6 t^3}{6} + \cdots\right)\right.$$

$$-\frac{1}{8}\left(\frac{\lambda}{(1-\lambda)}\right)^2\left(f'(0)\bar{\xi}^2 t+\frac{f''(0)\bar{\xi}^4 t^2}{2}+\cdots\right)+\cdots\Big]$$

$$\therefore\quad \bar{t}=(1-\lambda)^{\frac{1}{2}}\Big[t-\frac{\lambda}{2(1-\lambda)}\left(\frac{f'(0)\bar{\xi}^2 t^2}{2}+\frac{f''(0)\bar{\xi}^4 t^3}{6}+\frac{f'''(0)\bar{\xi}^6 t^4}{24}+\cdots\right)$$
$$-\frac{1}{8}\left(\frac{\lambda}{(1-\lambda)}\right)^2\left(\frac{\left(f'(0)\bar{\xi}^2\right)^2 t^3}{3}+\frac{f''(0)\bar{\xi}^6 t^4}{4}+\cdots\right)+\cdots\Big] \tag{41a}$$

That is

$$\bar{t}=(1-\lambda)^{\frac{1}{2}}\left[t+\bar{\xi}^2\Omega_1 t^2+\bar{\xi}^4\Omega_2 t^3+\bar{\xi}^6\Omega_3 t^4+\cdots\right] \tag{41b}$$

where

$$\Omega_1=\frac{-f'(0)\lambda}{4(1-\lambda)},\qquad \Omega_2=\frac{-f''(0)\lambda}{12(1-\lambda)}-\frac{1}{24}\left(\frac{\lambda}{(1-\lambda)}\right)^2(f'(0))^2 \tag{41c}$$

$$\Omega_3=-\left[\frac{1}{48}\left(\frac{\lambda}{(1-\lambda)}\right)f'''(0)+\frac{1}{64}\left(\frac{\lambda}{(1-\lambda)}\right)^2 f'(0)f''(0)\right] \tag{41d}$$

Evaluating (41b – d) at maximum values, gives

$$\bar{t}_a=(1-\lambda)^{\frac{1}{2}}\left[t_a+\bar{\xi}^2\Omega_1 t_a^2+\bar{\xi}^4\Omega_2 t_a^3+\bar{\xi}^6\Omega_3 t_a^4+\cdots\right] \tag{42a}$$

Expansion of $\bar{t}_a$ and t_a, using (42) and equating coefficients of $O(1)$ gives

$$\bar{t}_0=\pi=(1-\lambda)^{\frac{1}{2}}t_0$$
$$\therefore\qquad t_0=\pi(1-\lambda)^{-\frac{1}{2}} \tag{42b}$$

Similarly, evaluating the coefficient $O(\bar{\xi}^2)$, easily gives

$$\bar{t}_2=\frac{f'(0)(\lambda+4)}{4(1-\lambda)^{\frac{3}{2}}}=(1-\lambda)^{\frac{1}{2}}\left[t_2+\Omega_1 t_0^2\right]$$
$$\therefore\qquad t_2=\frac{f'(0)(\lambda+4)}{4(1-\lambda)^2}-\frac{\Omega_1\pi^2}{(1-\lambda)} \tag{42c}$$

etc.

To evaluate the maximum displacement η_a as in (40), it is necessary to note the following values as evaluated at $\bar{t}=\bar{t}_0$ and $\tau=0$:

$$\eta^{(1)}(\bar{t}_0,0)=2B(0),\quad \eta^{(3)}(\bar{t}_0,0)=\frac{\alpha B^3(0)}{1-\lambda}$$

$$\eta^{(5)}(\bar{t}_0,0)=\frac{2\beta B^5(0)\varphi_{12}}{1-\lambda},\quad \text{where } \varphi_{12}=\frac{27\varphi_5}{2}-5\varphi_6-\frac{\varphi_{10}}{3}$$

$$\eta_{,\tau}^{(3)}(\bar{t}_0,0)=\frac{\alpha B^4(0)\varphi_{14}}{1-\lambda},\quad \text{where } \varphi_{14}=f'(0)\left(\frac{5}{2}-\frac{6}{\lambda}\right)+f'(0)\left(\frac{2-\lambda-\lambda^2}{4\lambda}\right)-\frac{\varphi_{13}}{4}+\frac{f'(0)}{32}$$

On evaluating the maximum displacement after evaluating (34) at maximum, it follows that

$$\eta_a=2B(0)\bar{\xi}+\frac{\alpha B^3(0)\bar{\xi}^3 Q_1}{1-\lambda}+\frac{2\beta B^5(0)\bar{\xi}^5\varphi_{12}Q_2}{1-\lambda}+\cdots \tag{43a}$$

where

$$Q_1=1+\frac{f'(0)(\lambda+4)t_0}{4B(0)} \tag{43b}$$

$$Q_2=1+\frac{1-\lambda}{2\beta B^5(0)\varphi_{12}}\left\{\frac{B(0)f'(0)(\lambda+4)t_2}{4(1-\lambda)}+\frac{B(0)\bar{t}_2^2}{2}+\frac{t_0^2}{2}\left(B''(0)-\alpha_1''(0)\right)+\frac{t_0 B^4(0)\varphi_{14}}{1-\lambda}\right\} \tag{43c}$$

7. DYNAMIC BUCKLING LOAD, λ_D

For the purpose of determining the dynamic buckling load λ_D, it is necessary to rewrite (43a) simply as

$$\eta_a=C_1\bar{\xi}+C_3\bar{\xi}^3+C_5\bar{\xi}^5+\cdots \tag{44a}$$

where

$$C_1 = 2B(0), \qquad C_3 = \frac{\alpha B^3(0) Q_1}{1-\lambda}, \qquad C_3 = \frac{2\beta B^5(0)\varphi_{12} Q_2}{1-\lambda} \tag{44b}$$

The dynamic buckling load λ_D is supposed to be determined by using the equivalent form of (4), in the form

$$\frac{d\lambda}{d\eta_a} = 0. \tag{44c}$$

However, as noted by Amazigo [6, 7], the series (44a) becomes unbounded when $\eta_a > \eta_{aD}$, where η_{aD} is the maximum displacement at dynamic buckling. The difficulty is overcome by reversing the series (44a) so that

$$\bar{\xi} = d_1\eta_a + d_3\eta_a^3 + d_5\eta_a^5 + \cdots \tag{45a}$$

The coefficient d_1, d_3 and d_5 are obtained either by using Lagrange's formula for reversion of series [23] or by substituting in (45b) for η_a and equating coefficient of powers of $\bar{\xi}$.

Adopting the latter, it follows that

$$d_1 = \frac{1}{C_1} = \frac{1}{2B(0)}, \qquad d_3 = \frac{-C_3}{C_1^4} = \frac{\alpha Q_1}{16\lambda} \tag{45b}$$

$$d_5 = \frac{3C_3^2 C_1 C_5}{C_1^7} = \frac{C_5}{C_1^6}\left(1 - \frac{3C_3^2}{C_1 C_5}\right) = \frac{\beta\varphi_{12} Q_2 \varphi_{15}}{32B(0)}, \qquad \varphi_{15} = 1 - \frac{3\alpha^2 Q_1^2}{4\beta\varphi_{12} Q_2} \tag{45c}$$

It should, however, be noted that each of d_1, d_3 and d_5 depends on the load parameter λ. The maximization (44d) now yields

$$\frac{d\bar{\xi}}{d\eta_a} = 0 = \frac{d(d_1)}{d\lambda}\frac{d\lambda}{d\eta_a} + d_1 + \frac{d(d_3)}{d\lambda}\frac{d\lambda}{d\eta_a} + 3d_3\eta_{aD}^2 + \frac{d(d_5)}{d\lambda}\frac{d\lambda}{d\eta_a} + 5d_5\eta_{aD}^4.$$

This implies, through (44c)

$$d_1 + 3d_3\eta_{aD}^2 + 5d_5\eta_{aD}^4 = 0 \tag{46a}$$

where

$$\eta_{aD} = \eta_a(\lambda_D)$$

From (46a), it follows that

$$\eta_{aD}^2 = \frac{-3d_3 \pm \sqrt{9d_3^2 - 20d_1 d_5}}{10d_5}$$

Taking the negative square root sign, results to

$$\eta_{aD}^2 = \frac{6Q_1\left(\frac{\alpha}{\beta}\right)\varphi_{16}}{5\varphi_{12}\varphi_{15} Q_2 (1-\lambda)} \tag{46b}$$

$$\varphi_{16} = 1 + \left\{1 + \frac{80(1-\lambda)^2\left(\frac{\beta}{\alpha^2}\right)\varphi_{12}\varphi_{15} Q_2}{9Q_1^2}\right\}^{\frac{1}{2}} \tag{46c}$$

$$\therefore \qquad \eta_{aD} = \sqrt{\frac{6}{5}}(1-\lambda)^{-\frac{1}{2}}\left(\frac{Q_1\left(\frac{\alpha}{\beta}\right)\varphi_{16}}{\varphi_{12}\varphi_{15} Q_2}\right)^{\frac{1}{2}} \tag{46d}$$

The dynamic buckling load is next obtained by evaluating (45a) at dynamic buckling stage. This is now obtained by first multiplying (45a) by 5 to get

$$5\bar{\xi} = \eta_{aD}\left[(5d_1 + 5d_3\eta_{aD}^2) + 5d_5\eta_{aD}^4\right] \tag{47a}$$

Making $5d_5\eta_{aD}^4$ the subject in (46a) and substituting same in (47a), gives

$$5\bar{\xi} = 2\eta_{aD}(2d_1 + d_3\eta_{aD}^2) \tag{47b}$$

After simplifying (47b), it seen that

$$5\bar{\xi}\lambda_D = 2\sqrt{\frac{6}{5}}(1-\lambda_D)^{\frac{1}{2}}\left(\frac{Q_1\varphi_{16}}{\varphi_{12}\varphi_{15}Q_2}\right)^{\frac{1}{2}}\left[1+\frac{3Q_1^2\left(\frac{\alpha^2}{\beta}\right)\varphi_{16}}{40\,\varphi_{12}\varphi_{15}Q_2(1-\lambda_D)^2}\right] \tag{48a}$$

It has to be noted that each φ_i and Q_i depends on λ_D so that the result (48a), which determine λ_D is also implicit in λ_D.

Using (6f, g), it is possible to relate the dynamic and static buckling loads to get

$$\frac{\lambda_D}{\lambda_S} = \frac{2(1-\lambda)^{\frac{1}{2}}}{(1-\lambda_S)R_2(R_1-1)^{\frac{1}{2}}}\left(\frac{Q_1\varphi_{16}}{\varphi_{12}\varphi_{15}Q_2}\right)^{\frac{1}{2}}\left[1+\frac{3Q_1^2\left(\frac{\alpha^2}{\beta}\right)\varphi_{16}}{40\,\varphi_{12}\varphi_{15}Q_2(1-\lambda_D)^2}\right] \tag{48b}$$

8. ANALYSIS OF THE RESULT

The result (48a) is implicit in the load parameter λ_D while (6f, g) are implicit in the static load parameter λ_S. Similarly, (48b) is implicit in both λ_S and λ_D. As observed from (48b), the relationship between λ_D and λ_S is independent of the imperfection parameter $\bar{\xi}$ but once any of λ_S or λ_D is specified, then the other can easily be obtained. Generally, the results (6f, g) and (48a) are valid for small values of the imperfection parameter, $\bar{\xi}$. It is here demanded that $\alpha > 0, \beta > 0$ and observe that as far as the loading history $f(\tau)$ is concerned, the dynamic buckling λ_D depends on $f'(0)$, ${f'}^2(0)$ and $f''(0)$, all depending on the accuracy retained. For higher degrees of accuracy of the result, one may expect dependence of λ_D on higher derivatives of $f(\tau)$ evaluated at the initial time $t = 0$. As in equation (3), the analysis has tacitly required that $\left|f\left(\bar{\xi}^2 t\right)\right| < 1, \quad t > 0$. However, has long as the inequality $0 < \lambda < 1$ holds (which has been assumed in this work), this analysis equally holds for $\left|f\left(\bar{\xi}^2 t\right)\right| \le 1$, and so, equally holds for the case $f(t) = 1$, i.e., for the step loading case. Thus, by setting to zero in all the results, the derivatives of $f(\tau)$, such as $f'(0), f''(0), {f'}^2(0)$ etc., the result for step loading case can easily obtained.

Figure (1) and Figure (2) were drawn using $f(\tau) = e^{-\tau}, \quad \tau = \bar{\xi}^2 t$. This choice of $f(\tau)$ satisfies all the conditions stipulated in equation (3).

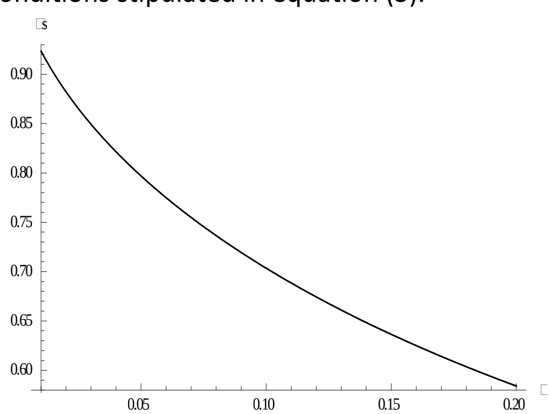

Fig.1: The graph of static buckling load λ_S using (6g) with $\alpha = 1, \beta = 1$.

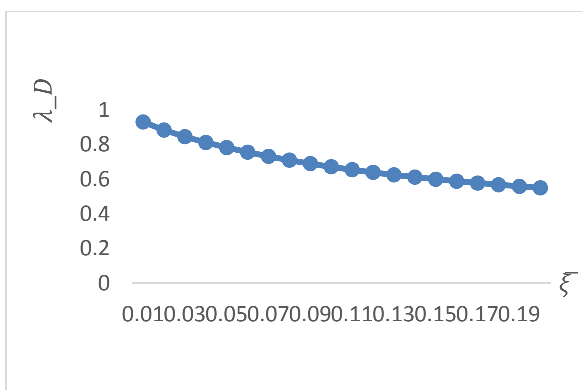

Fig.2: The graph of dynamic buckling load λ_D for various values of $\bar{\xi}$, from eqn. (48a), with $\alpha = 1, \beta = 1$.

9. CONCLUSION

This analysis has carried out a perturbation approach in analyzing the dynamic buckling load of a cubic-quintic nonlinear elastic model structure struck by an explicitly time-dependent slow-varying load. The results are asymptotic in nature. It has been shown that the dynamic buckling load λ_D depends among other things, on the first derivative of the load function evaluated at the initial time $t = 0$. It is observed that it is possible to relate the dynamic buckling load λ_D to its static equivalent λ_S and that relationship is independent of the imperfection parameter $\bar{\xi}$. Hence given either of the λ_S or λ_D, the other can easily be evaluated using the relationship.

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