

Statistical Measure of Second Order Response Surface Rotatability using an infinite class of supplementary Difference Sets.

ABSTRACT

Rotatability is a desirable feature of a response surface experimental design. In case a design is non rotatable or exhibit surface of constant prediction variances that are nearly spherical then an attempt is made to make the design rotatable. In this paper, a measure of rotatability of five level second order rotatable designs using an infinite class of supplementary difference sets is suggested. The variance function of a second-order response design and an infinite class of supplementary difference sets is used in coming up with the design

Keywords: Response surface Methodology; Rotatable designs; second order designs; five leve; Supplementary difference sets.

1. INTRODUCTION

The property of rotatability as a desirable quality of an experimental design was first advanced by [1] and requires that the variances of the estimated response are constant on circles or spheres about the center of the design.

Koukouvinos et al. [4] gave a general construction method for five level second order rotatable designs. . Further research was done by Mutiso, Kerich and Ng'eno [5&6] where they constructed five level rotatable designs using an infinite class of supplementary difference sets. Park et al. [9&10] developed a measure of rotatability that is invariant under rotation and Ng'eno [7] developed a measure of modified rotatability using an infinite class of supplementary difference sets by fixing $c=5$. This article presents a measure of Box- Hunter [1] rotatability by fixing $c = 3$. The measure will be investigated using an infinite class of supplementary difference sets.

The following symmetry conditions (Moments conditions) needs to be satisfied for a design to form a Second order rotatable arrangement [1]

$$\text{i) } \sum_{u=1}^N \prod_{i=1}^v x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd for } \sum \alpha_i \leq 4$$

$$\text{ii) } \sum_{u=1}^N x_{iu}^2 = \text{Constant} = N\lambda_2 \text{ for } i = 1, 2, \dots, v$$

$$\text{iii) } \sum_{u=1}^N x_{iu}^4 = \text{Constant} = cN\lambda_4 \text{ for } i = 1, 2, \dots, v$$

(1.1)

$$32 \quad \text{iv)} \quad \sum_{u=1}^N x_{iu}^2 x_{ju}^2 = \text{Constant} = N\lambda_4$$

$$33 \quad \text{v)} \quad \frac{\sum_{u=1}^N x_{iu}^4}{\sum_{u=1}^N x_{iu}^2 x_{ju}^2} = c \text{ for } i \neq j$$

34 where c , λ_2 and λ_4 are constants.

35 Using these symmetry conditions the variance and the covariance of the estimates are obtained and are
36 shown below

$$37 \quad \text{i)} \quad V(b_0) = \frac{(c+v-1) \lambda_4 \delta^2}{N[(c+v-1) \lambda_4 - v \lambda_2^2]}$$

$$38 \quad \text{ii)} \quad V(b_i) = \frac{\delta^2}{N \lambda_2}$$

$$39 \quad \text{iii)} \quad V(b_{ij}) = \frac{\delta^2}{N \lambda_4}$$

$$40 \quad \text{iv)} \quad V(b_{ii}) = \frac{\delta^2}{N(c-1) \lambda_4} \left[\frac{(c+v-2) \lambda_4 - \lambda_2^2 (v-1)}{(c+v-1) \lambda_4 - v \lambda_2^2} \right] \quad (1.2)$$

$$41 \quad \text{v)} \quad \text{Cov}(b_0, b_{ii}) = \frac{-\lambda_2 \delta^2}{N[(c+v-1) \lambda_4 - v \lambda_2^2]}$$

$$42 \quad \text{vi)} \quad \text{Cov}(b_{ii}, b_{ij}) = \frac{\delta^2}{N \lambda_4 (c-1)} \left[\frac{\lambda_2^2 - \lambda_4}{(c+v-1) \lambda_4 - v \lambda_2^2} \right]$$

43 and all other covariances are zero.

44 An inspection of the variances shows that a necessary condition for the existence of a non-singular
45 second order design is $(c + v - 1) \lambda_4 - v \lambda_2^2 > 0$ which leads to the following non-singularity condition first
46 developed by Box and Hunter [1]

$$47 \quad \frac{\lambda_4}{\lambda_2^2} > \frac{v}{c+v-1} \quad (1.3)$$

48 Harder and Park [3] studied estimates in response at two different points in the factor space. They
49 introduced the analogous form of Box Hunter rotatability and termed it as slope rotatability. Harder and
50 Park [6] and Park [8] stated that the necessary and sufficient condition for slope rotatability is $4V(b_{ii}) =$
51 $V(b_{ij})$. The condition was simplified by Victorbabu and Narasimham [12] and they developed equation
52 (1.4) below which is the necessary and sufficient condition for a 2^{nd} order design to be slope rotatable

$$53 \quad \lambda_4 [v(5-c) - (c-3)^2] + \lambda_2^2 [v(c-5) + 4] = 0 \quad (1.4)$$

54 2. METHODOLOGY

55 Seberry Wallis [11] defined supplementary difference sets and stated that the parameters of e - $[v;$
56 $k_1, k_2, \dots, k_e, \lambda]$ SDS satisfies $\lambda(v-1) = e k(k-1)$. Koukouvinos et al [4] came up with the following useful
57 relationships which we will utilize in this study.

$$58 \quad \text{i)} \quad \sum_{u=1}^N x_{1u} = \sum_{u=1}^N x_{2u} = \sum_{u=1}^N x_{3u} = \sum_{u=1}^N x_{4u} = 0$$

$$60 \quad \text{ii)} \quad \sum_{u=1}^N x_{iu}^2 = 2^{t(m)}(e-1) + 2b^2 = N\lambda_2$$

$$61 \quad \text{iii)} \quad \sum_{u=1}^N x_{iu}^4 = 2^{t(m)}(e-1) + 2b^4 = cN\lambda_4$$

$$62 \quad \text{iv)} \quad \sum_{u=1}^N x_{iu}^2 x_{ju}^2 = 2^{t(m)}(e-2) = N\lambda_4$$

63

Mutiso et al. [6] stated in a theorem that the Supplementary difference sets with parameters e -[v :2:1]

gives a five level second order rotatable design in $b^4 = \frac{2^{t(m)}[2e-5]}{2}$ and $n_0 = \frac{-[-2v+4][2^{t(m)}(e-1)+2b^2]^2}{2v[2^{t(m)}(e-2)]}$

$$-m2^{t(m)} - 2m$$

Using (iii) and (iv) above we obtain

$$b^4 = \frac{2^{t(m)}[c(e-2) - (e-1)]}{2} \quad (2.1)$$

We use equation (1.4) to get the number of centre points (n_0) to be added to the design i.e

$$\lambda_4[v(5-c) - (c-3)^2] + \lambda_2^2[v(c-5) + 4] = 0 \text{ where } \lambda_2 = \frac{2^{t(m)}(e-1)+2b^2}{N}; \lambda_4 = \frac{2^{t(m)}(e-2)}{N}$$

on simplification we obtain

$$N = - \frac{[v(c-5)+4][2^{t(m)}(e-1)+2b^2]^2}{[v(5-3)-(c-3)^2]2^{t(m)}(e-2)} \quad (2.2)$$

$$\text{therefore } n_0 = N - m2^{t(m)} - 2m$$

$$\therefore n_0 = \frac{-[v(c-5)+4][2^{t(m)}(e-1)+2b^2]^2}{[v(5-3)-(c-3)^2]2^{t(m)}(e-2)} - m2^{t(m)} - 2m$$

Basing on Box –Hunter rotatability criteria [1] we fix c to be 3 in 2.1 and 2.2 and therefore we have

$$b^4 = \frac{2^{t(m)}[2e-5]}{2} \quad (2.3)$$

and therefore,

$$n_0 = \frac{-[-2v+4][2^{t(m)}(e-1)+2b^2]^2}{2v[2^{t(m)}(e-2)]} - m2^{t(m)} - 2m \quad (2.4)$$

A design whose moments do not conform to the moment conditions of rotatability is said to be non-rotatable. If circumstances are such that exact rotatability is unattainable, it is still a good idea to make the design nearly rotatable [9&10]

The article utilizes the measure developed by Park et al [10] to access the degree of rotatability. If the region of interest is $0 \leq \rho \leq 1$, then the rotatability measure is expresses as

$$P_v[D] = \frac{1}{1+R_v[D]} \quad (2.5)$$

Where

$$R_v[D] = \left[\frac{N}{\delta^2} \right]^2 \left\{ \frac{6v[v(b_{ij}) + 2Cov(b_{ii}, b_{ij}) - 2V(b_{ii})]^2(v-1)}{(v+2)^2(v+4)(v+6)(v+8)g^8} \right\} \text{ and } g \text{ is a scaling factor.}$$

on simplification the numerator portion of $v(b_{ij}) + 2Cov(b_{ii}, b_{ij}) - 2V(b_{ii})$ become $\frac{(c-3)\delta^2}{(c-1)N\lambda_4}$ thus $R_v[D]$

$$\text{becomes, } R_v[D] = \left[\frac{N}{\delta^2} \right]^2 \left\{ \frac{6v[(c-3)\delta^2]^2(v-1)}{(c-1)N\lambda_4)^2 (v+2)^2(v+4)(v+6)(v+8)g^8} \right\} \quad (2.6)$$

For second order rotatable design, we have $c=3$. Substituting the value of $c=3$ in the above equation we get $R_v[D] = 0$. Hence $P_v[D]$ takes the value of 1 if and only if a design is rotatable and it's smaller than one for non rotatable design.

To compare the design, Park et al [10] considered the scaling of the design. In this article, the design is scaled in such a way that all the point's lie inside or on the unit sphere [2].

If we have a set of points $x'_i = (x_{1u}, x_{2u}, \dots, x_{vu})$ $i=1,2,\dots,N$ then the scaled point, gx_i should satisfy

$$0 \leq g\sqrt{(x_{1u})^2 + (x_{2u})^2 + \dots + (x_{vu})^2} \leq 1 \quad i=1,2,\dots,N \quad (2.7)$$

One advantage of this is that, when we add center point, the remaining points do not have to be rescaled.

To construct the measure we will use equation (2.5) developed by park et al [10] and replace v with m because in supplementary difference sets we have $m = \frac{v-1}{2}$ factors. We therefore obtain the following

$$\text{expressions } P_m[D] = \frac{1}{1+R_m[D]}$$

$$\text{Where } R_m[D] = \left[\frac{N}{\delta^2} \right]^2 \left\{ \frac{6m[(c-3)\delta^2]^2(m-1)}{(c-1)N\lambda_4)^2 (m+2)^2(m+4)(m+6)(m+8)g^8} \right\} \quad (2.8)$$

$$c = \frac{\sum_{u=1}^N x_{iu}^4}{\sum_{u=1}^N x_{iu}^2 x_{ju}^2} = \frac{2^{t(m)}(e-1) + 2b^4}{2^{t(m)}(e-2)}, \quad b^4 = \frac{2^{t(m)}[2e-5]}{2}, \quad \lambda_4 = \frac{2^{t(m)}(e-2)}{N}$$

$$g = \begin{cases} \frac{1}{b} & \text{if } b \leq \sqrt{2^{t(m)-1} + m} \\ \frac{1}{\sqrt{2^{t(m)-1} + m}} & \text{if } b > \sqrt{2^{t(m)-1} + m} \end{cases}$$

$$\text{On simplifying 2.8 we have } R_m[D] = \frac{6m(m-1)(c-3)^2}{(m+2)^2(m+4)(m+6)(m+8)\lambda_4^2(c-1)^2 g^8} \quad (2.9)$$

2. RESULTS, DISCUSION AND CONCLUSION

For a second order rotatable design, we have $c=3$. Substituting the value of $c=3$ in (4.6), we get $R_m[D] = 0$. Hence $P_m[D]$ takes the value of 1 if the design is rotatable and less than 1 if it is not rotatable.

Illustration

Let us consider 3-[7:2:1] SDS. In this case we have, $v = 7, m = e = \frac{v-1}{2} = 3$, and $\lambda_4 = \frac{2^{t(m)}(e-2)}{N} =$

$$\frac{2^2(3-2)}{21} = \frac{4}{21} = 0.1905$$

$$\text{Using (2.9) we have } R_m[D] = \frac{6m(m-1)(c-3)^2}{(m+2)^2(m+4)(m+6)(m+8)\lambda_4^2(c-1)^2 g^8} = 0.05727 \frac{(c-3)^2}{(c-1)^2 g^8}$$

For rotatable design, $c=3$ meaning that $R_m[D] = 0$ and therefore $P_m[D] = \frac{1}{1+R_m[D]} = 1$. Suppose $c = 3.92$,

We use (2.8) to get the value of g . In this case $g = \frac{1}{b} = 0.7143$

$$R_m[D] = 0.05727 \frac{(c-3)^2}{(c-1)^2 g^8} = 0.05727 \frac{(3.92-3)^2}{(3.92-1)^2 g^8} = 8.3886 \times 10^{-2}$$

114 Therefore $P_m[D] = \frac{1}{1+R_m[D]} = \frac{1}{1+8.3886 \times 10^{-2}} = 0.9226$.

115 The above value is less than one but very close to one meaning that the design is nearly rotatable.

116 Suppose $c=13.71$, $R_m[D] = 0.05727 \frac{(13.71-3)^2}{(13.71-1)^2 g^8} = 22.3328$

117 Therefore $P_m[D] = \frac{1}{1+R_m[D]} = \frac{1}{1+22.3328} = 4.285 \times 10^{-2}$

118 The above value is less than one but not close to one meaning that the design is not rotatable.

119 The expressions for calculating $R_m[D]$ for different class of supplementary difference sets are given in
120 the following equations. The results of the measure for different class of SDS are as shown in tables 1 to
121 10 in the appendix.

122 3-[7:2:1] SDS - $R_m[D] = 5.7258 \times 10^{-2} \frac{(c-3)^2}{(c-1)^2 g^8}$

123 4-[9:2:1] SDS- $R_m[D] = 7.9991 \times 10^{-3} \frac{(c-3)^2}{(c-1)^2 g^8}$

124 5-[11:2:1] SDS– Taking half replicate of factorial part- $R_m[D] = 1.1111 \times 10^{-2} \frac{(c-3)^2}{(c-1)^2 g^8}$

125 5-[11:2:1] SDS - $R_m[D] = 8.4266 \times 10^{-3} \frac{(c-3)^2}{(c-1)^2 g^8}$

126 6-[13:2:1] SDS – Taking $\frac{1}{4}$ replicate of factorial part - $R_m[D] = 7.5599 \times 10^{-3} \frac{(c-3)^2}{(c-1)^2 g^8}$

127 7-[15:2:1] SDS – Taking $\frac{1}{4}$ replicate of factorial part- $R_m[D] = 4.3792 \times 10^{-3} \frac{(c-3)^2}{(c-1)^2 g^8}$

128 8-[17:2:1] SDS – Taking $\frac{1}{8}$ replicate of factorial part- $R_m[D] = 3.3859 \times 10^{-3} \frac{(c-3)^2}{(c-1)^2 g^8}$

129 9-[19:2:1] SDS – Taking $\frac{1}{16}$ replicate of factorial part- $R_m[D] = 2.6592 \times 10^{-3} \frac{(c-3)^2}{(c-1)^2 g^8}$

130 10-[21:2:1] SDS – Taking $\frac{1}{32}$ replicate of factorial part - $R_m[D] = 2.1586 \times 10^{-3} \frac{(c-3)^2}{(c-1)^2 g^8}$

131 11-[23:2:1] SDS – Taking $\frac{1}{64}$ replicate of factorial part – $R_m[D] = 1.7802 \times 10^{-3} \frac{(c-3)^2}{(c-1)^2 g^8}$

132 **COMPETING INTERESTS**

133 Authors have declared that no competing interests exist.”.

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135 **REFERENCES**

136 [1] Box, G.E.P., Hunter, J.S.(1957). Multifactor experimental designs for exploring response surfaces.

137 Annals of Mathematical Statistics. 28 195-241.

138

139 [2] Draper, N.R., Pukelsheim, F. (1990). Another look at rotatability. Technometrics. 32 195-202.

- 140
- 141 [3] Hader, R.J., Park, S.H. (1978). Slope rotatable central composite designs. *Technometrics*. 20 413-
- 142 417.
- 143 [4] Koukiuvinos, C., Mylona, K., Skountzou, A., Goos, P. (2013). A general construction Method for five
- 144 level second order rotatable designs. *Communication in statistics- simulation and computation*. 42
- 145 1961-1969.
- 146 [5] Mutiso, J.M., Kerich, G.K., Ng'eno, H.M. (2016a). Construction of five level modified second order
- 147 rotatable designs using supplementary difference sets. *Far East Journal of Theoretical Statistics*. 52: 333-
- 148 343.
- 149
- 150 [6] Mutiso, J.M., Kerich, G.K., Ng'eno, H.M. (2016b). Construction of five level second order rotatable
- 151 designs using supplementary difference sets. *Advances and Application in statistics*. 49:21-30
- 152
- 153 [7] Ng'eno, M.N. (2018). Measure of rotatability of modified second-order rotatable design using
- 154 supplementary difference sets. *Statistics Theory and Related field*.
- 155
- 156 [8] Park, S.H. (1987). A class of multifactor designs for estimating the slope of response surfaces.
- 157 *Technometrics*. 29 449-453.
- 158 [9] Park, S.H., Kim, H.J. (1992). A measure of slope rotatability for second order response surface
- 159 experimental designs. *Journal of Applied Statistics*. 19(3) 391-404.
- 160
- 161 [10] Park, S.H., Lim, J.H., Baba, Y. (1993). A measure of rotatability for second order response surface
- 162 designs. *Ann. Inst. Statist. math*. 45 655-664
- 163
- 164 [11] Seberry W.J. (1973). Some remarks on supplementary difference sets. *Colloquia Mathematica*
- 165 *Societatis Janos Bolyai Hungary*. 10 1503-1526.
- 166 [12] Victorbabu, B.Re., Narasimham, V.L. (1991). Construction of second order slope rotatable designs
- 167 through balanced incomplete block designs. *Communication in Statistics- Theory and*
- 168 *Methods*. 20(8) 2467-2478.
- 169
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APPENDIX

Table 1 – Measure of rotatability for five level SORD for 3-[7:2:1] SDS

$v=7, m=3, N = 21 \lambda_4 = 0.1905$

b	c	g	$R_m[D]$	$P_m[D]$
1.00	2.50	1.0000	6.3621×10^{-3}	0.9937
1.19	3.00	0.8403	0.0000	1.0000
1.20	3.04	0.8333	9.4686×10^{-5}	0.9999
1.40	3.92	0.7143	8.3886×10^{-2}	0.9226
1.60	5.28	0.6250	0.6988	0.5889
1.80	7.25	0.5556	2.9158	0.2554
2.00	10.00	0.5000	8.8672	0.1013
2.20	13.71	0.4545	22.3283	4.287×10^{-2}
2.40	18.58	0.4472	28.1139	3.4235×10^{-2}
2.60	24.85	0.4472	30.0435	3.2213×10^{-2}
2.80	32.73	0.4472	31.4249	3.0834×10^{-2}
3.00	42.50	0.4472	32.4282	2.9915×10^{-2}
3.20	54.43	0.4472	33.1656	2.9269×10^{-2}
3.40	68.81	0.4472	33.7148	2.8806×10^{-2}
3.60	85.98	0.4472	34.1302	2.8466×10^{-2}
3.80	106.26	0.4472	34.4478	2.8210×10^{-2}
4.00	130.00	0.4472	34.6939	2.8016×10^{-2}
4.20	157.58	0.4472	34.8866	2.7866×10^{-2}
4.40	189.40	0.4472	35.0393	2.7748×10^{-2}
Rotatable b = 1.19				

Table 2 - Measure of rotatability for five level SORD for 4-[9:2:1] SDS

$m=4, v=9, N = 47, \lambda_4 = 0.5106$

b	c	g	$R_m[D]$	$P_m[D]$
1.00	1.63	1.0000	3.7788×10^{-2}	0.9636
1.20	1.76	0.8333	9.1496×10^{-2}	0.9161
1.40	1.98	0.7143	0.1277	0.8867
1.60	2.32	0.6250	9.1080×10^{-2}	0.9165
1.80	2.81	0.5556	9.6972×10^{-3}	0.9904
1.86	3.00	0.5376	0.0000	1.0000
2.00	3.50	0.5000	8.1827×10^{-2}	0.9244
2.20	4.43	0.4545	0.7628	0.5673
2.40	5.65	0.4167	2.8549	0.2594
2.60	7.21	0.3846	7.6719	0.1153
2.80	9.18	0.3571	17.2483	5.4799×10^{-2}
3.00	11.63	0.3536	21.5503	4.4345×10^{-2}
3.20	14.61	0.3536	23.7928	4.0334×10^{-2}
3.40	18.20	0.3536	25.5346	3.7687×10^{-2}
3.60	22.49	0.3536	26.8937	3.5851×10^{-2}
3.80	27.56	0.3536	27.9576	3.4533×10^{-2}
4.00	35.50	0.3536	29.0153	3.3316×10^{-2}
4.20	40.39	0.3536	29.4604	3.2829×10^{-2}
4.40	48.35	0.3536	29.9925	3.2266×10^{-2}
Rotatable b = 1.86				

187 **Table 3 - Measure of rotatability for five level SORD for 5-[11:2:1] SDS**

188 $m=5(\frac{1}{2} \text{ replicate}), v=11, N=58, \lambda_4=0.4138$

b	c	g	$R_m[D]$	$P_m[D]$
1.00	1.42	1.0000	0.1573	0.8641
1.20	1.51	0.8333	0.4079	0.7102
1.40	1.65	0.7143	0.7073	0.5857
1.60	1.88	0.6250	0.7731	0.5639
1.80	2.21	0.5556	0.5216	0.6571
2.00	2.67	0.5000	0.1110	0.9000
2.12	3.00	0.4717	0.0000	1.0000
2.20	3.29	0.4545	9.7877×10^{-2}	0.91088
2.40	4.09	0.4167	1.5212	0.3966
2.60	5.14	0.3846	6.2023	0.1388
2.80	6.16	0.3571	15.7609	5.9663×10^{-2}
3.00	8.08	0.3333	37.5668	2.5929×10^{-2}
3.20	10.07	0.3333	44.3371	2.2057×2
3.40	12.47	0.3333	49.7412	1.9708×2
3.60	15.33	0.3333	54.0227	1.8174×10^{-2}
3.80	18.71	0.3333	57.4194	1.7118×10^{-2}
4.00	22.67	0.3333	60.1221	1.6361×2
4.20	27.26	0.3333	62.2781	1.5803×10^{-2}
4.40	32.57	0.3333	64.0171	1.5380×10^{-2}
Rotatable b = 2.12				

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190 **Table 4 - Measure of rotatability for five level SORD for 5-[11:2:1] SDS**

191 $m=5, v=11, N=101, \lambda_4=0.4752$

b	c	g	$R_m[D]$	$P_m[D]$
1.00	1.38	1.0000	0.1531	0.8672
1.20	1.42	0.8333	0.5129	0.6609
1.40	1.49	0.7143	1.1808	0.4583
1.60	1.61	0.6250	1.8792	0.3473
1.80	1.77	0.5556	2.3680	0.2969
2.00	2.00	0.5000	2.1572	0.3167
2.20	2.31	0.4545	1.2839	0.4378
2.40	2.72	0.4167	0.2457	0.8028
2.50	2.96	0.4000	5.3552×10^{-2}	0.9947
2.52	3	0.3968	0.0000	1.0000
2.60	3.24	0.3846	0.2021	0.8319
2.80	3.89	0.3571	3.0221	0.2486
3.00	4.71	0.3333	11.7548	7.8402×10^{-2}
3.20	5.70	0.3125	30.5762	3.1669×10^{-2}
3.40	6.90	0.2941	65.7838	1.4974×10^{-2}
3.60	8.33	0.2778	125.6152	7.8980×10^{-3}
3.80	10.02	0.2778	143.89873	6.9010×10^{-3}
4.00	12.00	0.2778	159.0360	6.2490×10^{-3}
4.20	14.30	0.2778	173.4942	5.7970×10^{-3}
4.40	16.95	0.2778	181.7284	5.4730×10^{-3}
Rotatable b = 2.52				

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194 **Table 5 - Measure of rotatability for five level SORD for 6-[13:2:1] SDS**195 $m = 6(\frac{1}{4} \text{ replicate}), v=13, N=68, \lambda_4 = 0.4706$

b	c	g	$R_m[D]$	$P_m[D]$
1.00	1.31	1.0000	0.2247	0.8166
1.20	1.38	0.8333	0.5909	0.6286
1.40	1.49	0.7142	1.0604	0.4853
1.60	1.66	0.6250	1.3383	0.4277
1.80	1.91	0.5556	1.1944	0.4557
2.00	2.25	0.50	0.6967	0.5894
2.20	2.71	0.4545	0.1194	0.8933
2.30	3.00	0.4348	0.0000	1.0000
2.40	3.32	0.4167	0.1582	0.8634
2.60	4.11	0.3846	2.0115	0.3321
2.80	5.09	0.3571	7.4646	0.1181
3.00	6.31	0.3333	19.2869	4.9292×10^{-2}
3.20	7.80	0.3162	37.6919	2.5845×10^{-2}
3.40	9.60	0.3162	44.5528	2.1953×10^{-2}
3.60	11.75	0.3162	50.1168	1.9563×10^{-2}
3.80	14.28	0.3162	54.5766	1.7993×10^{-2}
4.00	17.25	0.3162	58.1711	1.6900×10^{-2}
4.20	20.70	0.3162	61.0659	1.6112×10^{-2}
4.40	24.68	0.3162	63.4073	1.5526×10^{-2}
Rotatable b = 2.30				

196

197 **Table 6 - Measure of rotatability for five level SORD for 7-[15:2:1] SDS**198 $m = 7(\frac{1}{4} \text{ replicate}), v=15, N=139, \lambda_4 = 0.5755$

b	c	g	$R_m[D]$	$P_m[D]$
1.00	1.23	1.0000	0.2593	0.7941
1.20	1.25	0.8333	0.9229	0.5200
1.40	1.30	0.7143	2.0773	0.3252
1.60	1.36	0.6250	3.9033	0.2039
1.80	1.46	0.5556	5.4054	0.1561
2.00	1.60	0.5000	6.1037	0.1408
2.20	1.79	0.4545	5.6421	0.1506
2.40	2.01	0.4177	4.5405	0.1805
2.60	2.34	0.3846	2.2192	0.3106
2.80	2.74	0.3571	0.3698	0.7301
2.90	2.97	0.3448	5.0840×10^{-3}	0.9949
2.91	3.00	0.3436	0.0000	1.0000
3.00	3.23	0.3333	0.3059	0.7658
3.20	3.82	0.3125	4.0712	0.1972
3.40	4.54	0.2941	14.8072	6.3263×10^{-2}
3.60	5.40	0.2778	36.7330	2.6502×10^{-2}
3.80	6.41	0.2632	75.5483	1.3064×10^{-2}
4.00	7.60	0.2582	107.6900	9.2005×10^{-3}
4.20	8.98	0.2582	124.4928	7.9686×10^{-3}
4.40	10.57	0.2582	138.7124	7.1576×10^{-3}
Rotatable b = 2.91				

199

200 **Table 7 - Measure of rotatability for five level SORD for 8-[17:2:1] SDS**201 $m = 8$ ($\frac{1}{8}$ replicate), $v=17$, $N=158$, $\lambda_4 = 0.6076$

b	c	g	$R_m[D]$	$P_m[D]$
1.00	1.19	1.0000	0.3073	0.7649
1.20	1.21	0.8333	1.0581	0.4859
1.40	1.25	0.7143	2.4508	0.2900
1.60	1.30	0.6250	4.6697	0.1764
1.80	1.39	0.5556	6.3548	0.1360
2.00	1.50	0.5000	7.8011	0.1136
2.20	1.65	0.4545	8.0213	0.1108
2.40	1.86	0.4167	6.5448	0.1325
2.60	2.12	0.3846	4.3665	0.1863
2.80	2.45	0.3571	1.8422	0.3518
3.00	2.85	0.3333	0.1462	0.8725
3.06	3.00	0.3268	0.0000	1.0000
3.20	3.35	0.3125	0.8258	0.5477
3.40	3.95	0.2941	6.2736	0.1375
3.60	4.67	0.2778	19.7659	4.8156×10^{-2}
3.80	5.51	0.2632	45.5388	2.1487×10^{-2}
4.00	6.50	0.2500	89.8597	1.1006×10^{-2}
4.20	7.65	0.2500	108.4967	9.1328×10^{-3}
4.40	8.98	0.2500	124.6094	7.9612×10^{-3}
Rotatable b = 3.06				

202

203 **Table 8 - Measure of rotatability for five level SORD for 9-[19:2:1] SDS**204 $m = 9$ ($\frac{1}{16}$ replicate), $v=19$, $N=176$, $\lambda_4 = 0.6364$

b	c	g	$R_m[D]$	$P_m[D]$
1.00	1.16	1.0000	0.3517	0.7398
1.20	1.18	0.8333	1.1693	0.4609
1.40	1.21	0.7143	2.8508	0.2597
1.60	1.26	0.6250	5.1152	0.1635
1.80	1.33	0.5556	7.4999	0.1176
2.00	1.43	0.5000	9.0751	9.9254×10^{-2}
2.20	1.56	0.4545	9.6567	9.3837×10^{-2}
2.40	1.74	0.4167	8.4809	0.1055
2.60	1.96	0.3846	6.5193	0.1329
2.80	2.24	0.3571	3.7776	0.2093
3.00	2.59	0.3333	1.1610	0.4627
3.10	2.79	0.3226	0.3120	0.7622
3.19	3.00	0.3135	0.0000	1.0000
3.20	3.02	0.3125	2.8662×10^{-3}	0.9971
3.40	3.53	0.2941	2.0849	0.3242
3.60	4.14	0.2778	9.8820	9.1894×10^{-2}
3.80	4.87	0.2632	26.9603	3.5765×10^{-2}
4.00	5.71	0.2500	57.6936	1.7037×10^{-2}
4.20	6.70	0.2425	93.6938	1.0560×10^{-3}
4.40	7.84	0.2425	111.3340	8.9019×10^{-3}
Rotatable b = 3.19				

205

206 **Table 1 - Measure of rotatability for five level SORD for 10-[21:2:1] SDS**207 $m = 10$ ($\frac{1}{32}$ replicate), $v=21$, $N=195$, $\lambda_4 = 0.6564$

b	c	g	$R_m[D]$	$P_m[D]$
1.00	1.14	1.0000	0.3810	0.7241
1.20	1.16	0.8333	1.2279	0.4488
1.40	1.19	0.7143	2.8905	0.2570
1.60	1.23	0.6250	5.4906	0.1541
1.80	1.29	0.5556	8.2655	0.1079
2.00	1.38	0.5000	10.0432	9.0553×10^{-2}
2.20	1.49	0.4545	11.2580	8.1579×10^{-2}
2.40	1.64	0.4167	10.7226	8.5330×10^{-2}
2.60	1.84	0.3846	8.5991	0.1042
2.80	2.09	0.3571	5.6896	0.1495
3.00	2.39	0.3333	2.7297	0.2681
3.20	2.76	0.3125	0.4413	0.6938
3.30	2.98	0.3030	3.0998×10^{-3}	0.9967
3.31	3.00	0.3021	0.0000	1.0000
3.40	3.21	0.2941	0.3482	0.7417
3.60	3.75	0.2778	4.5266	0.1809
3.80	4.38	0.2632	15.6246	6.0152×10^{-2}
4.00	5.13	0.2500	37.6280	2.5888×10^{-2}
4.20	5.99	0.2357	81.3648	1.2141×10^{-2}
4.40	6.98	0.2357	100.3828	9.8636×10^{-3}
Rotatable b = 3.31				

208

209 **Table 10 - Measure of rotatability for five level SORD for 11-[23:2:1] SDS**210 $m = 11$ ($\frac{1}{64}$ replicate), $v=23$, $N=214$, $\lambda_4 = 0.6729$

b	c	g	$R_m[D]$	$P_m[D]$
1.00	1.13	1.0000	0.3684	0.7308
1.20	1.14	0.8333	1.3515	0.4253
1.40	1.16	0.7143	3.4739	0.2235
1.60	1.20	0.6250	6.1932	0.1390
1.80	1.26	0.5556	8.7806	0.1022
2.00	1.33	0.5000	11.6712	7.8920×10^{-2}
2.20	1.44	0.4545	12.2897	7.5247×10^{-2}
2.40	1.57	0.4167	12.3255	7.5045×10^{-2}
2.60	1.75	0.3846	10.3298	8.8264×10^{-2}
2.80	1.96	0.3571	7.9009	0.1123
3.00	2.24	0.3333	4.3911	0.1855
3.20	2.57	0.3125	1.4683	0.4051
3.40	2.97	0.2941	7.3760×10^{-3}	0.9927
3.41	3.00	0.2933	0.0000	1.0000
3.60	3.44	0.2778	1.6321	0.3799
3.80	4.01	0.2632	8.7035	0.1031
4.00	4.67	0.2500	24.1574	3.9750×10^{-2}
4.20	5.43	0.2381	51.8560	1.8919×10^{-2}
4.40	6.32	0.2294	90.4012	1.0941×10^{-2}
Rotatable b = 3.41				

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