Original Research Article

THE THERMISTOR PROBLEM WITH HYPERBOLIC ELECTRICAL CONDUCTIVITY

ABSTRACT

This paper presents the one-dimensional, positive temperature coefficient (PTC) thermistor equation, using the hyperbolic-tangent function as an approximation to the electrical conductivity of the device. The hyperbolic-tangent function describes the qualitative behaviour of the evolving solution of the thermistor in the entire domain. The steady state solution using the new approximation yielded a distribution of device temperature over the spatial dimension and all the phases of temperature distribution of the device without having to look for a moving boundary. The analysis of the steady state solution and the numerical solution of the unsteady state is presented in the paper.

Keywords: [Thermistor, electrical –conductivity, hyperbolic-tangent, method of lines]

1. INTRODUCTION

Thermistors are thermo-electric devices made from ceramic materials. The electrical conductivity of the device varies strongly with temperature; this effect has enabled thermistors to be used as switching devices in many electronic circuits. The study of the thermistor problems in heat and current flow has a long history of applications in several areas of electronics and its related industries [1]. There are generally two kinds of thermistors; one is the positive temperature coefficient (PTC) thermistor in which the electrical conductivity decreases with increasing temperature, and the other is the negative temperature coefficient thermistor for which the electrical conductivity increases with increasing temperature [2].

The current flows through the PTC thermistor heating it to above a critical temperature, at which its conductivity decreases substantially. This leads to a steady state where the heat generated is balanced by the heat lost to the surroundings. For the device to be useful, the steady state current need to be much less than the original current.
Mathematical problems related to the heat and current flow in the thermistor under the title ‘‘the thermistor problem’’ have been studied by several authors. The aspects of modeling, existence, uniqueness, and behaviour of solutions have also been presented [4, 5, 6, and 7]. Wood and Kutluay [8] gave an approximate functional solution for the one-dimensional thermistor problem with a step function electrical conductivity, using the heat balance integral method. They showed that the solution exhibits all the correct physical characteristics and that the simple model also exhibits a possible mechanism by which the observed cracking of the thermistor might be initiated. Bahadir [9] solved the PTC thermistor problem numerically by finite element method using quadratic splines as shape functions and also obtained the steady state solutions. The result obtained was compared with analytical solution and found to exhibit correct physical characteristics of the PTC thermistor.

Kutluay[8] gave the description of the three phases of steady state solutions obtainable assuming monotonicity of the temperature profile such that the point $x = 0$ will always be the hottest and the first point to reach the critical temperature $U_c = 1$ above which σ drops. Due to the decrease in σ, the rate of heat loss at $x = 1$ will ultimately equal the internal heat generation and a steady-state will be reached [7, 8].

1.1. Mathematical Approximation of the Electrical Conductivity

Traditionally, the step function was used as an approximation for the electrical conductivity though it does not completely reflect its qualitative behavior. This has necessitated the search for a more representative approximation of the PTC conductivity characteristics for use in solving the PTC thermistor problem. Many researchers have therefore sought to find an approximate representation for the electrical conductivity.
Fowler et al [10] represented the variation of σ with u (electrical conductivity) as an exponential function which is continuous but with discontinuous derivatives at $u = 1$ and $u = 2$.
Kutluay et al [11] observed from the step function conductivity that the electrical conductivity in the warm phase drops sharply from 1 at the temperature $0 \leq u \leq 1$ to δ at the temperature $u > 1$ and that the decrease can cause oscillation in the predicted temperature when the finite difference methods are applied to the problem. In order to avoid unwanted oscillations in the numerical solution, they presented a modification to the electrical conductivity depending on the location of the interface unknown a priori.
Kutluay and Wood [12] introduced a slightly more realistic model for the electrical conductivity ($\sigma(u)$) whose value decreases linearly from 1 at the critical temperature $u_{crit} = 1$ to δ at a temperature $1+\varepsilon$ which is mathematically equivalent to a ramp function. In the limit as ε approaches zero the ramp model approaches the step model. In other words, its behaviour is a ‘‘mushy’’ form of the step function conductivity. In their analysis, they concluded that the ramp function is also not particularly a good model for electrical conductivity since it is of course a stretched form of the step one.

This paper presents a solution of the PTC thermistor problem using a hyperbolic-tangent approximation of the device conductivity which is a good representation of its qualitative behavior. The exact steady state solution of the problem, using this new approximation is presented as well as the numerical solution using the method of lines.

In the rest of the paper, a recollection of the PTC thermistor model is presented in section two of the paper. The steady state solution of the problem, using the method of asymptotic expansion and the numerical solution using the method of lines are shown.

2. MATERIAL AND METHODS

2.1.The Problem Statement

The typical thermistor model is an initial-boundary-value problem comprising of coupled non-linear differential equations for heat and current flow. The dimensionless temperature of the PTC thermistor $u(x,t)$ satisfies the following heat equation [13, 14]

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha\,\sigma\left(\frac{\partial \phi}{\partial x}\right)^2 \quad , \qquad 0<x<1, \quad t>0 \tag{1}$$

subject to boundary conditions

$$\frac{\partial u}{\partial x}=0\ ,\ x=0,\ t>0\,, \tag{2}$$

$$\frac{\partial u}{\partial x}+\ \beta u=0\ ,\ x=1,\ t>0 \tag{3}$$

And the initial condition

$$u(x,0)=0\ ,\ \ 0\le x\le 1 \tag{4}$$

in which β is a positive heat transfer coefficient and α is the ratio of electric heating to heat diffusion.

The electric potential $\phi(x,t)$ in the device is governed by

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial \phi}{\partial x}\right)=0 \quad , \quad 0<x<1, \quad t>0 \tag{5}$$

Subject to the boundary condition

$$\phi(0,t)=0,\ t>0\,, \qquad \phi(1,t)=0,\ t>0 \tag{6}$$

and the initial condition

$$\phi(x,0)=x,\ 0\le x\le 1 \tag{7}$$

In the traditional solution of the thermistor problem, $\sigma(u)$, the electrical conductivity is approximated by

$$\sigma(u)=\begin{cases}1 & 0\le u\le 1\\ \delta & u\ge 1\end{cases} \tag{8}$$

which is mathematically equivalent to a step function and with a typical value $\delta=10^{-5}$

However, the conductivity of a physical PTC device does not display the step-wise discontinuity exhibited by the approximation equation (8).

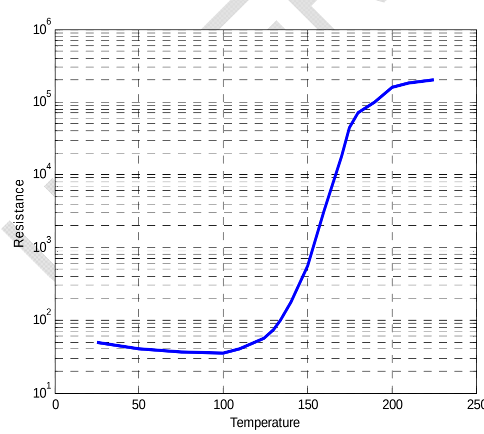

Figure 1.Typical variation of Resistance with Temperature for a PTC thermistor.

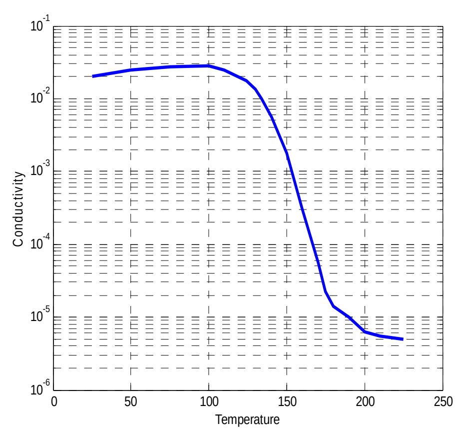

Figure 2.Typical variation of conductivity with Temperature for a PTC thermistor.

The typical Resistance/Temperature characteristic is shown in figure 1 [15].From this we obtain a proportional conductivity/resistance characteristics as shown in figure (2)

Following the disparity in the qualitative behavior of $\sigma(u)$ in the physical PTC characteristics and the approximation in equation (8), many researchers began to search for a more appropriate representation for the electrical conductivity.

2.2. A NEW APPROXIMATION OF THE ELCTRICAL CONDUCTIVITY

In this paper, we present a new approximation to the electrical conductivity as given below

$$\sigma(u) = \eta - (\eta - \delta)\tanh k(u - \varphi) \qquad 0 \le u \le 2 \qquad (9)$$

This is a hyperbolic tangent function where 2η is the initial conductivity, δ is the final conductivity, φ is the normalised critical temperature, u is the normalised temperature and k controls the slope. This approximation is so generic that by adjusting the slope it can be made to approximate the step function. For example taking $k \ge 500$, we have a step function approximation.

Consider an initial conductivity $2\eta = 1$, a critical temperature $u = 1$ and $k = 100$, the hyperbolic tangent approximation can be written as

$$\sigma(u) = 0.5 - (0.5 - \delta)\tan 100(u - 1) \qquad 0 \le u \le 2 \qquad (10)$$

A graph of a typical conductivity variation with temperature (normalized) alongside that of the hyperbolic tangent approximation is presented in figure (3).

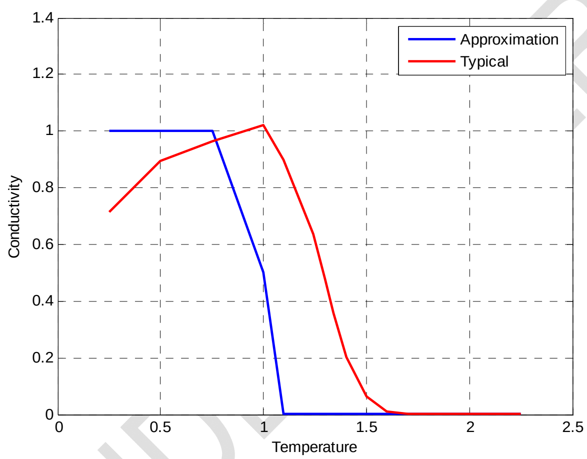

Figure 3. Graph of typical Conductivity variation with Temperature and that of the new approximation.

This electrical conductivity given by the hyperbolic tangent function is defined for the full range $0 \le u \le 2$ and covers the traditional points of discontinuities, assumed in most reported studies.

However our new approximation, when evaluated at $u << 1$ gives $\sigma(u) = 1$, which in related literature, corresponds to the cold phase; and when evaluated at $u >> 1$ gives $\sigma(u) = \delta$, which is traditionally referred to as the hot phase. In the same manner, the warm phase may be characterised by values of u near unity.

The exact solution of the electric potential problem (5), (6) and (7) is easily found to be $\phi(x,t) = x$ $(0 \le x \le 1$ and $t \ge 0)$ and the thermistor problem is reduced to a heat conduction description

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha\left(0.5-(0.5-\delta)\tan 100(u-1)\right)\ , \qquad 0 \le x \le 1,\ \ t>0 \qquad (11)$$

supplemented by boundary conditions (2) and (3) and the initial condition (4).

2.3. EXACT STEADY-STATE SOLUTIONS

At steady-state the time derivative in the model equation vanishes, we obtain the steady state solution for each phase as follows. For the cold and hot phases the steady state solution is obtained by standard analytical methods and results obtained are same with [7].

2.3.1. Cold phase ($0 < t \le t_0$)

In this phase $0 < U(x,t) \le U_c$ and $\sigma(U)=1$, so the steady state equation is

$$\frac{d^2 u}{dx^2}+\alpha=0, \qquad 0 < x < 1 \qquad (12)$$

subject to boundary conditions (2) and (3) and the solution is

$$u(x)=\alpha\left(\frac{1}{\beta}+\frac{1}{2}-\frac{x^2}{2}\right)$$

(13)

Enforcing the condition $u(0) \le 1$,we have

$$\alpha\delta \le \frac{2\beta}{2+\beta} \qquad (14)$$

2.3.2. Hot Phase ($U(x,t) > U_c$ and $\sigma(U)=\delta$).

The steady state equation is

$$\frac{d^2 u}{dx^2}+\alpha\delta=0, \qquad 0 < x < 1 \qquad (15)$$

subject to boundary conditions (2) and (3) and the solution is

$$u(x)=\alpha\delta\left(\frac{1}{\beta}+\frac{1}{2}-\frac{x^2}{2}\right) \qquad (16)$$

Enforcing the condition $u(1)>1$, we have

$$\alpha\delta > \beta \qquad (17)$$

2.3.3. Warm phase

The electrical conductivity is described by

$\sigma(u)=0.5-(0.5-\delta)\tan 100(u-1)$ and the steady state equation is given by

$$\frac{d^2 u}{dx^2}+\alpha\left(0.5-(0.5-\delta)\tan 100(u-1)\right)=0, \qquad 0 < x < 1 \qquad (18)$$

$$u_x = 0\ ,\ x = 0\ , \qquad u_x + \beta u = 0\ ,\ x = 1$$

we write (18) as

$$\frac{d^2u}{dx^2} + \alpha(0.5-\delta)\tanh 100(u-1) = -\frac{1}{2}\alpha$$

or

$$\frac{d^2u}{dx^2} + \varepsilon \tanh 100(u-1) = -\frac{1}{2}\alpha\ , \qquad (19)$$

where $\varepsilon = \alpha(0.5-\delta)$

We now solve (19) by the method of asymptotic expansion [16].

Assume a solution of the form

$$u(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots + \qquad (20)$$

Substituting in (19) and sorting yields

$$\frac{d^2u_0}{dx^2} = -\frac{1}{2}\alpha \qquad (21)$$

$$\frac{du_0}{dx} = 0\ ,\ x = 0\ , \qquad \frac{du_0}{dx} + \beta u_0 = 0\ ,\ x = 1$$

$$\frac{d^2u_1}{dx^2} = \tanh 100(u_0 - 1) \qquad (22)$$

$$\frac{du_1}{dx} = 0\ ,\ x = 0\ , \qquad \frac{du_1}{dx} + \beta u_1 = 0\ ,\ x = 1$$

From (21),

$$u_0(x) = -\frac{1}{4}\alpha x^2 + \frac{\alpha}{2}\left(\frac{1}{\beta} + \frac{1}{2}\right)\ ,\ x = 0 \qquad (23)$$

So that (22) can be written as

$$\frac{d^2u_1}{dx^2} = \tanh\left(-\frac{5}{2}\alpha x^2 + 10a\right) \qquad (24)$$

where $a = \frac{\alpha}{2\beta} + \frac{\alpha}{4} - 1$ (25)

then

$$u_1(x) = \iint \tanh\ (-25\ \alpha\ x^2 + 100\ a)\ dxdx\ + c_1 x + c_2$$

In polynomial form this can be written as

$$u_1(x) = \frac{1}{2} A\ x^2 - \frac{25}{3} B\ x^4 - \frac{250}{3} C\ x^6 + c_1\ x + c_2$$

Where $A = \frac{\left(e^{100\ a}\right)^2 - 1}{\left(e^{100\ a}\right)^2 + 1}$, $B = \frac{\left(e^{100a}\right)^2 \alpha}{\left(\left(e^{100a}\right)^2 + 1\right)^2}$, $C = \frac{\left(e^{100\ a}\right)^2\left(\left(e^{100\ a}\right)^2 - 1\right)\alpha^2}{\left(\left(e^{100\ a}\right)^2 + 1\right)^3}$ (26)

Applying the boundary conditions and simplifying, we have

$$A \approx 1,\ B \approx 0\ ,\ C \approx 0 \quad (27)$$

Substituting (27) we have

$$u(x) = \alpha\delta\left(\frac{1}{\beta} + \frac{1}{2} - \frac{x^2}{2}\right) \quad (28)$$

Enforcing the condition $u(1) < 1 < u(0)$ **,** we have

$$\frac{1}{\beta} \le \frac{1}{\alpha\delta} < \frac{2+\beta}{2\beta} \quad (29)$$

2.4. NUMERICAL SOLUTION (METHOD OF LINES)

The method of lines is regarded as a special finite difference method but more effective with respect to accuracy and computational time than the regular finite difference method. The method of lines (MOL) involves discretising the spatial domain and thus replacing the partial differential equation with a vector system of ordinary differential equations(ODEs), for which efficient and effective integrating packages have been developed [17,18,19]. The MATLAB package has strong vector and matrix handling capabilities, a good set of ODE solvers, and an extensive functionality which can be used to implement the MOL [19]. MOL has the merits of both the finite difference method and analytical method. Results on stability of the method are given by [20, 21].

We apply finite difference method to discretise the spatial domain $x \in (0,1]$ of equation (11).

Using the usual central difference approximation for $\frac{\partial^2 u}{\partial x^2}$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} + O(\Delta x^2)$$

Substituting in (11) gives

$$\frac{\partial u_i}{\partial t} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} + \alpha\,(0.5 - (0.5 - \delta)\tanh 100\,(u_i - 1)) \quad (30)$$

The second order approximation for u_x is given as

$$u_x = \frac{u_{i+1} - u_{i-1}}{2(\Delta x)} + O(\Delta x^2)$$

Applying this to the boundary condition (2) we have

$$u_{i+1} = u_{i-1} \qquad i = 1 \quad (31)$$

And to the boundary conditions (3) we have

$$u_{i+1} = u_{i-1} - 2\beta\Delta x u_i\ , \qquad i = N \quad (32)$$

substituting(31) and (32) in (30) gives a system of approximating ordinary differential equations.

For the warm phase, the system can be written as

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \vdots \\ \dot{u}_{N-1} \\ \dot{u}_N \end{bmatrix} = \frac{1}{(\Delta x)^2} \begin{bmatrix} -2 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & -2(1+\beta\Delta x) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} + \begin{bmatrix} \alpha\left(0.5-(0.5-\delta)\tanh 100(u_1-1)\right) \\ \alpha(0.5-(0.5-\delta)\tanh 100(u_2-1)) \\ \alpha(0.5-(0.5-\delta)\tanh 100(u_3-1)) \\ \vdots \\ \alpha(0.5-(0.5-\delta)\tanh 100(u_{N-1}-1)) \\ \alpha(0.5-(0.5-\delta)\tanh 100(u_N-1)) \end{bmatrix} \quad (33)$$

$$u_i\,(0)=0 \quad (34)$$

2.5. Stability Analysis

We apply the indirect method of Lyapunov to determine the local stability of the system. According to Lyapunov, if the linearization of the system exists, its stability determines the local stability of the original system [21].

Theorem1. (Lyapunov's indirect method)

Let $x=0$ be an equilibrium point for the nonlinear system $\dot{x}=f(x)$, where $f:D\to R^n$ is continuously differentiable and D is a neighborhood of the origin. Let the Jacobian matrix A at $x=0$ be:

$A=\left.\frac{\partial f}{\partial x}\right|_{x=0}$. Let $\lambda_i,\ i=1,\ldots,n$ be the eigenvalues of A. Then,

1. The origin is asymptotically stable if $\mathrm{Re}(\lambda_i)<0$ for all eigenvalue of A.
2. The origin is unstable if $\mathrm{Re}(\lambda_i)>0$ for any of the eigenvalues of A [23].

Evaluating the eigenvalues of the linearized equation for $\alpha=2000$, $\beta=0.2$ and $\Delta x=0.05$, shows that all eigenvalues are real and negative; hence the solution is stable.

This system of ordinary differential equations (ODEs) is then integrated using the Matlab integrator ode15s which is a stiff integrator since the ordinary differential equations in the system are sufficiently stiff. The values of α and β used are chosen to satisfy inequalities (14), (17) and (29) obtained from exact steady state solution.

3. Results

Results obtained are shown in table 1.

Table 1

Table of exact solution and numerical solutions by method of lines

x	COLD PHASE		WARM PHASE		HOT PHASE	
	$u(x)$ (Exact)	$u(x)$ (Numerical)	$u(x)$ (Exact)	$u(x)$ (Numerical)	$u(x)$ (Exact)	$u(x)$ (Numerical)
0.0	0.5500	0.550000	1.1	1.105563	5.500	5.50000
0.1	0.5495	0.549500	1.099	1.105102	5.495	5.49500
0.2	0.5480	0.548000	1.096	1.03707	5.480	5.48000
0.3	0.5455	0.545500	1.091	1.101377	5.455	5.45500
0.4	0.5420	0.542000	1.084	1.097925	5.420	5.42000
0.5	0.5375	0.537500	1.075	1.093381	5.375	5.37500
0.6	0.5320	0.532000	1.064	1.087693	5.320	5.32000
0.7	0.5255	0.525500	1.051	1.080428	5.255	5.25500
0.8	0.5180	0.518000	1.036	1.071730	5.180	5.18000
0.9	0.5095	0.509500	1.019	1.061253	5.095	5.09500
1.0	0.5000	0.500000	1.000	1.048011	5.000	5.00000

4. CONCLUSION

We have presented a mathematical model of the PTC thermistor problem with a new conductivity which is a hyperbolic-tangent approximation and describes the qualitative behaviour of the evolving solution of the thermistor in the entire domain. Result obtained for all the phases of temperature evolution shows that our approximation is a better representation for the electrical conductivity of the PTC thermistor. Moreover, for numerical techniques the absence of a discontinuity will improve stability and convergence properties, the new electrical conductivity is therefore a good improvement over the step function conductivity and the modified electrical conductivity in that it describes the conductivity and takes care of the discontinuities. We have also shown that the method of lines is a good method for solving the problem since results obtained are in good agreement with exact steady state solutions. In addition we showed that the solutions obtained by the method of lines are stable solutions.

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