Some Inequalities for the Extension of k-Gamma Function

Original Research Article

Abstract

In this paper, some inequalities involving the extension of k-gamma function are presented. Consequently, some previous results are recovered as particular cases of the present results.

Keywords: Gamma function; extension of k-gamma function; extension of k-digamma function; inequality

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1 Introduction

In recent years, some extensions of the well known Euler's classical gamma function have been considered by several authors. Also many properties and inequalities concerning these functions have been examined; see for example, [Askey (1978)], [Diaz and Pariguan (2007)], [Diaz and Teruel (2005)], [Kokologiannaki and Krasniqi (2013)], [Nantomah et al. (2016)] and [Krasniqi and Qi (2014)]. The Chaudhry-Zubair extension of the gamma function is defined as [Chaudhry et al. (1997)]

$$\Gamma_b(z) = \int_0^\infty t^{z-1} e^{-t-\frac{b}{t}} dt, \quad Re(b) > 0, Re(z) > 0, \tag{1.1}$$

and satisfies the recursion relation and reflection formula respectively as

$$\Gamma_b(z+1) = z\Gamma_b(z) + b\Gamma_b(z-1),$$
$$\Gamma_b(-z) = b^{-z}\Gamma_b(z).$$

In the case $b = 0$, The Chaudhry-Zubair extension of the gamma function conclude with the classical gamma function. Mubeen have introduced the following extension of k-gamma function [Mubeen et al. (2016)]

$$\Gamma_{b,k}(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}} dt, \quad Re(z) > 0, b \geq 0, k > 0. \tag{1.2}$$

Note that, when $b = 0$, $\Gamma_{b,k}(z)$ tends to the k-gamma function defined by [Diaz and Pariguan (2007)]

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} \, dt, \quad k > 0, Re(z) > 0. \tag{1.3}$$

Also, when $k = 1$, $\Gamma_{b,k}(z)$ tends to $\Gamma_b(z)$ and if both $b = 0$ and $k = 1$, then $\Gamma_{b,k}(z)$ tends to Euler's classical gamma function $\Gamma(z)$.

Some properties of the extended gamma k-function $\Gamma_{b,k}(x)$ are given in [Mubeen et al. (2016)] as follows:

$$\Gamma_{b,k}(x+k) = x\Gamma_{b,k}(x) + b^k \Gamma_{b,k}(x-k), \quad b \geq 0, k > 0 \quad \text{(difference formula)}, \tag{1.4}$$

$$b^x \Gamma_{b,k}(-x) = \Gamma_{b,k}(x), \quad Re(b) > 0, k > 0 \quad \text{(reflection formula)}. \tag{1.5}$$

Throughout of this work, $\mathbb{N}$ indicates the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
By differentiating repeatedly (1.3) with respect to z, one can obtain

$$\Gamma_{b,k}^{(m)}(z) = \int_0^\infty t^{z-1} (\ln t)^m e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}} \, dt, \quad Re(z) > 0, b \geq 0, k > 0 \tag{1.6}$$

where $m \in \mathbb{N}$.
In this paper our goal is to give some inequalities concerning the function $\Gamma_{b,k}^{(m)}(x)$ for $x > 0$ by using similar techniques as in [Nantomah et al. (2017)] and [Atugba and Nantomah (2019)]. Our results are also generalizations of some known results in the literature.

2 Main Results

In this section we present our main results by using Holder's, Minkowski's and Young's inequalities among other algebraic tools.

Lemma 1 (Mitrinovic (1970))**.** *(Holder's Inequality) Let $\alpha, \beta \in (0,1)$ and $\alpha + \beta = 1$. If $f(x)$ and $g(x)$ are integrable real valued functions on $[0, \infty)$, then the inequality*

$$\int_0^\infty |f(x)g(x)| \, dx \leq \left[\int_0^\infty |f(x)|^\alpha \, dx \right]^{\frac{1}{\alpha}} + \left[\int_0^\infty |g(x)|^\beta \, dx \right]^{\frac{1}{\beta}} \tag{2.1}$$

holds.

Theorem 1. *Let $x, y > 0$, $b \geq 0$, $k > 0$, $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$, m, n even, $m, n \in \mathbb{N}_0$ and $\alpha m + \beta n \in \mathbb{N}_0$. Then the extension of k-gamma function satisfies the inequality*

$$\Gamma_{b,k}^{(\alpha m + \beta n)}(\alpha x + \beta y) \leq \left[\Gamma_{b,k}^{(m)}(x) \right]^\alpha \left[\Gamma_{b,k}^{(n)}(y) \right]^\beta.$$

Proof. By using the equation 1.6, we obtain

$$\Gamma_{b,k}^{(\alpha m + \beta n)}(\alpha x + \beta y) = \int_0^\infty t^{\alpha x + \beta y - 1} (\ln t)^{\alpha m + \beta n} e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}} \, dt.$$

Then since $\alpha + \beta = 1$, and m, n are even we have

$$\begin{aligned} \Gamma_{b,k}^{(\alpha m + \beta n)}(\alpha x + \beta y) &= \int_0^\infty t^{\alpha(x-1)} (\ln t)^{\alpha m} e^{\left(-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}\right)\alpha} t^{\beta(y-1)} (\ln t)^{\beta n} e^{\left(-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}\right)\beta} \, dt \\ &\leq \left[\int_0^\infty t^{x-1} (\ln t)^m e^{\left(-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}\right)} \right]^\alpha \left[\int_0^\infty t^{x-1} (\ln t)^n e^{\left(-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}\right)} \right]^\beta, \end{aligned}$$

by using Holder's inequality (2.1) and the result follows. □

Remark 1. *By letting $k = 1$ in the theorem 1, we obtain the theorem 3.1 of [Atugba and Nantomah (2019)].*

The following definition is well known in the literature; see for example Zhang (2012).

Definition 1. *Let $f : [a, b] \subset \mathbb{R} \to (0, \infty)$. Then f is called a log-convex function, if*

$$f(\alpha x + (1-\alpha)y) \leq [f(x)]^{\alpha}[f(y)]^{1-\alpha}$$

holds for any $x, y \in [a, b]$ and $\alpha \in [0, 1]$.

Corollary 1. *Let $x > 0,\ b \geq 0,\ k > 0,\ \alpha, \beta \in (0, 1),\ \alpha + \beta = 1,\ m$ even and $m \in \mathbb{N}_0$. Then the function $\Gamma^{(m)}_{b,k}(x)$ is log-convex.*

Proof. From the theorem 1 by letting $m = n$ we get

$$\Gamma^{(m)}_{b,k}(\alpha x + \beta y) \leq \left[\Gamma^{(m)}_{b,k}(x)\right]^{\alpha}\left[\Gamma^{(m)}_{b,k}(y)\right]^{\beta},$$

which completes the proof. □

Corollary 2. *Let $x > 0,\ b \geq 0$ and $k > 0$. Then the function $\Gamma_{b,k}(x)$ satisfies the inequality*

$$\Gamma_{b,k}(x)\Gamma^{''}_{b,k}(x) \geq [\Gamma^{'}_{b,k}(x)]^2.$$

Proof. From the log convexity property of $\Gamma_{b,k}(x)$ we have $[\ln \Gamma_{b,k}(x)]^{''} \geq 0$. Then

$$[\ln \Gamma_{b,k}(x)]^{''} = \left[\frac{\Gamma^{'}_{b,k}(x)}{\Gamma_{b,k}(x)}\right] = \frac{\Gamma_{b,k}(x)\Gamma^{''}_{b,k}(x) - [\Gamma^{'}_{b,k}(x)]^2}{[\Gamma_{b,k}(x)]^2} \geq 0,$$

and the proof completes. □

Corollary 3. *Let $x > 0,\ b \geq 0,\ k > 0,\ m \in \mathbb{N}_0$ and m even. Then the inequality*

$$[\Gamma^{(m+1)}_{b,k}(x)]^2 \leq \Gamma^{(m)}_{b,k}(x)\Gamma^{(m+2)}_{b,k}(x)$$

holds.

Proof. Let $n = m + 2$, $\alpha = \beta = \frac{1}{2}$ and $x = y$ in the theorem 1. □

Definition 2. *We introduce the extended k-digamma (k-psi) function $\psi_{b,k}(x)$ as the logarithmic derivative of $\Gamma_{b,k}(x)$;*

$$\psi_{b,k}(x) = \frac{d}{dx}\ln \Gamma_{b,k}(x) = \frac{\Gamma^{'}_{b,k}(x)}{\Gamma_{b,k}(x)} = \frac{1}{\Gamma_{b,k}(x)}\int_0^{\infty} t^{x-1}\ln t\, e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}}\, dt$$

and more generally the extended k-polygamma function $\psi^{(m)}_{b,k}(x)$ by

$$\psi^{(m)}_{b,k}(x) = \frac{d^{m+1}}{dx}\ln \Gamma_{b,k}(x)$$

for $b, k > 0$, $m = 1, 2, \ldots$ and $x > 0$.

From the difference formula (1.4), we get

$$\ln \Gamma_{b,k}(x + k) = \ln x + \ln \Gamma_{b,k}(x) + k \ln b + \ln \Gamma_{b,k}(x - k).$$

Then,

$$\psi_{b,k}(x + k) = \frac{1}{x} + \psi_{b,k}(x) + \psi_{b,k}(x - k). \tag{2.2}$$

Theorem 2. *The function $\psi_{b,k}(x)$ is increasing for $x > 0$.*

Proof. Since $\Gamma_{b,k}(x)$ is log-convex function we have $[\ln \Gamma_{b,k}(x)]^{''} \geq 0$ for all $x > 0$. Then,

$$\psi_{b,k}^{'}(x) = [\ln \Gamma_{b,k}(x)]^{''} = \frac{\Gamma_{b,k}(x)\Gamma_{b,k}^{''}(x) - \left(\Gamma_{b,k}^{'}(x)\right)^2}{[\Gamma_{b,k}(x)]^2} \geq 0$$

by using the corollary 2. □

Theorem 3. *The following reflection formulas hold true for $b, k > 0$, $m = 1, 2, \ldots$ and $x > 0$,*

$$\psi_{b,k}(x) + \psi_{b,k}(-x) = \ln b, \tag{2.3}$$

$$\psi_{b,k}^{(m)}(x) = (-1)^{m+1}\psi_{b,k}^{(m)}(-x). \tag{2.4}$$

Proof. By using the reflection formula 1.5, we have

$$x \ln b + \ln \Gamma_{b,k}(-x) = \ln \Gamma_{b,k}(x),$$

and taking the derivative of both sides in the last equation, we obtain the equation 2.3. Also taking the derivatives of the equation 2.3 repeatedly, we get the equation 2.4. □

Theorem 4. *Let $x, y > 0$, $b \geq 0$, $k > 0$, $m \in \mathbb{N}_0$, m even, $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$ and $s \geq 0$. Then the inequality*

$$\Gamma_{b,k}^{(m)}(\alpha x + \beta y + s) \leq [\Gamma_{b,k}^{(m)}(x + s)]^{\alpha}[\Gamma_{b,k}^{(m)}(y + s)]^{\beta}$$

is valid.

Proof. By using the equation (1.6) and Holder's inequality, we have

$$\begin{aligned}
\Gamma_{b,k}^{(m)}(\alpha x + \beta y + s) &= \int_0^{\infty} t^{(\alpha x+\beta y+s)-1}(\ln t)^m e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}\,dt \\
&= \int_0^{\infty} t^{\alpha x+s\alpha-\alpha}(\ln t)^{\alpha m}\, e^{\left(-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}\right)\alpha}\, t^{\beta y+s\beta-\beta}(\ln t)^{\beta m}\, e^{\left(-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}\right)\beta} \\
&\leq \left[\int_0^{\infty} t^{x+s-1}(\ln t)^m e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}\,dt\right]^{\alpha}\left[\int_0^{\infty} t^{y+s-1}(\ln t)^m e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}\,dt\right]^{\beta} \\
&= [\Gamma_{b,k}^{(m)}(x + s)]^{\alpha}[\Gamma_{b,k}^{(m)}(y + s)]^{\beta}.
\end{aligned}$$

Hence;

$$\Gamma_{b,k}^{(m)}(\alpha x + \beta y + s) \leq [\Gamma_{b,k}^{(m)}(x + s)]^{\alpha}[\Gamma_{b,k}^{(m)}(y + s)]^{\beta}.$$

□

Theorem 5. *Let $x, y > 0$, $b \geq 0$, $a, c, k > 0$, $m \in \mathbb{N}_0$, m even, $\alpha, \beta \in (0, 1)$ and $\alpha + \beta = 1$. Then the function $\Gamma_{b,k}(x)$ satisfy the inequality*

$$\Gamma_{b,k}^{(m)}(ax + cy) \leq [\Gamma_{b,k}^{(m)}(\frac{ax}{\alpha})]^{\alpha}[\Gamma_{b,k}^{(m)}(\frac{cy}{\beta})]^{\beta}.$$

Proof. Similarly, by the Holder's inequality, we obtain

$$\Gamma_{b,k}^{(m)}(ax+cy) = \int_0^\infty t^{ax+cy-1}(\ln t)^m\, e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}\,dt$$
$$= \int_0^\infty t^{ax-\alpha}(\ln t)^{\alpha m}\, e^{\left(-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}\right)\alpha}\, t^{cy-\beta}(\ln t)^{\beta m}\, e^{\left(-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}\right)\beta}$$
$$\leq \Big[\int_0^\infty t^{\frac{ax}{\alpha}-1}(\ln t)^m\, e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}\,dt\Big]^\alpha \Big[\int_0^\infty t^{\frac{cy}{\beta}-1}(\ln t)^m\, e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}\,dt\Big]^\beta,$$

establishing the result. □

Lemma 2. *[Mitrinovic (1970)](Young's Inequality) If a and b are nonnegative, $\alpha,\beta \in (0,1)$ and $\alpha+\beta = 1$, then the inequality*

$$a^\alpha b^\beta \leq \alpha a + \beta b \tag{2.5}$$

holds.

Corollary 4. *Let $x, y > 0$, $b \geq 0$, $a, c, k > 0$, $m \in \mathbb{N}_0$, m even, $\alpha, \beta \in (0,1)$ and $\alpha+\beta = 1$. Then the following inequality holds*

$$\Gamma_{b,k}^{(m)}(ax+cy) \leq \alpha\Gamma_{b,k}^{(m)}\big(\frac{ax}{\alpha}\big) + \beta\Gamma_{b,k}^{(m)}\big(\frac{cy}{\beta}\big).$$

Proof. The proof follows immediately by using the theorem 5 and the lemma 2.5. □

Remark 2. *Let $m = n = 0$ and $a = b = 1$ in the theorem 5. Then we obtain the theorem 3.9 and corollary 3.10 in [Atugba and Nantomah (2019)].*

Lemma 3 (Mitrinovic (1970))**.** *(Minkowski's Inequality) Let $1 \leq p < \infty$. If $f(x)$ and $g(x)$ are integrable real valued functions on $[0,\infty)$, then the inequality*

$$\Big[\int_0^\infty |f(x)+g(x)|^p\,dx\Big]^{\frac{1}{p}} \leq \Big[\int_0^\infty |f(x)|^p\,dx\Big]^{\frac{1}{p}} + \Big[\int_0^\infty |g(x)|^p\,dx\Big]^{\frac{1}{p}} \tag{2.6}$$

holds.

Theorem 6. *Let $x, y > 0$, $b \geq 0$, $k > 0$, $m, n \in \mathbb{N}_0$, m, n even, $\alpha, \beta \in (0,1)$ and $u \geq 1$. Then the inequality*

$$[\Gamma_{b,k}^{(m)}(x) + \Gamma_{b,k}^{(n)}(y)]^{\frac{1}{u}} \leq (\Gamma_{b,k}^{(m)}(x))^{\frac{1}{u}}(\Gamma_{b,k}^{(n)}(y))^{\frac{1}{u}}$$

holds for $x, y > 0$.

Proof. Since $x^k + y^k \leq (x+y)^k$, for $x, y \geq 0$ and $k \geq 1$, by using Minkowski's inequality we obtain that

$$[\Gamma_{b,k}^{(m)}(x) + \Gamma_{b,k}^{(n)}(y)]^{\frac{1}{u}} = \Big[\int_0^\infty t^{x-1}(\ln t)^m e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}\,dt + \int_0^\infty t^{y-1}(\ln t)^n e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}\,dt\Big]^{\frac{1}{u}}$$
$$\leq \Big[\int_0^\infty t^{\frac{x-1}{u}}(\ln t)^{\frac{m}{u}} e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}\frac{1}{u}} + t^{\frac{y-1}{u}}(\ln t)^{\frac{n}{u}} e^{\left(-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}\right)\frac{1}{u}}\Big]^{\frac{1}{u}}$$
$$\leq \Big[\int_0^\infty t^{x-1}(\ln t)^m e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}\,dt\Big]^{\frac{1}{u}} + \Big[\int_0^\infty t^{y-1}(\ln t)^n e^{-\frac{t^k}{k}-\frac{b^k t^{-k}}{k}}\,dt\Big]^{\frac{1}{u}},$$

and the proof completes. □

Remark 3. *By letting $k = 1$ in the theorem 6, we obtain the theorem 3.12 of [Atugba and Nantomah (2019)].*

Theorem 7. *The inequality*

$$\Gamma_{b,k}^{(m)}(x) \leq \frac{\Gamma_{b,k}^{(m-r)}(x) + \Gamma_{b,k}^{(m+r)}(x)}{2}$$

is valid for $x > 0$, $m, r \in \mathbb{N}_0$, m, r even such that $m \geq r$.

Proof. By direct computation, we obtain the result since we have

$$\begin{aligned}\Gamma_{b,k}^{(m-r)}(x) + \Gamma_{b,k}^{(m+r)}(x) - 2\Gamma_{b,k}^{(m)}(x) &= \int_0^\infty \Big[\frac{1}{(\ln t)^r} + (\ln t)^r - 2\Big](\ln t)^m t^{x-1} e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}}\, dt \\ &= \int_0^\infty \big[1 - (\ln t)^r\big]^2 (\ln t)^{m-r} t^{x-1} e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}}\, dt \geq 0.\end{aligned}$$

□

Theorem 8. *Let $b, k > 0$, $m \in \mathbb{N}_0$ and m even. Then for $0 < a \leq 1$, the inequalities*

$$[\Gamma_{b,k}^{(m)}(k)]^{a-1} \leq \frac{\Gamma_{b,k}^{(m)}(k+x)}{\Gamma_{b,k}^{(m)}(k+ax)} \leq \frac{\Gamma_{b,k}^{(m)}(2k)}{\Gamma_{b,k}^{(m)}(k+ak)} \tag{2.7}$$

hold true for $x \in [0, k]$. If $a \geq 1$, then the inequalities (2.7) are reversed.

Proof. From the corollary 1 we have $\Gamma_{b,k}^{(m)}(x+k)$ is log-convex. Then logarithmic derivative of $\Gamma_{b,k}^{(m)}(x+k)$ is increasing. Let $f(x) = [\ln \Gamma_{b,k}^{(m)}(x+k)]'$ and $g(x) = \frac{[\Gamma_{b,k}^{(m)}(x+k)]^a}{\Gamma_{b,k}^{(m)}(ax+k)}$. Then

$$\ln g(x) = a \ln \Gamma_{b,k}^{(m)}(x+k) - \ln \Gamma_{b,k}^{(m)}(ax+k).$$

Now, taking derivatives of both sides of the last equation, we get

$$\frac{g'(x)}{g(x)} = a[f(x) - f(ax)].$$

If $0 < a \leq 1$ then $g'(x) \geq 0$, since $f(x)$ is increasing and $g(x) > 0$. Then the equation (2.7) follows for $x \in [0, k]$. Similarly, for $a \geq 1$ reverse of the equation (2.7) is satisfied. □

Theorem 9. *Suppose that $s \in (0, 1)$, $b, k > 0$, $m \in \mathbb{N}_0$ and m even. Then the inequality*

$$\Gamma_{b,k}^{(m)}(x+s) \leq [\Gamma_{b,k}^{(m)}(x)]^{1-s}[\Gamma_{b,k}^{(m)}(x+1)]^s$$

is valid for $x > 0$.

Proof. Let $a = \frac{1}{1-s}$, $b = \frac{1}{s}$, $f(t) = t^{(1-s)(x-1)}(\ln t)^{m(1-s)}\, e^{-(1-s)\left(\frac{t^k}{k} - \frac{b^k t^{-k}}{k}\right)}$ and $g(t) = t^{sx}(\ln t)^{ms}\, e^{-s\left(\frac{t^k}{k} - \frac{b^k t^{-k}}{k}\right)}$. Then by using Holder's inequality we get

$$\begin{aligned}\Gamma_{b,k}^{(m)}(x+s) &\leq \Big[\int_0^\infty \Big(t^{(1-s)(x-1)}(\ln t)^{m(1-s)} e^{-(1-s)\left(\frac{t^k}{k} - \frac{b^k t^{-k}}{k}\right)}\Big)^{\frac{1}{1-s}}\, dt\Big]^{1-s} \times \\ &\quad \times \Big[\int_0^\infty \Big(t^{sx}(\ln t)^{ms} e^{-s\left(\frac{t^k}{k} - \frac{b^k t^{-k}}{k}\right)}\Big)^{\frac{1}{s}}\, dt\Big]^s \\ &= \Big[\int_0^\infty t^{x-1}(\ln t)^m e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}}\, dt\Big]^{1-s} \Big[\int_0^\infty (t^x (\ln t)^m e^{-\frac{t^k}{k} - \frac{b^k t^{-k}}{k}}\, dt\Big]^s dt \\ &= [\Gamma_{b,k}^{(m)}(x)]^{1-s}[\Gamma_{b,k}^{(m)}(x+1)]^s,\end{aligned}$$

completes the proof of the theorem. □

3 CONCLUSIONS

In this study, we establish some inequalities for the extension of k-gamma function by using the classical Holder's and Minkowski's inequalities and other algebraic tools. The established results are recovered as particular cases of some previous results.

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