DYNAMIC BUCKLING ANALYSIS OF A CLAMPED AND VISCOUSLY DAMPED COLUMN RESTING ON A QUADRATIC – CUBIC ELASTIC FOUNDATION BUT PRESSURIZED BY A STEP LOAD

ABSTRACT: This work discussed the analysis of the dynamic buckling of a clamped finite imperfect viscously damped column that is subjected to a step load lying on a quadratic-cubic elastic foundation, using the methods of asymptotics and perturbation technique. The formulation of the governing equation contains two small independent parameters (δ and ϵ) which are used in asymptotic expansions of the relevant variables. The results of the analysis show that: (a) the dynamic buckling load decreases with increased imperfections. (b) in the case of the column considered, the dynamic buckling load decreases with increase in damping. The results obtained are strictly asymptotic and therefore valid as the small parameters become increasingly small relative to unity.

Keywords: Dynamic Buckling, Viscous damping, asymptotics and perturbation technique, Column-like elastic structures

1.0 INTRODUCTION

Buckling is a phenomenon associated with failure of column-like structures. Structures on non-linear elastic foundations are commonly used in engineering applications and occupy a prominent place in structural mechanics. These structures can also serve as simplified models for complex non-linear systems such as columns, shells and plates. Globally, collapse of buildings, bridges and other material structures are issues of concern. Structural failures are forms of material failures which are dangerous in nature and should be prevented by all cost. Series of investigations and studies have been done by Engineers and Applied Mathematicians to determine the maximum loads structures can carry before buckling occurs, yet buckling of elastic structures remain inevitable. Structural elastic materials normally display certain tendencies of failures and instability when loaded either statically or dynamically and one of the pre-occupations of the Structural Engineers and Applied Mathematicians is the determination of the load which a given elastic material can support prior to buckling. A vast quantum of insights on dynamic stability of elastic structure has been achieved by subjecting these materials to diverse dynamic loading conditions. These loads include, step loading, impulsive loading, rectangular loading, triangular loading [1] and even periodic loading [1] and [2]. From these findings, it has become firmly established that initial imperfections, and to a lesser extent, the loading duration, are some of the main factors that have been seriously implicated as causative agents of reduction of the elastic strength of these materials. [3] investigated the dynamic response of columns under impulsive axial compression. The investigation has been carried out on clamped specimens, made of metals and composite materials, loaded impulsively by a striking mass. In the theoretical study Rayleigh-type beam equations were assumed for a geometrically imperfect column of a linear-elastic anisotropic material, and the numerical solution, yielded buckling behaviour that correlated well with the experimental results. The results have shown that initial geometrical imperfections, duration of impulse and effective slenderness have a major influence on the buckling loads, whereas the effect of the material is secondary. Recent studies on dynamic buckling have been directed principally on columns, beams, plates, spherical shells and cylindrical shells, and so, extensive literatures (most often numerical approach), have since come to limelight. In this regard, mention must be made of [4], who studied some important parameters in dynamic buckling analysis of plated structures subjected to pulse loading, while [5] equally investigated the buckling of impulsively loaded prismatic cores. In the same token, [6] studied the dynamic buckling of thin-walled composite plates with varying width-wise material properties while [7] also investigated interactive dynamic buckling of thin-walled columns. We now mention [8], who studied the dynamic buckling of thin-walled viscoplastic columns, while [9] similarly investigated some aspects of dynamic buckling of plates under in-plane pulse compression. A study on longitudinal step-wise loading was undertaken by [10], while [11] investigated triply coupled vibrations of axially loaded thin-walled composite beams. An investigation on computational nonlinear stochastic dynamics was undertaken by [12], while [13] discussed nonlinear stochastic dynamical post buckling analysis of uncertain cylindrical shells. Similarly, [14] as well as [15], and [16] made excellent contributions to the dynamics of dynamic buckling. An investigation into the dynamic effect of lateral buckling of high temperature/high pressure offshore pipeline was carried out by [17. In the same token, [18] investigated the dynamic buckling and fragmentation in brittle rods, while a study on the vibration of nonlocal Kelvin-Voight viscoelastic damped Timoshenko beams was undertaken by [19]. The study by [20] on non-linear analysis of viscoelastic rectangular plates subjected to in-plane compression was insightful. [21] also investigated the static buckling of infinitely column lying on quadratic-cubic elastic foundations using asymptotic approach, similarly [22] analyzed the dynamic stability of a simple quadratic elastic model structure that is pre-statically loaded but trapped by a step load using asymptotic approach.

The dynamic buckling load of a viscously damped elastic structure trapped by a step load is a real life problem and the governing equation is the mathematical generalization of some of the physical structures encountered in engineering practice. This work aims at investigating, using asymptotic and perturbation procedures, the dynamic buckling of a viscously damped but clamped finite column lying on a quadratic-cubic nonlinear foundation. In addition, the effects of light viscous damping as well as imperfection on the dynamic stability of the structure are discussed. This work aims at determining the dynamic buckling load of a finite imperfect elastic structure namely a viscously damped but clamped column trapped by a step load by means of approximate analytic approach namely, the asymptotic and perturbation methods.

The dynamic buckling load λ_D is defined as the maximum load parameter for which the displacement or solution of the governing equation remains bounded for all time and is obtained from the maximization [1],

$$\frac{d\lambda}{dU_a} = 0 \qquad (1.1)$$

where λ is the load parameter and U_a is the maximum value of the displacement of the column.

2.0 FORMULATION OF THE PROBLEM

The usual dimensional differential equation satisfied by the deflection $W(X,T)$of the column under consideration satisfies the following partial differential equation, as in [23] and [24],

$$m_0 W_{,TT} + c_0 W_{,T} + EIW_{,XXXX} + 2P(T)W_{,XX} + Wk_1 - k_2W^2 - k_3W^3 = -2P(T)\frac{d^2\overline{W}}{dX^2}, T > 0 \quad (2.2a)$$

$$0 < X < \pi \qquad (2.2b)$$

$$W(X,0) = 0 = W_{,T}(X,0) = 0, \; 0 < X < \pi \qquad (2.3)$$

$$W = W_{,X} = 0 \text{ at } X = 0, \pi \qquad (2.4)$$

where, m_0 is the mass per unit length ,c_0 is the damping coefficient, EI is the bending stiffness where, E and I are the Young's modulus and I is the moment of inertia respectively.

Here the nonlinear elastic foundation exerts a force per unit length given by $Wk_1 - k_2W^2 - k_3W^3$ on the column where k_1, k_2 and $k_3$are constants such that k_1>0, k_2>0 , k_3>0. In this formulation, all nonlinearities higher than cubic are excluded, while all nonlinear derivatives of W (X,T) are also excluded. Here, $\overline{W}$ is the stress-free time independent twice-differentiable initial imperfection displacement and all aspects of axial inertia are neglected.

3.0 PERTURBATION PROCEDURE

To reduce equation (2.2) to (2.4) to non-dimensional form, we adopt the following quantities:

$$x = \left(\frac{k_1}{EI}\right)^{\frac{1}{4}} X, \; \omega = \left(\frac{k_2}{k_1}\right)^{\frac{1}{2}} W, \; \lambda f(t) = \frac{P(T)}{2(EIk_1)^{\frac{1}{2}}}, \; t = \left(\frac{k_1}{m_0}\right)^{\frac{1}{2}} T, \; \epsilon\overline{\omega} = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} \overline{W}, \; 2\delta = \frac{c_0}{(m_0k_1)^{\frac{1}{2}}}, \; \alpha = \frac{k_2}{\sqrt{k_1k_2}},$$

$$\beta = \left(\frac{k_3}{k_1}\right)^{\frac{3}{2}} \qquad (3.5a)$$

Here, we shall assume the following inequalities

$$0 < \delta << 1, \; 0 < \epsilon << 1. \qquad (3.5b)$$

On substituting (3.5a) in (3.2) and simplifying, the following is obtained

$$\omega_{,tt} + 2\delta\omega_{,t} + \omega_{,xxxx} + 2\lambda f(t)\omega_{,xx} + \omega - \alpha\omega^2 - \beta\omega^3 = -2\epsilon\lambda f(t)\frac{d^2\overline{\omega}}{dx^2} \qquad (3.6)$$

$t > 0,\ 0 < x < \pi$ (3.7a)

$\omega\ (x, 0) = 0 = \omega_{,t}\ (x,0) = 0\ ,\quad 0 < x < \pi$ (3.7b)

$\omega = \omega_{,x} = 0$ at $x = 0, \pi$ (3.7c)

where, ω is the displacement, t is the time variable, δ is the damping coefficient, α and β are the imperfection – sensitivity parameters, ϵ is the amplitude of the imperfection, $\overline{\omega}$ is a stress-free time independent twice-differentiable imperfection and f(t) is a time dependent loading function while λ is the nondimentional amplitude (or magnitude) of the loading.

Here, a subscript following a comma indicates partial differentiation while $\overline{\omega}$ is a twice-differentiable stress-free imperfection and $f(t)$ is a step load such that,

$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \quad (3.8)$$

Here, it is assumed that $\delta\ and\ \epsilon$ are two small but unrelated parameters that satisfy the inequalities as in (3.5b). Our ultimate aim is to determine the dynamic buckling load λ_D which is obtained by using the maximization (3.1).

Let,

$$\tau = \delta t \quad (3.9a)$$

$$\hat{t} = t + \frac{1}{\delta}\left[\omega_1(\tau)\epsilon + \omega_2(\tau)\epsilon^2 + \omega_3(\tau)\epsilon^3 + \omega_4(\tau)\epsilon^4 + \cdots\right. \quad (3.9b)$$

where,

$$\omega_i(0) = 0,\ i = 1,2,3, \ldots . \quad (3.10a)$$

Let,

$$\omega\ (x, t) = U\ (x, t, \tau, \epsilon, \delta) \quad (3.10b)$$

From equation (3.10b); we have;

$$\omega_{,t} = \left(\frac{\partial u}{\partial \hat{t}}.\frac{\partial \hat{t}}{\partial t}\right) + \left(\frac{\partial u}{\partial \hat{t}}.\frac{\partial \hat{t}}{\partial \tau}.\frac{\partial \tau}{\partial t}\right) + \left(\frac{\partial u}{\partial \tau}.\frac{\partial \tau}{\partial t}\right) \quad (3.11)$$

$$= U_{,\hat{t}} + (\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \ldots\ldots.)U_{,t} + \delta U_{,\tau} \quad (3.12)$$

The following also follows:

$$\omega_{,tt} = U_{,\hat{t}\hat{t}} + (\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \ldots)^2 U_{,\hat{t}\hat{t}} + \delta^2 U_{,\tau\tau} + 2(\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \cdots)U_{,\hat{t}\hat{t}} + 2\delta U_{,\hat{t}\tau} + 2\delta(\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \ldots)U_{,\hat{t}\tau} + \delta(\omega_1''\epsilon + \omega_2''\epsilon^2 + \omega_3''\epsilon^3 + \ldots)U_{,\hat{t}} \quad (3.13)$$

Substituting (3.12) and (3.13) into equation (3.6) results to;

$$U_{,\hat{t}\hat{t}} + (\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \ldots)^2 U_{,\hat{t}\hat{t}} + \delta^2 U_{,\tau\tau} + 2(\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \cdots)U_{,\hat{t}\hat{t}} + 2\delta U_{\hat{t},\tau} + 2\delta(\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \ldots)U_{,\hat{t}\tau} + \delta(\omega_1''\epsilon + \omega_2''\epsilon^2 + \omega_3''\epsilon^3 + \ldots)U_{,\hat{t}} + 2\delta\left[U_{,\hat{t}} + (\omega_1'\epsilon + \omega_2'\epsilon^2 + \omega_3'\epsilon^3 + \ldots)U_{,\hat{t}} + \delta U_{,\tau}\right] + U_{,xxxx} + 2\lambda U_{,xx} + U + \alpha U^2 - \beta U^3 = -2\lambda\epsilon\frac{d^2\overline{\omega}}{dx^2} \quad (3.14)$$

Let,

$$U(x, \epsilon. \tau) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} U_n^{(i,j)}(x, t, \tau)\epsilon^i\delta^j \quad (3.15)$$

$$= \epsilon\left(U^{(10)} + \delta U^{(11)} + \delta^2 U^{(12)} + \cdots\right) + \epsilon^2\left(U^{(20)} + \delta U^{(21)} + \delta^2 U^{(22)} + \cdots\right) + \epsilon^3\left(U^{(30)} + \delta U^{(31)} + \delta^2 U^{(32)} + \cdots\right) + \cdots \quad (3.16)$$

Here, the ij in $U^{(ij)}$are not powers but superscripts. Therefore, the following orders of equations are obtained

$$O(\epsilon):\ U_{,\hat{t}\hat{t}}^{(10)} + U_{,xxxx}^{(10)} + 2\lambda U_{,xx}^{(10)} + U^{(10)} = -2\lambda\frac{d^2\tilde{\omega}}{dx^2}, \quad (3.17)$$

$$O(\epsilon\delta):\ U_{,\hat{t}\hat{t}}^{(11)} + U_{,xxxx}^{(11)} + 2\lambda U_{,xx}^{(11)} + U^{(11)} = -2U_{,\hat{t}\tau}^{(10)} - 2U_{,\hat{t}}^{(10)} \quad (3.18)$$

$$O(\epsilon\delta^2):\ U_{,\hat{t}\hat{t}}^{(12)} + U_{,xxxx}^{(12)} + 2\lambda U_{,xx}^{(12)} + U^{(12)} = -2U_{,\hat{t}\tau}^{(11)} - 2U_{,\hat{t}}^{(11)} - U_{,\tau\tau}^{(10)}$$

$$O(\epsilon^2):\ U_{,\hat{t}\hat{t}}^{(20)} + U_{,xxxx}^{(20)} + 2\lambda U_{,xx}^{(20)} + U^{(20)} = -\left(\alpha U^{(10)}\right)^2 - 2\omega_1' U_{,\hat{t}\hat{t}}^{(10)} \quad (3.20)$$

$$O(\epsilon^2\delta):\ U_{,\hat{t}\hat{t}}^{(21)} + U_{,xxxx}^{(21)} + 2\lambda U_{,xx}^{(21)} + U^{(21)} = -2\alpha U^{(10)}U^{(11)} - 2U_{,\hat{t}\tau}^{(20)} - 2U_{,\hat{t}}^{(20)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(11)} - \omega_1'' U_{,\hat{t}}^{(10)} - 2\omega_1' U_{,\hat{t}}^{(10)} \quad (3.21)$$

$$O(\epsilon^2\delta^2):\ U^{(22)}_{,\hat{t}\hat{t}} + U^{(22)}_{,xxxx} + 2\lambda U^{(22)}_{,xx} + U^{(22)} = -U^{(20)}_{,\tau\tau} - 2\omega_1' U^{(12)}_{,\hat{t}\hat{t}} - 2U^{(21)}_{,\hat{t}\,\tau} -$$
$$2\omega_1' U^{(12)}_{,\hat{t}\hat{t}} - 2\omega_1'' U^{(11)}_{,\hat{t}} - 2U^{(21)}_{,\hat{t}} - 2\omega_1' U^{(11)}_{,\hat{t}} - \alpha\left\{\left(\mathrm{U}^{(11)}\right)^2 + \mathrm{U}^{(10)}\mathrm{U}^{(12)}\right\} \qquad (3.22)$$

$$O(\epsilon^3)\ :\ U^{(30)}_{,\hat{t}\hat{t}} + U^{(30)}_{,xxxx} + 2\lambda U^{(30)}_{,xx} + U^{(30)} =$$
$$-(\omega_1')^2 U^{(10)}_{,\hat{t}\hat{t}} - 2\left(\omega_1' U^{(20)}_{,\hat{t}\hat{t}} + \omega_2' U^{(10)}_{,\hat{t}\hat{t}}\right) - 2\alpha \mathrm{U}^{(20)}\mathrm{U}^{(12)} + \beta\left(\mathrm{U}^{(10)}\right)^3 \qquad (3.23)$$

$$\begin{aligned} O(\epsilon^3\delta)\ :\ & U^{(31)}_{,\hat{t}\hat{t}} + U^{(31)}_{,xxxx} + 2\lambda U^{(31)}_{,xx} + U^{(31)} \\ &= -(\omega_1')^2 U^{(10)}_{,\hat{t}\hat{t}} - 2\left(\omega_1' U^{(21)}_{,\hat{t}\,\tau} + \omega_2' U^{(11)}_{,\hat{t}\,\tau}\right) - 2U^{(30)}_{,\hat{t}\,\tau} + 2\left(\omega_1' U^{(20)}_{,\hat{t}\hat{t}} + \omega_2' U^{(10)}_{,\hat{t}\hat{t}}\right) \\ &\quad - \left(\omega_1'' U^{(20)}_{,\hat{t}} + \omega_2''^{+} U^{(10)}_{,\hat{t}}\right) - 2\left\{U^{(30)}_{\hat{t}} + \left(\omega_1' U^{(20)}_{,\hat{t}} + \omega_2'^{+} U^{(10)}_{,\hat{t}}\right)\right\} \\ &\quad - \alpha\left(\mathrm{U}^{(10)}\mathrm{U}^{(21)} + \mathrm{U}^{(11)}\mathrm{U}^{(20)}\right) + 3\beta\left(\mathrm{U}^{(10)}\right)^2\left(\mathrm{U}^{(11)}\right) \qquad (3.24) \end{aligned}$$

$$\begin{aligned} O(\epsilon^3\delta^2)\ :\ & U^{(32)}_{,\hat{t}\hat{t}} + U^{(32)}_{,xxxx} + 2\lambda U^{(32)}_{,xx} + U^{(32)} \\ &= -(\omega_1')^2 U^{(12)}_{,\hat{t}\hat{t}} - U^{(30)}_{,\tau\tau} - 2\left(\omega_1' U^{(22)}_{,\hat{t}\hat{t}} + \omega_2' U^{(12)}_{,\hat{t}\hat{t}}\right) - 2U^{(31)}_{,\hat{t}\,\tau} - 2\left(\omega_1' U^{(21)}_{,\hat{t}\,\tau} + \omega_2' U^{(11)}_{,\hat{t}\tau}\right) \\ &\quad - \left(\omega_1'' U^{(21)}_{,\hat{t}} + \omega_2'' U^{(11)}_{,\hat{t}\,\tau}\right) - 2\left(U^{(31)}_{,\hat{t}} + \omega_1' U^{(21)}_{,\hat{t}} + \omega_2' U^{(11)}_{,\hat{t}}\right) - 2U^{(30)}_{,\tau} \\ &\quad - 2\alpha\left(\mathrm{U}^{(10)}\mathrm{U}^{(32)} + \mathrm{U}^{(11)}\mathrm{U}^{(21)} + \mathrm{U}^{(12)}\mathrm{U}^{(20)}\right) \\ &\quad + \beta\left[\left(\mathrm{U}^{(10)}\right)^2\mathrm{U}^{(12)} + 3\mathrm{U}^{(10)}\left(\mathrm{U}^{(10)}\right)^2\right] \qquad (3.25) \end{aligned}$$

The associated initial conditions are as follows:

$$O(\epsilon): U^{(ij)}(x,0,0) = 0;\ i = 1,2,3\ \dots,\ j = 1,2,3\ \dots \qquad (3.26)$$
$$O(\epsilon\delta): U^{(11)}_{,\hat{t}}(x,0,0) + U^{(10)}_{,\tau}(x,0,0) = 0 \qquad (3.27)$$
$$O(\epsilon\delta^2): U^{(12)}_{,\hat{t}}(x,0,0) + U^{(11)}_{,\tau}(x,0,0) = 0 \qquad (3.28)$$
$$O(\epsilon^2): U^{(20)}_{,\hat{t}}(x,0,0) + \omega_1'(0) U^{(10)}_{,\hat{t}}(x,0,0) = 0 \qquad (3.29)$$
$$O(\epsilon^2\delta): U^{(21)}_{,\hat{t}}(x,0,0) + \omega_1'(0) U^{(11)}_{,\hat{t}}(x,0,0) + U^{(20)}_{,\tau}(x,0,0) = 0 \qquad (3.30)$$

$$O(\epsilon^2\delta^2): U^{(22)}_{,\hat{t}}(x,0,0) + \omega_1'(0) U^{(12)}_{,\hat{t}}(x,0,0) + U^{(21)}_{,\tau}(x,0,0) = 0 \qquad (3.31)$$

$$O(\epsilon^3): U^{(30)}_{,\hat{t}}(x,0,0) + \omega_1'(0) U^{(20)}_{,\hat{t}}(x,0,0) + \omega_2'(0) U^{(10)}_{,\tau}(x,0,0) = 0 \qquad (3.32)$$

$$O(\epsilon^3\delta): U^{(31)}_{,\hat{t}}(x,0,0) + \omega_1'(0) U^{(21)}_{,\hat{t}}(x,0,0) + \omega_2'(0) U^{(11)}_{,\hat{t}}(x,0,0) + U^{(30)}_{,\tau}(x,0,0) = 0 \qquad (3.33)$$

$$O(\epsilon^3\delta^2): U^{(32)}_{,\hat{t}}(x,0,0) + \omega_1'(0) U^{(22)}_{,\hat{t}}(x,0,0) + \omega_2'(0) U^{(12)}_{,\hat{t}}(x,0,0) + U^{(31)}_{,\tau}(x,0,0) = 0 \qquad (3.34)$$

The associated Boundary Conditions are

$$U^{(ij)} = U^{(ij)}_{,x} = 0;\ x = 0, \pi \qquad (3.35)$$

4.0 DYNAMIC DEFORMATION OF THE COLUMN

Let

$$\bar{\omega} = \bar{a}_m(1 - cos2mx),\ \text{where } \bar{a}_m \text{ is a constant,} \qquad (4.1)$$

And let

$$U^{(ij)}(t,\tau,x) = \sum_{n=1}^{\infty} U^{(ij)}_n(\hat{t}\ \tau)(1 - cos2nx) \qquad (4.2)$$

<u>Solution of equation of order $\epsilon\delta^j$, j=0,1,2</u>

Substituting (4.1) and (4.2) into (3.17) gives

$$\sum_{n=1}^{\infty}(1 - cos2nx) U^{(10)}_{n,\hat{t}\hat{t}} + \{-16n^4 + 8\lambda n^2 + (1 - cos2nx)\} U^{(10)}_n$$
$$= -8\lambda m^2 \bar{a}_m cos2mx \qquad (4.3)$$

Multiplying (4.3) through by cos2mx and integrating from 0 to π and for n = m, the result is,

$$\int_0^{\pi}\sum_{n=1}^{\infty}\Big[\{(1-cos2nx)cos2mx\}U_{n,\hat{t}\hat{t}}^{(10)} + U_n^{(10)}\{(-16n^4+8\lambda n^2)cos2nxcos2mx + (1-cos2nx)\}cos2mx\,\Big]dx$$
$$= -\int_0^{\pi} 8\lambda m^2\bar{a}_m cos2mxdx = -\,8\lambda m^2\bar{a}_m\int_0^{\pi}\frac{(1+cos4mx)}{2}dx = \frac{-8\lambda m^2\bar{a}_m\,\pi}{2}$$
$$= -4\lambda m^2\bar{a}_m\,\pi \qquad (4.4)$$

The left hand side vanishes for all n except where n =m. Thus, for n=m, it easily follows that

$$\int_0^{\pi}\sum_{n=1}^{\infty}\Big[\{(1-cos2nx)cos2mx\}U_{n,\hat{t}\hat{t}}^{(10)} + \Big\{U_n^{(10)}(-16n^4+8\lambda n^2)cos2nx + (1-cos2nx)U_n^{(10)}\Big\}cos2mx\,\Big]dx \qquad (4.5)$$

It is to be noted that, when n=m, then

$$\int_0^{\pi} U_n^{(10)}(-16n^4+8\lambda n^2)cos2nxcos2mxdx = U_m^{(10)}(-16m^4+8\lambda m^2)\int_0^{\pi} cos^2 2mxdx = \frac{\pi}{2}U_m^{(10)}(-16m^4+8\lambda m^2) \qquad (4.6)$$

Thus, substituting (4.6) into (4.4), gives,

$$-\tfrac{\pi}{2}U_{m,\hat{t}\hat{t}}^{(10)} + \tfrac{\pi}{2}(-16m^4+8\lambda m^2)U_m^{(10)} - \tfrac{\pi}{2}U_m^{(10)} = -8\lambda m^2\bar{a}_m\left(\tfrac{\pi}{2}\right) \qquad (4.7a)$$

And this yields,

$$U_{m,\hat{t}\hat{t}}^{(10)} + (16m^4-8\lambda m^2+1)U_m^{(10)} + U_m^{(10)} = \quad 8\lambda m^2\bar{a}_m \qquad (4.7b)$$

Let,

$$16m^4 - 8\lambda m^2 + 1 = \theta^2 \qquad (4.7c)$$

Then (4.7b) becomes

$$U_{m,\hat{t}\hat{t}}^{(10)} + \theta^2 U_m^{(10)} + U_m^{(10)} = 8\lambda m^2\bar{a}_m \qquad (4.7d)$$

Initial conditions are

$$U_m^{(10)}(0,0) = 0;\; U_{m,\hat{t}}^{(10)}(0,0) = 0$$

Therefore, the solutions of (4.7d) is

$$U_m^{(10)} = \alpha_1(\tau)cos\theta\hat{t} + \beta_1(\tau)sin\theta\hat{t} + B \qquad (4.7e)$$

where, $B = \frac{8\lambda m^2\bar{a}_m}{\theta^2}$ (4.7f)

The use of initial conditions gives

$$\alpha_1(0) = -\frac{8\lambda m^2\bar{a}_m}{\theta^2}\;,\beta_1 = 0 \qquad (4.7g)$$

Thus

$$U^{(10)} = U_m^{(10)}(1-cos2mx) \qquad (4.8)$$

From (3.18), we have,

$$O(\epsilon\delta):\; U_{,\hat{t}\hat{t}}^{(11)} + U_{,xxxx}^{(11)} + 2\lambda U_{,xx}^{(11)} + U^{(11)} = -2U_{,\hat{t}\,\tau}^{(10)} - 2U_{,\hat{t}}^{(10)}$$

Let

$$U^{(11)} = \sum_{n=1}^{\infty} U_n^{(11)}(\hat{t},\tau)(1-cos2nx)$$

$$\sum_{n=1}^{\infty}\big[U_{n,\hat{t}\hat{t}}^{(11)}(1-cos2nx) + (-16n^4+8\lambda n^2)U_n^{(11)}cos2nx + (1-cosnx)\,U_n^{(11)}\big]U_n^{(10)} = -2\big[U_{m,\hat{t}\,\tau}^{(10)} + U_{m,\hat{t}}^{(10)}\big](1-cos2mx) \qquad (4.9a)$$

Multiplying both sides of (4.9a) through by cos2mx and integrating from 0 to π and for n=m, gives

$$\int_0^\pi \sum_{n=1}^{\infty} [\{(1-cos2nx)cos2mx\}U^{(11)}_{n,\hat{t}\hat{t}}$$
$$+ U^{(11)}_n\{(-16n^4+8\lambda n^2)cos2nxcos2mx + (1-cos2nx)\}cos2mx\,]dx$$
$$= -2\big[U^{(10)}_{m,\hat{t}\,\tau} + U^{(10)}_{m,\hat{t}}\big]\int_0^\pi (1-cos2mx)cos2mxdx$$

$$= -\tfrac{\pi}{2}U^{(11)}_{m,\hat{t}\hat{t}} + \tfrac{\pi}{2}(-16m^4+8\lambda m^2)U^{(11)}_m - \tfrac{\pi}{2}U^{(11)}_m = -2\big(U^{(10)}_{m,\hat{t}\,\tau} + U^{(10)}_{m,\hat{t}}\big)\left(-\tfrac{\pi}{2}\right) \quad (4.9b)$$

Further simplification gives

$$U^{(11)}_{m,\hat{t}\hat{t}} + (16m^4 - 8\lambda m^2 + 1)U^{(11)}_m = -2\big(U^{(10)}_{m,\hat{t}\,\tau} + U^{(10)}_{m,\hat{t}}\big) \quad (4.9c)$$

i.e.

$$U^{(11)}_{m,\hat{t}\hat{t}} + \theta^2 U^{(11)}_m = -2\big(U^{(10)}_{m,\hat{t}\,\tau} + U^{(10)}_{m,\hat{t}}\big) \quad (4.10)$$

The initial conditions are

$$U^{(11)}_m(0,0) = 0;\; U^{(11)}_{m,\hat{t}}(0,0) + U^{(10)}_{m,\tau}$$

Substituting for $U^{(10)}_m$on the right hand side (RHS) of (4.10), from (4.7e) gives

$$U^{(11)}_{m,\hat{t}\hat{t}} + \theta^2 U^{(11)}_m = -2[-\theta\alpha_1' sin\theta\hat{t} + \theta\beta_1' cos\theta\hat{t} + (-\theta\alpha_1 sin\theta\hat{t} + \theta\beta_1 cos\theta\hat{t})]$$
$$= -2\theta[-(\alpha_1' + \alpha_1)sin\theta\hat{t} + (\beta_1' + \beta)cos\theta\hat{t}] \quad (4.11a)$$

To ensure a uniformly valid solution in $\hat{t}$, implies equating to zero the coefficients of $cos\theta\hat{t}$ and $sin\theta\hat{t}$ on the RHS of (4.11a). Therefore, the coefficient of $cos\theta\hat{t}$ gives

$$\beta_1' + \beta = 0 \quad (4.11b)$$

The integrating factor is e^{τ} , $then$,

$$\frac{d(e^{\tau}\beta_1)}{d\tau} = 0 \quad (4.11c)$$

This gives,

$$\beta_1(\tau) = Ae^{-\tau} \;\; and\; \beta_1(\tau) = 0 \quad (4.11d)$$

Similarly, the coefficient of $sin\theta\hat{t}$ gives,

$$\alpha_1' + \alpha_1 = 0 \quad (4.11e)$$

This gives,

$$\alpha_1'(0) = -\alpha_1(0) = B\; and\; \alpha_1(\tau) = -Be^{-\tau} \quad (4.11f)$$

$$\therefore\; U^{(10)}_m = \alpha_1(\tau)cos\theta\hat{t} + B \quad (4.11g)$$

The remaining equation in (4.11a) is;

$$U^{(11)}_{m,\hat{t}\hat{t}} + \theta^2 U^{(11)}_m = 0 \quad (4.11h)$$

$$U^{(11)}_m = \alpha_2(\tau)cos\theta\hat{t} + \beta_2(\tau)sin\theta\hat{t} \quad (4.12a)$$

From $U^{(11)}_m(0,0) = 0$,

$$\alpha_2(0) = 0 \quad (4.12b)$$

From $U^{(11)}_{m,\hat{t}}(0,0) + U^{(10)}_{m,\tau} = 0$,

$$\beta_2(0)\theta + \alpha_1'(0) = 0 \text{ and } \beta_2(0) = -\frac{\alpha_1'(0)}{\theta} = \frac{-B}{\theta} \quad (4.12c)$$

$$\therefore U^{(11)}_m = U^{(11)}_m(1 - cos2mx) \quad (4.12d)$$

From (3.19); the next equation is

$$O(\epsilon\delta^2): U^{(12)}_{,\hat{t}\hat{t}} + U^{(12)}_{,xxxx} + 2\lambda U^{(12)}_{,xx} + U^{(12)} = -2U^{(11)}_{,\hat{t}\,\tau} - 2U^{(11)}_{,\hat{t}} - U^{(10)}_{,\tau\tau}$$

Substituting for $U^{(11)}_m$ and $U^{(10)}_m$from (4.11g) and (4.12a) respectively on the RHS of (3.19), gives

$$U^{(12)}_{m,\hat{t}\hat{t}} + \theta^2 U^{(12)}_m = -2[-\theta\alpha_2'(\tau)sin\theta\hat{t} + \theta\beta_2'(\tau)cos\theta\hat{t} + (-\theta\alpha_2(\tau)sin\theta\hat{t} + \theta\beta_2(\tau)cos\theta\hat{t})]$$
$$- \alpha_1''(\tau)cos\theta\hat{t} \quad (4.13a)$$

$$= 2\theta\alpha_2'(\tau)sin\theta\hat{t} - 2\theta\beta_2'(\tau)cos\theta\hat{t} - \theta\alpha_2(\tau)sin\theta\hat{t} + \theta\beta_2(\tau)cos\theta\hat{t} - \alpha_1''(\tau)cos\theta\hat{t}$$
$$= \big(2\theta\alpha_2'(\tau) - 2\theta\alpha_2\big)sin\theta\hat{t} + \big(2\theta\beta_2(\tau) - 2\theta\beta_2'(\tau) - \alpha_1''(\tau)\big)cos\theta\hat{t}\big) \quad (4.13b)$$

To remove secular terms in the solution of$U^{(12)}_m$, ie to ensure a uniformly valid solution in$\hat{t}$ implies equating to zero the coefficients of $cos\theta\hat{t}\; and\; sin\theta\hat{t}$ on the RHS. These respectively give

$$cos\theta\hat{t}: -2(\theta\beta_2' + \theta\beta_2) - \alpha_1'' = 0$$

And

$$sin\theta\hat{t} : -2(-\theta\alpha_2' - \theta\alpha_2) = 0$$

$$\therefore \beta_2' + \beta_2 = \frac{-\alpha_1''}{2\theta}\; and\; \left[\beta_2'(0) = \frac{3B}{2\theta}\right] \quad (4.13c)$$

$$\therefore \alpha_2' + \alpha_2 = 0 \quad (4.13d)$$

Therefore, from (4.13),

$$\alpha_2(\tau) \equiv 0, \quad (4.13e)$$

And from (4.13d),

$$\beta_2(\tau) = e^{-\tau}\left[-\int_0^{\tau} \frac{e^s \alpha''}{2\theta} ds \; + \beta_2(0)\right] \quad (4.13f)$$

i.e

$$\beta_2(\tau) = e^{-\tau}\left[-\int_0^{\tau} \frac{e^s \alpha''}{2\theta} ds - \frac{B}{\theta}\right] \quad (4.13g)$$

$$\therefore U_m^{(11)} = \beta_2(\tau) sin\theta \hat{t} \quad (4.13h)$$

Equating the left hand side (LHS) of (4.13a) to zero,

i.e

$$U_{m,\hat{t}\hat{t}}^{(12)} + \theta^2 U_m^{(12)} = 0$$

The Initial conditions are

$$U_m^{(12)}(0,0) = 0, \qquad U_{m,\hat{t}}^{(12)} + U_{m,\tau}^{(11)}(0,0) = 0$$

$$\therefore U_m^{(12)}(\hat{t}, \tau) = \alpha_3(\tau)\cos\theta\, \hat{t} + \beta_3(\tau) sin\theta \hat{t} \quad (4.13i)$$

Applying the initial conditions,

$$\alpha_3(0) = 0, \quad \beta_3(0) = 0 \quad (4.13j)$$

$$U_{m,\hat{t}}^{(12)}(0,0) = \theta\beta_3(0) = 0$$

$$\beta_3(0) = 0 \quad (4.13k)$$

$$\therefore U^{12} = U_m^{(12)}(1 - cos2mx) \quad (4.13l)$$

From (3.20),

$$O(\epsilon^2):\; U_{,\hat{t}\hat{t}}^{(20)} + U_{,xxxx}^{(20)} + 2\lambda U_{,xx}^{(20)} + U^{(20)} = -\left(\alpha U^{(10)}\right)^2 - 2\omega_1' U_{,\hat{t}\hat{t}}^{(10)}$$

Let,

$$U^{(20)} = \sum_{\mathrm{n}=1}^{\infty} U_n^{(20)}(\hat{t}\;\tau)(1 - \cos 2\mathrm{nx}) \quad (4.14)$$

Substituting (4.14)into (3.20)*gives* ;

$$\sum_{n=1}\left[\begin{array}{c} U_{n,\hat{t}\hat{t}}^{(20)}(1 - \cos 2\mathrm{nx}) + (-16n^4 + 8\lambda n^2)U_n^{(20)}\; cos2nx \\ +(1 - cos2nx)U_n^{(20)} \end{array}\right]$$
$$= \quad -\alpha\left(U_m^{(10)}\right)^2\left[\frac{3}{2} - 2cos2mx + \frac{1}{2}cos4mx\right]$$
$$- 2\omega_1' U_{m,\hat{t}\hat{t}}^{(10)}(1 - cos2mx) \quad (4.15a)$$

Multiplying both sides of (4.15a) through by cos2mx and integrating from 0 to π and for n=m, the result gives;

$$\left[-\frac{\pi}{2}U_{m,\hat{t}\hat{t}}^{(20)} + (-16m^4 + 8\lambda m^2)U_m^{(20)}\left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}U_m^{(20)}\right)\right]$$
$$= -\alpha\left(U_m^{(10)}\right)^2\left[-2\left(\frac{\pi}{2}\right) - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(10)}\left(\frac{-\pi}{2}\right)\right] \quad (4.15b)$$

i.e,

$$\frac{\pi}{2}\left[-U_{m,\hat{t}\hat{t}}^{(20)} + (-16m^4 + 8\lambda m^2)U_m^{(20)} - U_m^{(20)}\right] = \frac{\pi}{2}\left[2\alpha\left(U_m^{(10)}\right)^2 + 2\omega_1' U_{m,\hat{t}\hat{t}}^{(10)}\right] \quad (4.15c)$$

Simplification of (4.15c) gives,

$$U_{m,\hat{t}\hat{t}}^{(20)} + (16m^4 - 8\lambda m^2 + 1)U_m^{(20)} = -\left[2\alpha\left(U_m^{(10)}\right)^2 + 2\omega_1' U_{m,\hat{t}\hat{t}}^{(10)}\right] (4.15d)$$

And this further gives,

$$U_{m,\hat{t}\hat{t}}^{(20)} + \theta^2 U_m^{(20)} = -\left[2\alpha\left(U_m^{(10)}\right)^2 + 2\omega' U_{m,\hat{t}\hat{t}}^{(10)}\right] \quad (4.16a)$$

The initial conditions are,

$$U_m^{(20)}(0,0) = 0, U_{m,\hat{t}}^{(20)}(0,0) + \omega'(0)U_{m,\hat{t}}^{(10)}(0,0) = 0$$

Next multiplying equation (4.15a) by cos4mx and integrating from 0 to π and for n=m, the result gives;

$$-\frac{\pi}{2}U_{2m,\hat{t}\hat{t}}^{(20)} + (-256m^4 + 32\lambda m^2)\left(\frac{\pi}{2}\right)U_{2m}^{(20)} - \left(\frac{\pi}{2}\right)U_{2m}^{(20)} = -\alpha\left(U_m^{(10)}\right)^2\left(\frac{1}{2}.\frac{\pi}{2}\right) \quad (4.16b)$$

Simplifying (4.16b) gives;

$$U^{(20)}_{2m,\hat{t}\hat{t}} + (256m^4 - 32\lambda m^2 + 1)U^{(20)}_{2m} = \frac{\alpha}{2}\left(U^{(10)}_m\right)^2 \qquad (4.16c)$$

Let,

$$\varphi^2 = (256m^4 + 32\lambda m^2 + 1) > 0 \qquad (4.16d)$$

Therefore, (4.16c) becomes

$$U^{(20)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(20)}_m = \frac{\alpha}{2}\left(U^{(10)}_m\right)^2 \qquad (4.17)$$

The initial conditions are,

$$U^{(20)}_{2m}(0,0) = 0;\ U^{(20)}_{2m,\hat{t}}(0,0) + \omega_1' U^{(10)}_{2m,\hat{t}}(0,0) = 0$$

On substituting for $U^{(10)}_m$on the RHS of (4.16a), the simplification is

$$U^{(20)}_{2m,\hat{t}\hat{t}} + \theta^2 U^{(20)}_m = -[\{2\alpha(\alpha_1 cos\theta\hat{t} + B)^2\} + \{2\omega_1'(-\alpha_1 cos\theta\hat{t})\}] =$$

$$-\left[2\alpha\left\{\left(\frac{\alpha_1^2}{2} + B^2\right) + 2B\alpha_1 cos\theta\hat{t} + \frac{\alpha_1^2}{2}\alpha_1 cos2\theta\hat{t}\right\} + 2\omega'_1(-\alpha_1\theta^2 cos\theta\hat{t})\right] \quad (4.18a)$$

To ensure a uniformly valid solution in$\hat{t}$, we equate to zero, the coefficients of $cos2\theta\hat{t}$on the RHS of (4.18a). That is,

$$-[2B\alpha_1 - 2\omega'_1\theta^2\alpha_1] = 0 \qquad (4.18b)$$

$$\therefore \omega'_1 = \frac{B}{\theta^2};\ \omega_1 = \int \frac{B}{\theta^2} d\tau \qquad (4.18c)$$

The remaining part of equation (4.18a) for $U^{(20)}_m$ is

$$U^{(20)}_{m,\hat{t}\hat{t}} + \theta^2 U^{(20)}_m = r_0 + r_1 cos2\theta\hat{t} \qquad (4.19a)$$

where, $r_0 = -2\alpha\left(\frac{\alpha_1^2}{2} + B^2\right)$, $r_0(0) = -3\alpha B^2$

$$r_1 = -\alpha\alpha_1^2, r_1(0) = -\alpha B^2, r_0'(0) = 2\alpha B^2, r_1'(0) = 2\alpha B^2$$

$$\therefore U^{(20)}_m(\hat{t},\tau) = \alpha_4(\tau)cos\theta\hat{t} + \beta_4(\tau)sin\theta\hat{t} + \frac{r_0}{\theta^2} - \frac{r_1 cos2\theta\hat{t}}{3\theta^2} \qquad (4.19b)$$

From the initial condition,

$U^{(20)}_m(0,0) = 0$; i.e, $\alpha_4(0) + \frac{r_0(0)}{\theta^2} - \frac{r_1}{3\theta^2} = 0$

$$\therefore \alpha_4(0) = \frac{r_1}{3\theta^2} - \frac{r_0(0)}{\theta^2} = \frac{8\alpha B^2}{3\theta^2} \qquad (4.19c)$$

Applying the initial condition, $U^{(20)}_{m,\hat{t}}(0,0) + \omega'(0) + U^{(10)}_{m,\hat{t}}(0,0)$ yields,

$$\beta_4(0) = 0 \qquad (4.19d)$$

Simplification of (4.17) yields,

$$U^{(20)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(20)}_m = \frac{\alpha}{2}\left[\left(\frac{\alpha_1^2}{2} + B^2\right) + 2B\alpha_1 cos\theta\hat{t} + \frac{\alpha_1^2}{2}cos\theta\hat{t}\right] (4.20a)$$

$$\therefore U^{(20)}_{2m}(\hat{t},\tau) = \alpha_5(\tau)cos\varphi\hat{t} + \beta_5(\tau)sin\varphi\hat{t} + \frac{\alpha}{2}\left[\frac{(\frac{\alpha_1^2}{2} + B^2)}{\varphi^2} + \frac{2B\alpha_1 cos\theta\hat{t}}{(\varphi^2 - \theta^2)} + \frac{\alpha_1^2 cos2\theta\hat{t}}{2(\varphi^2 - \theta^2)}\right] \qquad (4.20b)$$

From the initial conditions,

$$U^{(20)}_{2m}(0,0) = 0;\ U^{(20)}_{2m,\hat{t}}(0,0) + \omega_1' U^{(10)}_{2m,\hat{t}}(0,0) = 0$$

$$\therefore \alpha_5(0) + \frac{\alpha}{2}\left[\frac{(\frac{\alpha_1^2}{2} + B^2)}{\varphi^2} + \frac{2B\alpha_1}{(\varphi^2 - \theta^2)} + \frac{\alpha_1^2}{2(\varphi^2 - 4\theta^2)}\right] = 0$$

$$\therefore \alpha_5(0) = -\frac{\alpha}{2}\left[\frac{(\frac{\alpha_1^2}{2}+B^2)}{\varphi^2} + \frac{2B\alpha_1}{(\varphi^2-\theta^2)} + \frac{\alpha_1^2}{2(\varphi^2-4\theta^2)} = 0\right] \text{ at } \tau=0$$

i.e

$\alpha_5(0) = -\frac{\alpha}{2}\left[\frac{3B^2}{2\varphi^2} - \frac{2B^2}{(\varphi^2-\theta^2)} + \frac{B^2}{2(\varphi^2-4\theta^2)}\right] = B^2\alpha S_0$ *and* $\beta_5(0) = 0$ (4.20c)

where, $S_0 = \left(-\frac{3\alpha}{2\varphi^2} + \frac{\alpha}{(\varphi^2-\theta^2)} - \frac{\alpha}{4(\varphi^2-4\theta^2)}\right)$

$$\therefore U^{(20)} = U^{(20)}_m(1 - cos2mx) + U^{(20)}_{2m}(1 - cos4mx) \qquad (4.20d)$$

From (3.21),

$$O(\epsilon^2\delta):\ U^{(21)}_{,\hat{t}\hat{t}} + U^{(21)}_{,xxxx} + 2\lambda U^{(21)}_{,xx} + U^{(21)}$$

$$= -2\alpha U^{(10)}U^{(11)} - 2U^{(20)}_{,\hat{t}\tau} - 2U^{(20)}_{,\hat{t}} - 2\omega_1' U^{(11)}_{,\hat{t}\hat{t}} - \omega_1'' U^{(10)}_{,\hat{t}} - 2\omega_1' U^{(10)}_{,\hat{t}}$$

i.e

$U^{(21)}_{,\hat{t}\hat{t}} + U^{(21)}_{,xxxx} + 2\lambda U^{(21)}_{,xx} + U^{(21)} = -2\alpha U^{(10)}_m(1-cos2mx)U^{(11)}_m(1-cos2mx) - 2U^{(20)}_{m,\hat{t}\,\tau}(1-cos2mx) - 2U^{(20)}_{m,\hat{t}}(1-cos2mx) - 2\omega_1' U^{(11)}_{m,\hat{t}\hat{t}}(1-cos2mx) - \omega_1''\, U^{(10)}_{m,\hat{t}}(1-cos2mx) - 2\omega_1' U^{(10)}_{m,\hat{t}}(1-cos2mx)$ (4.21)

Let

$$U^{(21)} = \sum_{\mathrm{n}=1}^{\infty} U^{(21)}_n(\hat{t}\,\tau)(1-\cos 2\mathrm{nx})$$

Substituting into (4.21) gives,

$$\sum_{n=1}\Big[U^{(21)}_{n,\hat{t}\hat{t}}(1-\cos 2\mathrm{nx}) + (-16n^4+8\lambda n^2)U^{(21)}_n\, cos2nx + \quad (1-cos2nx)U^{(21)}_n\Big]$$
$$= -2\alpha U^{(10)}_m U^{(11)}_m\left[\frac{3}{2} - 2cos2mx + \frac{1}{2}cos4mx\right] - 2U^{(20)}_{m,\hat{t}\tau}(1-cos2mx) - 2U^{(20)}_{m,\hat{t}}(1-cos2mx) \quad - 2\omega_1'\, U^{(11)}_{m,\hat{t}\hat{t}}(1-cos2mx) - \omega_1'' U^{(10)}_{m,\hat{t}}(1-cos2mx) - 2\omega_1' U^{(10)}_{m,\hat{t}}(1-cos2mx) \qquad (4.22a)$$

Multiplying both sides of (4.22) through by cos2mx and integrating from 0 to π and for n=m, gives;

$$\left[-\frac{\pi}{2}U^{(21)}_{m,\hat{t}\hat{t}} + (-16m^4+8\lambda m^2)U^{(21)}_m\left(\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}U^{(21)}_m\right)\right]$$
$$= \begin{bmatrix} -2\alpha U^{(10)}_m U^{(11)}_m\left(-\frac{\pi}{2}\right) - 2U^{(20)}_{m,\hat{t}\tau}\left(-\frac{\pi}{2}\right) - 2U^{(20)}_{m,\hat{t}} \\ (-\frac{\pi}{2}) - 2\omega_1' U^{(11)}_{m,\hat{t}\hat{t}}\left(-\frac{\pi}{2}\right) - \omega_1'' U^{(10)}_{m,\hat{t}}(-\frac{\pi}{2}) - 2\omega_1' U^{(10)}_{m,\hat{t}}\left(-\frac{\pi}{2}\right)\end{bmatrix} \qquad (4.22b)$$

Further simplification of (4.22b) yields,

$$U^{(21)}_{m,\hat{t}\hat{t}} + (16m^4 - 8\lambda m^2 + 1)U^{(21)}_m$$
$$= -2\alpha U^{(10)}_m U^{(11)}_m - 2U^{(20)}_{m,\hat{t}\tau} - 2U^{(20)}_{m,\hat{t}} - 2\omega_1' U^{(11)}_{m,\hat{t}\hat{t}} - \omega_1'' U^{(10)}_{m,\hat{t}} - 2\omega_1' U^{(10)}_{m,\hat{t}} \qquad (4.22c)$$

The above finally yields,

$U^{(21)}_{m,\hat{t}\hat{t}} + \theta^2 U^{(21)}_m = -2\alpha U^{(10)}_m U^{(11)}_m - 2U^{(20)}_{m,\hat{t}\tau} - 2U^{(20)}_{m,\hat{t}} - 2\omega_1' U^{(11)}_{m,\hat{t}\hat{t}} - \omega_1'' U^{(10)}_{m,\hat{t}} - 2\omega_1' U^{(10)}_{m,\hat{t}}$ (4.23a)

The initial conditions for (4.33a) are,

$$U^{(21)}_m(0{,}0) = 0;\ U^{(21)}_{m,\hat{t}}(0{,}0) + \omega_1'(0)U^{(11)}_{m,\hat{t}}(x,0{,}0) + U^{(20)}_{m,\tau}(0{,}0) = 0$$

Next, multiplying (4.22a) by cos4mx and integrating from 0 to π for n =m, gives

$\left[-\frac{\pi}{2}U^{(21)}_{2m,\hat{t}\hat{t}} + (-256m^4+32\lambda m^2)U^{(21)}_{2m}\left(\frac{\pi}{2}\right) - \frac{\pi}{2}U^{(21)}_{2m}\right] = -2\alpha U^{(10)}_m U^{(11)}_m\left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right) - 2\left(U^{(20)}_{2m,\hat{t}\tau} + U^{(20)}_{m,\hat{t}}\right)$ (4.23b)

$$U^{(21)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(21)}_{2m} = \alpha U^{(10)}_m U^{(11)}_m + 2\left(U^{(20)}_{2m,\hat{t}\tau} + U^{(20)}_{2m,\hat{t}}\right) \qquad (4.24)$$

The initial conditions for (4.33b) are,

$$U^{(21)}_{2m}(0{,}0) = 0;\ U^{(21)}_{2m,\hat{t}}(0{,}0) = 0$$

Substituting for $U^{(10)}_m, U^{(11)}_m and\ U^{(20)}_m$ in (4.24) yields

$U^{(21)}_{m,\hat{t}\hat{t}} + \theta^2 U^{(21)}_m = -2\alpha U^{(10)}_m U^{(11)}_m - 2U^{(20)}_{m,\hat{t}\tau} - 2U^{(20)}_{m,\hat{t}} - 2\omega_1' U^{(11)}_{m,\hat{t}\hat{t}} - \omega_1'' U^{(10)}_{m,\hat{t}} - 2\omega_1' U^{(10)}_{m,\hat{t}}$ (4.25a)

i.e,

$-2\alpha\left(\frac{\alpha_1\beta_2}{2}sin\theta\hat{t} + B\beta_2 sin\theta\hat{t}\right) - 2\left(-\theta\alpha_4' sin\theta\hat{t} + \theta\beta_4' cos\theta\hat{t} + \frac{2\theta r_1' sin2\theta\hat{t}}{3\theta^2}\right) - 2\left(-\theta\alpha_4 sin\theta\hat{t} + \theta\beta_4 cos\theta\hat{t} + \frac{2\theta r_1 sin2\theta\hat{t}}{3\theta^2}\right) - 2\omega_1'(-\theta^2)\beta_2 sin\theta\hat{t} - \omega_1''(-\theta\alpha_1 sin\theta\hat{t}) - 2\omega_1'(-\alpha_1\theta sin\theta\hat{t})$ (4.25b)

To ensure a uniformly valid solution in , $\hat{t}$, we equate to zero the coefficients of $cos\theta\hat{t}\ and\ sin\theta\hat{t}$. This yields respectively,

$$-2\theta\beta_4' - 2\theta\beta_4 = 0 \qquad (4.25c)$$

$$\alpha B\beta_2 + 2\theta\alpha_4' + 2\theta\alpha_4 + 2\theta^2\omega_1'\beta_2 + \omega_1''\theta\alpha_1 + 2\omega_1'\alpha_1\theta \qquad (4.26d)$$

$$\therefore \beta_4' + \beta_4 = 0 \qquad (4.26e)$$

Solving (4.26e) yields,

$$\beta_4(\tau) = \beta_4(0)e^{-\tau} = 0 \; since \; \beta_4(0) = 0 \qquad (4.26f)$$

Solving (4.25d) yields,

$$\alpha_4' + \alpha_4 = \rho_1(\tau) = \frac{1}{2\theta}(\alpha B\beta_2 - 2\theta^2\omega_1'\beta_2 - \omega_1''\theta\alpha_1 - 2\omega_1'\alpha_1\theta) \qquad (4.26g)$$

$$\alpha_4(\tau) = e^{-\tau}\left[\int_0^\tau e^s\rho_1(s)ds + \alpha_4(0)\right]$$

$$\therefore \alpha_4'(0) = \rho_1(0) - \alpha_4(0) \qquad (4.26h)$$

where

$$\alpha_4(0) = \frac{8\alpha B^2}{3\theta^2} \qquad (4.26i)$$

$$\alpha_4'(0) = \frac{-13\alpha B^2}{3\theta^2} + \frac{4B^2}{\theta} \qquad (4.26j)$$

$$\therefore \rho_1(0) = \frac{-5\alpha B^2}{3\theta^2} + \frac{4B^2}{\theta} = B^2\left(\frac{-5\alpha}{3\theta^2} + \frac{4}{\theta}\right) = B^2V \qquad (4.26k)$$

where, $V = \left(\frac{-5\alpha}{3\theta^2} + \frac{4}{\theta}\right)$

The remaining equation in (4.25a) becomes,

$$U_{m,\hat{t}\hat{t}}^{(21)} + \theta^2U_m^{(21)} = r_2 + r_3cos2\theta\hat{t} + r_4sin2\theta\hat{t} \qquad (4.26l)$$

with the initial conditions,

$$U_m^{(21)}(0,0) = 0; \;\; U_{m,\hat{t}}^{(21)}(0,0) + \omega_1'(0)U_{m,\hat{t}}^{(11)} + U_{m,\hat{\tau}}^{(20)}(0,0) = 0$$

where,

$$r_2 = \alpha_1\alpha_2; \;\; r_2(0) = \alpha_1(0)\alpha_2(0) = 0$$

$$r_3 = \alpha\alpha_1\alpha_2; \; r_3(0) = \alpha\alpha_1(0)\alpha_2(0) = 0; \;\; r_3'(0) = 0,$$

$$r_4 = \left[\alpha\alpha_1\beta_2 + \frac{8\alpha}{3\theta}(\alpha_1\alpha_1' + \alpha_1^2)\right]; \; r_4(0) = \left[\alpha(-B)\left(\frac{-B}{\theta}\right) + \frac{8\alpha}{3\theta}(-B.B + B^2)\right] = \frac{\alpha B^2}{\theta}, r_4'(0) = \frac{-5\alpha B^2}{2\theta}$$

The solution of (4.26l) becomes

$$U_m^{(21)}(\hat{t},\tau) = \alpha_6cos\theta\hat{t} + \beta_6sin\theta\hat{t} + \frac{r_2}{\theta^2} - \left(\frac{r_3cos2\theta\hat{t} + r_4sin2\theta\hat{t}}{3\theta^2}\right) \qquad (4.27a)$$

with the initial conditions, $\alpha_6(0) + \frac{r_2}{\theta^2} - \frac{r_3}{3\theta^2} = 0$

$$\therefore \alpha_6(0) = \frac{r_3 - 3r_2}{3\theta^2} = 0, \qquad \beta_6(0) = 0 \qquad (4.27b)$$

From (4.24),

$$\begin{aligned} U_{2m,\hat{t}\hat{t}}^{(21)} + \varphi^2U_{2m}^{(21)} &= \alpha\left[\frac{\alpha_1\beta_2}{2}sin2\theta\hat{t} + B\beta_2sin\theta\hat{t}\right] + 2\left(U_{2m,\hat{t}\tau}^{(20)} + U_{2m,\hat{t}}^{(20)}\right) \\ &= \alpha\left[\frac{\alpha_1\beta_2}{2}sin2\theta\hat{t} + B\beta_2sin\theta\hat{t}\right] \\ &+ 2\left[-\varphi\alpha_5'sin\varphi\hat{t} + \beta_5'\varphi cos\varphi\hat{t} + \frac{\alpha}{2}\left\{\frac{-2\theta\alpha_1'Bsin\theta\hat{t}}{\varphi^2 - \theta^2} - \frac{2\theta(\alpha_1^2)'sin2\theta\hat{t}}{2(\varphi^2 - 4\theta^2)}\right\}\right. \\ &+ \left\{\{-\varphi\alpha_5sin\varphi\hat{t} + \beta_5cos\varphi\hat{t}\right. \\ &\left.\left.+ \frac{\alpha}{2}\left\{\frac{-2\theta\alpha_1Bsin\theta\hat{t}}{\varphi^2 - \theta^2} - \frac{2\theta\alpha_1^2sin2\theta\hat{t}}{2(\varphi^2 - 4\theta^2)}\right\}\right\}\right\} \end{aligned} \qquad (4.28a)$$

To ensure a uniformly valid solution in , $\hat{t}$, we equate to zero the coefficient of $cos\varphi\hat{t} and \; sin\varphi\hat{t}$

$$2\varphi\beta_5' + 2\varphi\beta_5 = 0 \;\Rightarrow\; \beta_5' + \beta_5 = 0 \;\Rightarrow\; \beta_5'(0) = -\beta_5(0) \qquad (4.28b)$$

$$-2\varphi\alpha_5' - 2\varphi\alpha_5 = 0 \;\Rightarrow\; \alpha_5' + \alpha_5 = 0 \Rightarrow \alpha_5'(0) = -\alpha_5(0) \qquad (4.28c)$$

$$\beta_5 = \beta_5(0)e^{-\tau} = 0, \; \alpha_5 = \alpha_5(0)e^{-\tau} = 0 \qquad (4.28d)$$

The remaining equation of (4.27a) is:

$$U_{2m,\hat{t}\hat{t}}^{(21)} + \varphi^2U_{2m}^{(21)} = \left[\alpha B\beta_2 + \frac{\alpha}{2}\left(\frac{-2\theta B}{\varphi^2 - \theta^2}\right)(\alpha_1' + \alpha_1)\right]sin\theta\hat{t} + \left[\frac{\alpha\alpha_1\beta_2}{2} + \frac{\alpha}{2}\left\{\frac{-\theta(\alpha_1^2)' + \alpha_1^2}{2(\varphi^2 - 4\theta^2)}\right\}\right]sin2\theta\hat{t} =$$

$$r_5sin\theta\hat{t} + r_6sin2\theta\hat{t} \qquad (4.29a)$$

where,

$$r_5 = \left[\alpha B\beta_2 - \frac{-2\theta B}{(\varphi^2 - \theta^2)}(\alpha_1' + \alpha)\right] = \alpha B\beta_2, \text{ since } \alpha_1' + \alpha = 0,$$

$$\alpha_1' = B; \; r_5(0) = \frac{-\alpha B^2}{\theta}, r_6 = \left[\frac{\alpha\alpha_1\beta_2}{2} + \frac{\alpha}{2}\left\{\frac{-\theta(\alpha_1^2)' + \alpha_1^2}{2(\varphi^2 - 4\theta^2)}\right\}\right]$$

$$\therefore r_6(0) = \left[\frac{\alpha\alpha_1(0)\beta_2(0)}{2} + \frac{\alpha}{2}\left\{\frac{-\theta(\alpha_1^2(0))' + \alpha_1^2(0)}{2(\varphi^2 - 4\theta^2)}\right\}\right] = \frac{B^2\alpha}{2\theta} + \frac{B^2\theta\alpha}{4(\varphi^2 - 4\theta^2)} = B^2S_1 \qquad (4.29b)$$

where, $S_1 = \left(\frac{\alpha}{2\theta} + \frac{\alpha\theta}{4(\varphi^2 - 4\theta^2)}\right)$

$$\therefore U_{2m}^{(21)} = \alpha_7(\tau)cos\varphi\hat{t} + \beta_7(\tau)sin\varphi\hat{t} + \frac{r_5 cos\theta\hat{t}}{\varphi^2-\theta^2} + \frac{r_6 cos\theta\hat{t}}{\varphi^2-4\theta^2} \qquad (4.30)$$

The initial conditions for (4.30) are

$$U_{2m}^{(21)}(0,0) = 0;\ \ U_{2m,\hat{t}}^{(21)}(0,0) + U_{2m,\hat{\tau}}^{(20)}(0,0) = 0;$$

$$\Rightarrow -\varphi\alpha_7(0)sin\varphi\hat{t} + \varphi\beta_7(0)cos\varphi\hat{t} + \frac{\theta r_5(0)cos\theta\hat{t}}{\varphi^2-\theta^2} + \frac{2\theta r_6(0)cos\theta\hat{t}}{\varphi^2-4\theta^2} + \alpha_5'(0)cos\varphi\hat{t} + \frac{\alpha}{2}\left[\frac{\alpha_1'\alpha_1}{\varphi^2} + \frac{2B\theta\alpha_1' cos\theta\hat{t}}{\varphi^2-\theta^2} + \frac{2\,\alpha_1'\alpha_1 cos2\theta\hat{t}}{2(\varphi^2-4\theta^2)}\right] = 0 \qquad (4.31a)$$

$$\therefore \alpha_7(0) = 0 \qquad (4.31b)$$

Similarly, the following is obtained

$$\varphi\beta_7(0) + \frac{\theta r_5(0)}{\varphi^2-\theta^2} + \frac{2\theta r_6(0)}{\varphi^2-4\theta^2} + \alpha_5'(0) + \frac{\alpha}{2}\left[\frac{\alpha_1'(0)\alpha_1(0)}{\varphi^2} + \frac{2B\theta\alpha_1'(0)}{\varphi^2-\theta^2} + \frac{\alpha_1'(0)\alpha_1(0)}{(\varphi^2-4\theta^2)}\right] = 0 \quad (4.32a)$$

$$\beta_7(0) = -\frac{1}{\varphi}\left[\frac{\theta r_5(0)}{\varphi^2-\theta^2} + \frac{2\theta r_6(0)}{\varphi^2-4\theta^2} + \alpha_5'(0) + \frac{\alpha}{2}\left(\frac{\alpha_1'(0)\alpha_1(0)}{\varphi^2} + \frac{2B\theta\alpha_1'(0)}{\varphi^2-\theta^2} + \frac{\alpha_1'(0)\alpha_1(0)}{(\varphi^2-4\theta^2)}\right)\right] \qquad (4.32b)$$

i.e

$$\beta_7(0) = B^2\left(\frac{\alpha S_0}{\varphi} + \frac{\alpha}{2\varphi^3} + \frac{\alpha}{2\alpha(\varphi^2-4\theta^2)} - \frac{\alpha}{\alpha(\varphi^2-\theta^2)} - \frac{2\theta\alpha S_1}{\varphi(\varphi^2-4\theta^2)}\right) \quad (4.32c)$$

So far, it follows that

$$U^{(21)} = U_m^{(21)}(1-cos2mx) + U_{2m}^{(21)}(1-cos4mx) \qquad (4.33)$$

From (3.23),

$$O(\epsilon^2\delta^2):\ U_{,\hat{t}\hat{t}}^{(22)} + U_{,xxxx}^{(22)} + 2\lambda U_{,xx}^{(22)} = -U_{,\tau\tau}^{(20)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(12)} - 2U_{,\hat{t}\,\tau}^{(21)} - 2\omega_1' U_{,\hat{t}\hat{t}}^{(12)} - 2\omega_1'' U_{,\hat{t}}^{(11)} - 2U_{,\hat{t}}^{(21)} - 2\omega_1' U_{,\hat{t}}^{(11)} - \alpha\left\{\left(U^{(11)}\right)^2 + U^{(10)}\right\}$$

$$\Rightarrow U_{,\hat{t}\hat{t}}^{(22)} + U_{,xxxx}^{(22)} + 2\lambda U_{,xx}^{(22)} = -\Big[U_{m,\tau\tau}^{(20)}(1-\cos 2mx) + U_{2m,\tau\tau}^{(20)}(1-\cos 4mx) + 2\omega_1' U_{m,\hat{t}\hat{t}}^{(12)}(1-cos2mx) + 2\left\{U_{m,\hat{t}\,\tau}^{(21)}(1-cos2mx) + U_{2m,\hat{t}\,\tau}^{(21)}(1-cos2mx)\right\} + 2\omega_1'' U_{m,\hat{t}}^{(11)}(1-cos2mx) + 2\left\{U_{m,\hat{t}}^{(21)}(1-cos2mx) + U_{2m,\hat{t}}^{(21)}(1-cos4mx)\right\} + + 2\omega_1' U_{m,\hat{t}\hat{t}}^{(11)}(1-cos2mx) + \alpha\left(U_m^{(11)}\right)^2\left\{\frac{3}{2} - 2\cos 2mx + \frac{1}{2}\cos 4mx\right\} + 2\left\{U_m^{(10)}U_m^{(12)}\right\}\left\{\frac{3}{2} - 2\cos 2mx + \frac{1}{2}\cos 4mx\right\}\Big] \ (4.34)$$

Let

$$U^{(22)} = \sum_{n=1}^{\infty} U_n^{(22)}(\hat{t}\ \tau)(1-cos2nx)$$

The LHS of (4.34) simplifies to,

$$\sum_{n=1}\left[U_{n,\hat{t}\hat{t}}^{(22)}(1-\cos 2nx) + (-16n^4 + 8\lambda n^2)U_n^{(22)} + U_n^{(22)}(1cos2nx)\right] = RHS\ of\ (4.34)$$

Multiplying (4.30) through by cos2mx and integrating from 0 to π and for n=m, we have,

$$-\frac{\pi}{2}U_{m,\hat{t}\hat{t}}^{(22)} + (-16m^4 + 8\lambda m^2)U_m^{(22)}\left(\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}U_m^{(22)}\right)$$
$$= -\Big[\left(-\frac{\pi}{2}\right)U_{m,\tau\tau}^{(20)} + 2\omega_1' U_{m,\hat{t}\hat{t}}^{(12)}\left(-\frac{\pi}{2}\right) + 2U_{m,\hat{t}\tau}^{(12)}\left(-\frac{\pi}{2}\right) + 2\omega_1'' U_{m,\hat{t}}^{(11)}\left(-\frac{\pi}{2}\right) + 2U_{m,\hat{t}}^{(21)}\left(-\frac{\pi}{2}\right) + + 2\omega_1' U_{m,\hat{t}}^{(11)}\left(-\frac{\pi}{2}\right) + \alpha\left(U_m^{(11)}\right)^2\left(-2.\frac{-\pi}{2}\right) + \alpha U_m^{(10)}U_m^{(12)}\left(-2.\frac{-\pi}{2}\right)\Big] \qquad (4.35a)$$

Further simplification of (4.35a) gives

$$-\frac{\pi}{2}U_{m,\hat{t}\hat{t}}^{(22)} + (16m^4 - 8\lambda m^2 + 1)U_m^{(22)}$$
$$= -\frac{\pi}{2}\Big[-U_{m,\tau\tau}^{(20)} - 2\omega_1' U_{m,\hat{t}\hat{t}}^{(12)} - 2U_{m,\hat{t}\tau}^{(21)} - 2\omega_1'' U_{m,\hat{t}}^{(11)} - 2U_{m,\hat{t}}^{(21)} - 2\omega_1' U_{m,\hat{t}}^{(11)} + 2\alpha(U_m^{(11)})^2 + 2\alpha U_m^{(10)}U_m^{(12)}\Big] \qquad (4.35b)$$

Further simplification of (4.35b) yields

$$U_{m,\hat{t}\hat{t}}^{(22)} + \theta^2 U_m^{(22)} = -\Big[U_{m,\tau\tau}^{(20)} + 2\omega_1' U_{m,\hat{t}\hat{t}}^{(12)} + 2U_{m,\hat{t}\tau}^{(21)} + 2\omega_1'' U_{m,\hat{t}}^{(11)} + 2U_{m,\hat{t}}^{(21)} + 2\omega_1' U_{m,\hat{t}}^{(11)} - 2\left\{(U_m^{(11)})^2 + U_m^{(10)}U_m^{(12)}\right\}\Big] \qquad (4.35c)$$

The initial conditions for (4.35c) are

$$U_m^{(22)}(0,0) = 0;\; U_{m,\hat{t}}^{(22)}(0,0) + \omega_1'(0)U_{m,\hat{t}}^{(12)} + U_{m,\tau}^{(21)}(0,0) = 0$$

Next from equation (4.34) for n=2m, let

$$U^{(22)} = \sum_{n=1}^{\infty} U_n^{(22)}(1 - cos4mx)$$

Multiplying (4.34) through by cos4mx and integrating from 0 to π and for n = 2m, gives

$$-\frac{\pi}{2}U_{2m,\hat{t}\hat{t}}^{(22)} + \frac{\pi}{2}(-256m^4 + 32\lambda m^2)U_{2m}^{(22)} - \frac{\pi}{2}U_{2m}^{(22)}\frac{\alpha}{2}\left\{\left(U_m^{(11)}\right)^2 + \left(U_m^{(10)}U_m^{(12)}\right)\left(\frac{\pi}{2}\right)\right\} \quad (4.36a)$$

This further gives

$$U_{2m,\hat{t}\hat{t}}^{(22)} + \varphi^2 U_{2m}^{(22)} = -\frac{\alpha}{2}\left\{\left(U_m^{(11)}\right)^2 + \left(U_m^{(10)}U_m^{(12)}\right)\right\} \quad (4.36b)$$

The initial conditions are

$$U_m^{(22)}(0,0) = 0;\; U_{2m,\hat{t}}^{(22)}(0,0) + \omega_1'(0)U_{2m,\hat{t}}^{(12)} + U_{m,\tau}^{(21)}(0,0) = 0$$

On substituting for terms in (4.35c) and simplifying, the result is

$$U_{2m,\hat{t}\hat{t}}^{(22)} + \theta^2 U_m^{(22)} =$$
$$-\left[\left\{\alpha_4''cos\theta\hat{t} + \frac{r_0''}{\theta^2} - \frac{r_1''cos2\theta\hat{t}}{3\theta^2}\right\} + 2\omega_1'\{-\theta^2\alpha_3 sin\theta\hat{t} + \beta_3\theta^2 cos\theta\hat{t}\} + 2\left\{-\theta\alpha_6' sin\theta\hat{t} + \theta\beta_6' cos\theta\hat{t} - \frac{2\theta r_3' sin2\theta\hat{t} + 2\theta r_4' cos2\theta\hat{t}}{3\theta^2}\right\} + 2\omega_1'(\theta\beta_2 cos\theta\hat{t}) + 2\left\{-\alpha_6\theta sin\theta\hat{t} + \beta_6\theta cos\theta\hat{t} + \left(\frac{2\theta r_3 sin\theta\hat{t} - 2\theta r_4 cos\theta\hat{t} -}{3\theta^2}\right)\right\} + 2\omega_1'\{\theta\beta_2 cos\theta\hat{t}\} + 2\alpha\left\{\frac{\beta_2}{2}(1 - cos2\theta\hat{t}) + \left(\frac{\alpha_1\beta_2}{2}sin2\theta\hat{t} + B\beta_2 sin2\theta\hat{t}\right)\right\}\right] \quad (4.37a)$$

To ensure a uniformly valid solution in$\hat{t}$; equate to zero the coefficients of $cos\theta\hat{t}$ *and* $sin\theta\hat{t}$ of (4.37a) and this yields respectively

$$-\alpha_4'' + 2\omega_1'\theta\beta_3 - 2\theta\beta_6' - 2\omega_1''\theta\beta_2 - 2\beta_6\theta - 2\omega_1'\theta\beta_2 = 0 \quad (4.37b)$$

and

$$2\omega_1'\theta^2\alpha_3 + 2\theta\alpha_6' + 2\alpha_6\theta - 2\alpha B\beta_2 = 0 \quad (4.37c)$$

Simplification of (4.37b) gives

$$\beta_6' + \beta_6 = \frac{1}{2\theta}[\alpha_4'' - 2\omega_1'\theta^2\beta_3 + 2\omega_1''\beta_2 + 2\omega_1'\theta\beta_2] = \rho_2(\tau) \quad (4.37d)$$

$$\beta_6(\tau) = e^{-\tau}[\int e^{\tau}\rho_2(\tau)d\tau + \beta_6(0)] = e^{-\tau}[\int e^{\tau}\rho_2(\tau)d\tau] \quad (4.37e)$$

Similarly, simplification of (4.37c) yields

$$\alpha_6' + \alpha_6 = \frac{1}{2\theta}[-2\omega_1'\theta^2\alpha_3 + 2\alpha B\beta_2] = \rho_3(\tau) \quad (4.37f)$$

Therefore

$$\alpha_6 = e^{-\tau}[\int e^{\tau}\rho_3(\tau)d\tau + \alpha_6(0)] \quad (4.37g)$$

The remaining part of equation (4.37a) is

$$U_{m,\hat{t}\hat{t}}^{(22)} + \theta^2 U_m^{(22)} = r_7 + r_8 cos2\theta\hat{t} + r_9 sin2\theta\hat{t} \quad (4.38)$$

$$r_7 = -\left[\frac{r_0''}{\theta^2} - \frac{2r_4'}{3\theta} + \alpha\beta_2\right],\; r_8 = \left[\frac{r_1''}{3\theta^2} - \frac{4r_4}{3\theta} - \alpha\beta_2\right],\; r_9 = -\left[\frac{2r_3'}{3\theta^2} - \frac{4r_3}{3\theta} - \alpha\alpha_1\beta_2\right]$$

It is to be recalled that, $r_0 = -2\alpha(\frac{\alpha_1^2}{2} + B^2)$

$$° \therefore r_0{}' = -2\alpha\alpha_1\alpha_1', r_0'' = -2\alpha(\alpha_1'^2 + \alpha_1\alpha_1''),\; r_0{}'(0) = 2\alpha B^2, r_0''(0) = -4\alpha B^2$$

$$r_4 = \alpha\alpha_1\beta_2 + \frac{8\alpha}{3\theta}(\alpha_1\alpha_1' + \alpha_1'^2), r_4{}' = \alpha(\alpha_1'\beta_2 + \alpha_1'\beta_2{}') + \frac{8\alpha}{3\theta}(\alpha_1'\alpha_1' + \alpha_1\alpha_1'' + 2\alpha_1\alpha_1')$$

$$r_4{}'(0) = \frac{-5\alpha B^2}{2\theta}, r_7(0) = \frac{17\alpha B^2}{3\theta^2} - \frac{\alpha B}{\theta},\; r_1 = -\alpha_1\alpha_1^2;\; r_1' = 2\alpha\alpha_1\alpha_1';\; r_1'' = -2\alpha(\alpha_1'^2 + \alpha_1\alpha_1'')$$

$$r_1'(0) = -2\alpha\alpha_1(0)\alpha_1'(0) = 2\alpha B^2;\; r_1''(0) = -4\alpha B^2,\; r_8(0) = \left(\frac{-4\alpha B^2}{3\theta^2} + \frac{4\alpha B^2}{3\theta^2} + \frac{\alpha B}{3\theta}\right) = \frac{\alpha B}{\theta}, r_3 = \alpha\alpha_1\alpha_2;\; r_3(0) = 0,\; r_3' = \alpha(\alpha_1'\alpha_1 + \alpha_1\alpha_2');\; r_3'(0) = 0,\; \beta_2' = \frac{3B}{2\theta},\; r_9(0) = \frac{3\theta B^2}{2\theta}$$

$$\therefore U_m^{(22)} = \alpha_8 cos\theta + \beta_8 sin\theta + \frac{r_7}{3\theta^2} + \frac{r_8 cos2\theta}{\theta^2} + \frac{r_9 sin2\theta\hat{t}}{\theta^2} \quad (4.39a)$$

The initial conditions are

$$U_m^{(22)}(0,0) = 0;\; U_m^{(22)}(0,0) + \omega_1'(0)U_{m,\hat{t}}^{(12)}(0,0) + U_{m,\tau}^{(21)}(0,0) = 0$$

$$\alpha_8(0) = \left(\frac{r_8(0)}{3\theta^2} - \frac{r_7(0)}{\theta^2}\right) = \frac{4\alpha B}{3\theta^3} - \frac{17\alpha B^2}{3\theta^4} \quad (4.39b)$$

Similarly, $\theta\beta_8(0) + \frac{2\theta r_9(0)}{\theta^2} = 0$

$$\therefore \beta_8(0) = \frac{-2\theta r_9(0)}{\theta^2} = \frac{-3\alpha B^2}{\theta^3} \qquad (4.39c)$$

From equation (3.23),

$$O(\epsilon^3)\ :\ U^{(30)}_{,\hat{t}\hat{t}} + U^{(30)}_{,xxxx} + 2\lambda U^{(30)}_{,xx} + U^{(30)} = -(\omega_1')^2 U^{(10)}_{,\hat{t}\hat{t}} - 2\left(\omega_1' U^{(20)}_{,\hat{t}\hat{t}} + \omega_2' U^{(20)}_{,\hat{t}\hat{t}}\right) - 2\alpha \mathrm{U}^{(20)}\mathrm{U}^{(10)} + \beta\left(\mathrm{U}^{(10)}\right)^3$$

Then, substituting on the RHS of (3.23) yields,

$$\begin{aligned} U^{(30)}_{,\hat{t}\hat{t}} + U^{(30)}_{,xxxx} + 2\lambda U^{(30)}_{,xx} + U^{(30)} &= -(\omega_1')^2 U^{(10)}_{m,\hat{t}\hat{t}}(1-\cos 2mx) \\ &- 2\left[\omega_1' U^{(20)}_{m,\hat{t}\hat{t}} + (1-\cos 2mx) + \omega_1' U^{(20)}_{2m,\hat{t}\hat{t}}(1-\cos 2mx)\right] - 2\omega_2' U^{(10)}_{,\hat{t}\hat{t}}(1-\cos 2mx) \\ &- 2\alpha\left[U^{(10)}_m(1-\cos 2mx)\left\{U^{(20)}_m(1-\cos 2mx)\right.\right. \\ &\left.\left.+ U^{(20)}_{2m}(1-\cos 4mx)\right\}\right] + \beta\left(\mathrm{U}_m{}^{(10)}\right)^3(1-\cos 2mx)^3 \qquad (4.40) \end{aligned}$$

Therefore, on further simplifications, (4.40) becomes

$$U^{(30)}_{,\hat{t}\hat{t}} + U^{(30)}_{,xxxx} + 2\lambda U^{(30)}_{,xx} + U^{(30)} = -(\omega_1')^2 U^{(10)}_{m,\hat{t}\hat{t}}(1-\cos 2mx) - 2\left[\omega_1' U^{(20)}_{m,\hat{t}\hat{t}}(1-\cos 2mx) + \omega_1' U^{(20)}_{2m,\hat{t}\hat{t}}(1-\cos 4mx) + \omega_2' U^{(10)}_m(1-\cos 2mx)\right] - 2\alpha\left[U^{(10)}_m U^{(20)}_m\left\{\frac{3}{2} - 2\cos 2mx + \frac{1}{2}\cos 4mx\right\} + U^{(10)}_m U^{(20)}_m\left\{1 - \frac{1}{2}\cos 2mx - \cos 4mx + \frac{1}{2}\cos 6mx\right\}\right] + \beta\left(\mathrm{U}_m{}^{(10)}\right)^3\left[\frac{5}{2} - \frac{15}{4}\cos 2mx + \frac{3}{2}\cos 4mx + \frac{1}{4}\cos 6mx\right] \qquad (4.41)$$

Let

$$U^{(30)} = \sum_{n=1}^{\infty} U^{(30)}_n(1-\cos 2nx)$$

Therefore, (4.41) becomes

$$\sum_{n=1}^{\infty}\left[U^{(30)}_{n,\hat{t}\hat{t}}(1-\cos 2nx) + (-16n^4 + 8\lambda n^2 + 1)U^{(30)}_n \cos 2nx\right] = RHS\ (4.41)$$

Multiplying (4.41) through by cos2mx and integrating from 0 to π and for n =m, the result is

$$-\frac{\pi}{2}U^{(30)}_{m,\hat{t}\hat{t}} + (-16m^4 + 8\lambda m^2 + 1)U^{(30)}_m\left(-\frac{\pi}{2}\right) = -\left[(\omega_1')^2 U^{(10)}_{m,\hat{t}\hat{t}}\left(-\frac{\pi}{2}\right) + 2\omega' U^{(20)}_{m,\hat{t}\hat{t}}\left(-\frac{\pi}{2}\right) + 2\omega_2' U^{(10)}_{m,\hat{t}\hat{t}}\left(-\frac{\pi}{2}\right) + 2\alpha U^{(10)}_m U^{(20)}_m\left(-2.-\frac{\pi}{2}\right) - \alpha U^{(10)}_m U^{(20)}_{2m}\left(-\frac{\pi}{2}\right) - \frac{15}{4}\left(\mathrm{U}_m{}^{(10)}\right)^3\left(-\frac{\pi}{2}\right)\right] \qquad (4.42a)$$

i.e,

$$-\frac{\pi}{2}\left[U^{(30)}_{m,\hat{t}\hat{t}} + (16m^4 - 8\lambda m^2 + 1)U^{(30)}_m\right] = -\frac{\pi}{2}\left[-(\omega_1')^2 U^{(10)}_{m,\hat{t}\hat{t}} - 2\omega'{}_1 U^{(20)}_{m,\hat{t}\hat{t}} - 2\omega_2' U^{(10)}_{m,\hat{t}\hat{t}} - 2\alpha\left[2U^{(10)}_m U^{(20)}_m + U^{(10)}_m U^{(20)}_{2m}\right] - \frac{15}{4}\beta\left(\mathrm{U}_m{}^{(10)}\right)^3\right] \qquad (4.42b)$$

$$\therefore\ U^{(30)}_{m,\hat{t}\hat{t}} + \theta^2 U^{(30)}_m = -(\omega_1')^2 U^{(10)}_{m,\hat{t}\hat{t}} - 2\omega'{}_1 U^{(20)}_{m,\hat{t}\hat{t}} - 2\omega_2' U^{(10)}_{m,\hat{t}\hat{t}} - 2\alpha\left[2U^{(10)}_m U^{(20)}_m + U^{(10)}_m U^{(20)}_{2m}\right] - \frac{15}{4}\beta\left(\mathrm{U}_m{}^{(10)}\right)^3 \qquad (4.43)$$

The initial conditions are

$$U^{(30)}_m(0,0) = 0;\quad U^{(30)}_{m,\hat{t}}(0,0) + \omega'(0)U^{(20)}_{m,\hat{t}}(0,0) + \omega_2'(0)U^{(10)}_{\tau}(0,0) = 0$$

Multiplying (4.41) through by cos4mx and integrating from 0 to π and for n=2m, the result gives

$$\begin{aligned} -\frac{\pi}{2}\left[U^{(30)}_{2m,\hat{t}\hat{t}} + (256m^4 - 32\lambda m^2 + 1)U^{(30)}_{2m}\right] &= 2\left[\omega_1' U^{(20)}_{2m,\hat{t}\hat{t}}\left(-\frac{\pi}{2}\right)\right] + 2\alpha\left[U^{(10)}_m U^{(20)}_m . \frac{1}{2}\left(-\frac{\pi}{2}\right) + U^{(10)}_m U^{(20)}_{2m}\left(-\frac{\pi}{2}\right)\right] \\ &- \beta\left(\mathrm{U}_m{}^{(10)}\right)^3 . \frac{3}{2}\left(-\frac{\pi}{2}\right) \end{aligned}$$

$$\therefore\ U^{(30)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(30)}_{2m} = -2\left[-\omega_1' U^{(20)}_{2m,\hat{t}\hat{t}}\right] + 2\alpha\left[U^{(10)}_m U^{(20)}_m - U^{(10)}_m U^{(20)}_{2m}\right] - \frac{3}{2}\beta\left(\mathrm{U}_m{}^{(10)}\right)^3 \qquad (4.44)$$

The initial conditions are

$$U_{2m}^{(30)}(0,0)=0;\ U_{2m}^{(30)}(0,0)+\omega_1'(0)U_{2m,\hat{t}}^{(20)}(0,0)=0$$

Multiplying (4.41) through by cos6mx and integrating from 0 to π and for n=3m and get,

$$U_{3m,\hat{t}\hat{t}}^{(30)}+(1296m^4-72\lambda m^2+1)U_{3m}^{(30)}=\alpha U_m^{(10)}U_{2m}^{(20)}-\frac{1}{4}\beta\left(\mathrm{U}_m{}^{(10)}\right)^3 \qquad (4.45a)$$

Let

$\Omega^2=1296m^4-72\lambda m^2+1>0$ for all m

$$\therefore U_{3m,\hat{t}\hat{t}}^{(30)}+\Omega^2U_{3m}^{(30)}=\alpha U_m^{(10)}U_{2m}^{(20)}-\tfrac{1}{4}\beta\left(\mathrm{U}_m{}^{(10)}\right)^3 \qquad (4.45b)$$

The initial conditions for (4.45b) are

$$U_{3m}^{(30)}(0,0)=0;\ U_{3m,\hat{t}}^{(30)}(0,0)=0$$

Further simplification of (4.43) gives

$$U_{m,\hat{t}\hat{t}}^{(30)}+\theta^2U_m^{(30)}=-(\omega_1')^2(-\theta^2\alpha_1 cos\theta\hat{t})-2\omega'_1\left(-\alpha_4\theta^2cos\theta\hat{t}+\tfrac{4r_1cos2\theta\hat{t}}{3}\right)-2\omega_2'(-\theta^2\alpha_1cos\theta\hat{t})-$$
$$2\alpha\left[\left(\tfrac{\alpha_1\alpha_4}{4}+\tfrac{Br_0}{\theta^2}\right)+\left(\tfrac{\alpha_1r_0}{\theta^2}-\tfrac{\alpha_1r_1}{6\theta^2}+B\alpha_4\right)cos\theta\hat{t}+\left(\tfrac{\alpha_1\alpha_4}{4}-\tfrac{Br_1}{3\theta^2}\right)cos2\theta\hat{t}-\tfrac{\alpha_1r_1}{6\theta^2}cos3\theta\hat{t}\right]-2\alpha\left[\tfrac{\alpha\alpha_1^2B}{2(\varphi^2-\theta^2)}+\right.$$
$$\left\{\tfrac{\alpha\alpha_1^2\left(\frac{\alpha_1^2}{2}+B^2\right)}{2\varphi^2}+\tfrac{\alpha\alpha_1^3}{8(\varphi^2-4\theta^2)}\right\}cos\theta\hat{t}+\tfrac{\alpha_1\alpha_5}{2}cos(\varphi+\theta)\hat{t}+\tfrac{\alpha\alpha_1^2Bcos2\theta\hat{t}}{2(\varphi^2-\theta^2)}+\tfrac{\alpha_1\beta_5}{2}sin(\varphi+\theta)\hat{t}+\tfrac{\alpha_1\alpha_5}{2}cos(\varphi-$$
$$\left.\theta)\hat{t}+\tfrac{\alpha_1\beta_5}{2}sin(\varphi-\theta)\hat{t}+\tfrac{\alpha\alpha_1^3cos3\theta\hat{t}}{8(\varphi^2-4\theta^2)}\right]-\tfrac{15\beta}{4}\left[\left(B^3+\tfrac{3\alpha_1^2B}{2}\right)\right]+3\left(\tfrac{\alpha_1^3}{4}+\alpha_1B^2\right)cos\theta\hat{t}+\tfrac{3\alpha_1^2}{2}Bcos2\theta\hat{t}+$$
$$\tfrac{\alpha_1^3}{4}cos3\theta\hat{t} \qquad (4.45c)$$

To ensure a uniformly valid solution in $\hat{t}$, equate to zero the coefficients of $cos\theta\hat{t}$ and this yields

$$(\omega_1')^2\theta^2+2\omega_1'\alpha_4\theta^2+2\omega_2'\theta^2\alpha_1-2\alpha\left(\tfrac{\alpha_1r_0}{\theta^2}-\tfrac{\alpha_1r_1}{6\theta^2}+B\alpha_4\right)-\left\{\tfrac{\alpha^2\alpha_1\left(\frac{\alpha_1^2}{2}+B^2\right)}{\varphi^2}+\tfrac{\alpha_1^3\alpha_1}{4(\varphi^2-4\theta^2)}\right\}-$$
$$\tfrac{45\beta}{4}\left(\tfrac{\alpha_1^3}{4}+\alpha_1B^2\right)=0$$

$$\therefore \omega_2'=-\tfrac{1}{2\theta^2\alpha_1}\left[(\omega_1')^2\theta^2+2\omega_1'\alpha_4\theta^2-2\alpha\left(\tfrac{\alpha_1r_0}{\theta^2}-\tfrac{\alpha_1r_1}{6\theta^2}+B\alpha_4\right)-\left\{\tfrac{\alpha^2\alpha_1\left(\frac{\alpha_1^2}{2}+B^2\right)}{\varphi^2}+\tfrac{\alpha_1^3\alpha_1}{4(\varphi^2-4\theta^2)}\right\}-\right.$$
$$\left.\tfrac{45\beta}{4}\left(\tfrac{\alpha_1^3}{4}+\alpha_1B^2\right)\right] \qquad (4.46)$$

The remaining equation in (4.45c) is

$$U_{m,\hat{t}\hat{t}}^{(30)}+\theta^2U_m^{(30)}=r_{10}+r_{11}cos2\theta\hat{t}+r_{12}cos3\theta\hat{t}+r_{13}cos(\varphi+\theta)\hat{t}+r_{14}sin(\varphi+\theta)\hat{t}+r_{15}cos(\varphi-$$
$$\theta)\hat{t}+r_{16}sin(\varphi-\theta)\hat{t} \qquad (4.47)$$

Solving (4.47) gives

$$U_m^{(30)}(\hat{t},\tau)=\alpha_9(\tau)cos\theta\hat{t}+\beta_9(\tau)sin\theta\hat{t}+\frac{r_{10}}{\theta^2}-\frac{r_{11}cos\theta\hat{t}}{3\theta^2}-\frac{r_{12}cos3\theta\hat{t}}{8\theta^2}$$
$$-\frac{1}{\varphi(2\theta+\varphi)}[r_{13}cos(\varphi+\theta)\hat{t}+r_{14}sin(\varphi+\theta)\hat{t}]$$
$$+\frac{1}{\varphi(2\theta-\varphi)}[r_{15}cos(\varphi-\theta)\hat{t}+r_{16}sin(\varphi-\theta)\hat{t}] \qquad (4.48)$$

The initial conditions are

$$U_m^{(30)}(0,0)=0\ ,\quad U_{m,\hat{t}}^{(30)}(0,0)+\omega_1'(0)U_{m,\hat{t}}^{(20)}(0,0)+\omega_2'(0)U_{m,\tau}^{(10)}(0,0)=0$$

where

$$\alpha_9(0)=\left[-\frac{r_{10}}{\theta^2}+\frac{r_{11}}{3\theta^2}+\frac{r_{12}}{8\theta^2}+\frac{r_{13}}{\varphi(2\theta+\varphi)}-\frac{r_{15}}{\varphi(2\theta-\varphi)}\right]at\ \tau=0$$

and

$$\beta_9(0)=\frac{1}{\theta}\left[\frac{r_{14}(\varphi+\theta)}{\varphi(2\theta+\varphi)}-\frac{r_{16}(\varphi-\theta)}{\varphi(2\theta-\varphi)}\right]\ at\ \tau=0$$

and where,

$$r_{10} = -\left[2\alpha\left(\frac{\alpha_1\alpha_4}{4} + \frac{Br_0}{\theta^2}\right) + \frac{\alpha^2\alpha_1^2 B}{\varphi^2-\theta^2} - \frac{15\beta}{4}\left(B^3 + \frac{3\alpha_1^2 B}{2}\right)\right]$$

$$r_{10}(0) = B^3\left(\frac{10\alpha^2}{3\theta^2} - \frac{\alpha^2}{\varphi^2-\theta^2} + \frac{75\beta}{8}\right)$$

$$r'_{10}(0) = B^3\left[\frac{\alpha S_5}{2} - \frac{8\alpha^2}{3\theta^2} - \frac{4\alpha^2}{\theta} - \frac{2\alpha^2}{(\varphi^2-\theta^2)} - \frac{45\beta}{4}\right]$$

$$r_{11} = -\left(\frac{8r_1\omega'_1}{3} + 2\alpha\left(\frac{\alpha_1\alpha_4}{4} + \frac{Br_0}{3\theta^2}\right) + \frac{\alpha^2\alpha_1^2 B}{\varphi^2-\theta^2} + \frac{45\beta\alpha_1^2 B}{2}\right)$$

$$r_{11}(0) = B^3\left(\frac{8\alpha}{3\theta^2} + \frac{2\alpha^2}{3\theta^2} - \frac{45\beta}{8} + \frac{\alpha^2}{\varphi^2-\theta^2}\right)$$

$$r'_{11}(0) = B^3\left[\frac{4\alpha^2}{3\theta^2} - \frac{16\alpha}{3\theta^2} - \frac{\alpha S_5}{2} - \frac{2\alpha}{(\varphi^2-\theta^2)} - \frac{45\beta}{4}\right]$$

$r_{12} = \frac{\alpha\alpha_1 r_1}{3\theta^2} - \frac{\alpha_1^3\alpha^2}{4(\varphi^2-\theta^2)} - \frac{15\beta\alpha_1^3}{16}$, $r_{12}(0) = B^3\left[\frac{\alpha^2}{3\theta^2} + \frac{\alpha^2}{4(\varphi^2-\theta^2)} + \frac{15\beta}{16}\right]$ $r'_{12}(0) = B^3\left[\frac{3\alpha^2}{4(\varphi^2-\theta^2)} - \frac{\alpha^2}{3\theta^2} + \frac{45\beta}{4}\right]$,

$r_{13} = -\alpha\alpha_1\alpha_5 = r_{15},\ r_{13}(0) = r_{15}(0) = B^3\alpha^2 S_0$, $r'_{13}(0) = r'_{15}(0) = -2\alpha S_0 B^3$

$r_{14} = -\alpha\alpha_1\beta_5 = r_{16}$, $r_{14}(0) = r_{16}(0) = 0\ since\ \beta_5(0) = 0$, $r'_{16}(0) = r'_{14}(0) = 0$

Substituting in (4.44) gives

$$U^{(30)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(30)}_{2m} =$$

$$2\omega'_1\left[-\varphi^2\alpha_5 cos\varphi\hat{t} - \varphi^2\beta_5 sin\varphi\hat{t} + \frac{\alpha}{2}\left\{\frac{-2\theta^2 B\alpha_1 cos\theta\hat{t}}{\varphi^2-\theta^2} - \frac{2\alpha_1^2\theta^2 cos2\theta\hat{t}}{\varphi^2-4\theta^2}\right\}\right] +$$

$$2\alpha\left[\begin{matrix}\left(\frac{\alpha_1\alpha_4}{4} + \frac{Br_0}{\theta^2}\right) + \left(\frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B\alpha_4\right)cos\theta\hat{t} + \left(\frac{\alpha_1\alpha_4}{4} + \frac{Br_1}{3\theta^2}\right)cos2\theta\hat{t} \\ -\frac{\alpha_1 r_1}{6\theta^2}cos3\theta\hat{t}\end{matrix}\right] - 2\alpha\left[\frac{\alpha\alpha_1^2 B}{2(\varphi^2-\theta^2)} + \left\{\frac{\alpha\alpha_1\left(\frac{\alpha_1^2}{2}+B^2\right)}{2\varphi^2} + \right.\right.$$

$$\left.\frac{\alpha_1^3\alpha}{8(\varphi^2-4\theta^2)}\right\}cos\theta\hat{t} + \frac{\alpha_1\alpha_5}{2}cos(\varphi+\theta)\hat{t} + \frac{\alpha\alpha_1^2 Bcos2\theta\hat{t}}{2(\varphi^2-\theta^2)} + \frac{\alpha_1\beta_5}{2}sin(\varphi+\theta)\hat{t} + \frac{\alpha_1\alpha_5}{2}cos(\varphi-\theta)\hat{t} +$$

$$\left.\frac{\alpha_1\beta_5}{2}sin(\varphi-\theta)\hat{t} + \frac{\alpha_1^3\alpha cos3\theta\hat{t}}{8(\varphi^2-4\theta^2)}\right] - \frac{3\beta}{2}\left[\left(B^3 + \frac{3\alpha_1^2 B}{2}\right) + 3\left(\frac{\alpha_1^3}{4} + \alpha_1 B^2\right)cos\theta\hat{t} + \frac{3\alpha_1^2 B}{2}cos2\theta\hat{t} +\right.$$

$$\left.\frac{\alpha_1^3}{4}cos3\theta\hat{t}\right] \qquad (4.49)$$

To ensure a uniformly valid solution in $\hat{t}$, needs equating to zero the coefficients of $cos\varphi\hat{t}$ and $sin\varphi\hat{t}$. A further simplification of (4.49) gives

$$U^{(30)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(30)}_{2m} = r_{17} + r_{18}cos\theta\hat{t} + r_{19}cos2\theta\hat{t} + r_{20}cos3\theta\hat{t} \qquad (4.50)$$

where,

$$r_{17} = \left[2\alpha\left(\frac{\alpha_1\alpha_4}{4} + \frac{Br_0}{\theta^2}\right) - \frac{\alpha^2\alpha_1^2 B}{\varphi^2-\theta^2} - \frac{3\beta}{2}\left(B^3 + \frac{3\alpha_1^2 B}{2}\right)\right]$$

$$r_{17}(0) = B^3\left(-\frac{22\alpha^2}{3\theta^2} + \frac{\alpha^2}{\varphi^2-\theta^2} + \frac{15\beta}{4}\right)$$

$$r'_{17}(0) = B^3\left[\frac{-S_5\alpha}{2} + \frac{20\alpha^2}{3\theta^2} + \frac{2\alpha^2}{2(\varphi^2-\theta^2)} + \frac{9\beta}{2}\right]$$

$$r_{18} = \left[\frac{-2\theta^2\omega'_1 B\alpha_1\alpha}{\varphi^2-\theta^2} + 2\alpha\left(\frac{\alpha_1 r_0}{\theta^2} - \frac{\alpha_1 r_1}{6\theta^2} + B\alpha_4\right) - \frac{\alpha^2\alpha_1\left(\frac{\alpha_1^2}{2} + B^2\right)}{\varphi^2} - \frac{\alpha_2\alpha_1^3}{4(\varphi^2-4\theta^2)} - \frac{9}{2}\left(\frac{\alpha_1^3}{4} + \alpha_1 B^2\right)\right]$$

$$r_{18}(0) = B^3\left(\frac{2\alpha}{\varphi^2-\theta^2} + \frac{18\alpha^2}{6\theta^2} + \frac{3\alpha^2}{\varphi^2} + \frac{\alpha^2}{4(\varphi^2-4\theta^2)} + \frac{45}{2}\right)$$

$$r'_{18}(0) = B^3\left[2\alpha S_5 - 2 - \frac{43\alpha^2}{\theta^2} - \frac{5\alpha^2}{2\varphi^2} - \frac{3\alpha^2}{4(\varphi^2-\theta^2)} - \frac{63}{8}\right]$$

$$r_{19} = \left[\frac{-2\omega'_1\alpha_1^2\alpha\theta^2}{\varphi^2-\theta^2} + 2\alpha\left(\frac{\alpha_1\alpha_4}{4} - \frac{Br_1}{3\theta^2}\right) + \frac{\alpha^2\alpha_1^2 B}{2(\varphi^2-\theta^2)} - -\frac{9\alpha_1^2 B\beta}{4}\right]$$

$$r_{19}(0) = B^3\left(\frac{-2\alpha}{\varphi^2-\theta^2} - \frac{4\alpha^2}{3\theta^2} + \frac{2\alpha^2}{3\theta^2} + \frac{\alpha^2}{2(\varphi^2-\theta^2)} - \frac{9\beta}{4}\right)$$

$$r'_{19}(0) = B^3\left[\frac{4\alpha}{B(\varphi^2-4\theta^2)} - \frac{S_5}{2} + \frac{4\alpha^2}{3\theta^2} - \frac{\alpha^2}{(\varphi^2-4\theta^2)} + \frac{9\beta}{2}\right]$$

$r_{20} = \left[-\frac{\alpha\alpha_1 r_1}{3\theta^2} - \frac{\alpha_1^3\alpha^2}{4(\varphi^2-4\theta^2)} - \frac{3\beta\alpha_1^3}{8}\right], r_{20}(0) = B^3\left(-\frac{\alpha^2}{3\theta^2} + \frac{\alpha^2}{4(\varphi^2-\theta^2)} + \frac{3\beta}{8}\right)$

$$r'_{20}(0) = B^3\left[\frac{\alpha^2}{\theta^2} + \frac{3\alpha^2}{4(\varphi^2-4\theta^2)} - \frac{9\beta}{8}\right]$$

The solution of (4.50) is

$$U_{2m}^{(30)} = \alpha_{10}cos\varphi\hat{t} + \beta_{10}sin\varphi\hat{t} + \frac{r_{17}}{\varphi^2} + \frac{r_{18}cos\theta\hat{t}}{(\varphi^2-\theta^2)} + \frac{r_{19}cos2\theta\hat{t}}{(\varphi^2-4\theta^2)} + \frac{r_{20}cos3\theta\hat{t}}{(\varphi^2-9\theta^2)} \qquad (4.51a)$$

The initial conditions for (4.51) are

$$U_{2m}^{(30)}(0,0) = 0;\ U_{2m,\hat{t}}^{(30)}(0,0) + \omega'_1(0)U_{2m,\hat{t}}^{(20)}(0,0) = 0$$

$$\therefore \alpha_{10}(0) = -\left[\frac{r_{18}}{(\varphi^2-\theta^2)} + \frac{r_{19}}{(\varphi^2-4\theta^2)} + \frac{r_{20}}{(\varphi^2-9\theta^2)}\right] at\ \tau = 0,\ \ \beta_{10}(0) = 0 \qquad (4.51b)$$

Substituting in (4.45b) the following is obtained

$$U_{3m,\hat{t}\hat{t}}^{(30)} + \Omega^2 U_{3m}^{(30)} = \alpha\left[\frac{\alpha\alpha_1^2 B}{2(\varphi^2-\theta^2)} + \left\{\frac{\alpha\alpha_1\left(\frac{\alpha_1^2}{2}+B^2\right)}{2\varphi^2} + \frac{\alpha_1^3\alpha}{8(\varphi^2-4\theta^2)}\right\}cos\theta\hat{t} + \frac{\alpha_1\alpha_5}{2}\cos(\varphi+\theta)\,\hat{t} + \frac{\alpha\alpha_1^2 Bcos2\theta\hat{t}}{2(\varphi^2-\theta^2)} + \right.$$

$$\left.\frac{\alpha_1\beta_5}{2}\sin(\varphi+\theta)\,\hat{t} + \frac{\alpha_1\alpha_5}{2}\cos(\varphi-\theta)\,\hat{t} + \frac{\alpha_1\beta_5}{2}\sin(\varphi-\theta)\,\hat{t} + \frac{\alpha_1^3\alpha cos3\theta\hat{t}}{8(\varphi^2-4\theta^2)}\right] - \frac{\beta}{4}\left[\left(B^3 + \frac{3\alpha_1^2 B}{2}\right) + \right.$$

$$\left.3\left(\frac{\alpha_1^3}{4} + \alpha_1 B^2\right)cos\theta\hat{t} + \frac{3\alpha_1^2}{2}Bcos2\theta\hat{t} + \frac{\alpha_1^3}{4}cos3\theta\hat{t}\right] \qquad (4.52a)$$

Rewriting (4.52a) gives

$$U_{3m,\hat{t}\hat{t}}^{(30)} + \Omega^2 U_{3m}^{(30)} = r_{21} + r_{22}cos\theta\hat{t} + r_{23}cos2\theta\hat{t} + r_{24}cos3\theta\hat{t} + r_{25}\cos(\varphi+\theta)\,\hat{t} + r_{26}\sin(\varphi+\theta)\,\hat{t} + r_{27}\cos(\varphi-\theta)\,\hat{t} + r_{28}\sin(\varphi-\theta)\,\hat{t} \qquad (4.52b)$$

The initial conditions are

$$U_{3m}^{(30)}(0,0) = 0;\ U_{3m,\hat{t}}^{(30)}(0,0) = 0$$

where,

$r_{21} = \left\{\frac{\alpha^2\alpha_1^2 B}{2(\varphi^2-\theta^2)} - \frac{\beta}{4}\left(B^3 + \frac{3\alpha_1^2 B}{2}\right)\right\}, r_{21}(0) = B^3\left(\frac{\alpha^2\alpha_1^2}{2(\varphi^2-\theta^2)} - \frac{5\beta}{8}\right)$

$$r_{22} = \left\{\frac{\alpha^2\alpha_1\left(\frac{\alpha_1^2}{2}+B^2\right)}{2\varphi^2} + \frac{\alpha_1^3\alpha^2}{8(\varphi^2-4\theta^2)} - \frac{3\beta}{4}\left(\frac{\alpha_1^3}{4} + \alpha_1 B^2\right)\right\};$$

$r_{22}(0) = B^3\left(\frac{15\beta}{16} - \frac{3\alpha^2}{4\varphi^2} - \frac{\alpha^2}{8(\varphi^2-4\theta^2)}\right),\quad r'_{22}(0) = B^3\left(\frac{5\alpha^2}{4\varphi} + \frac{3\alpha^2}{8(\varphi^2-4\theta^2)} - \frac{21\beta}{16}\right)$

$r_{23} = \left\{\frac{\alpha^2\alpha_1^2 B}{2(\varphi^2-\theta^2)} - \frac{3\alpha_1^2 B}{8}\right\}, r_{23}(0) = B^3\left(\frac{\alpha^2}{2(\varphi^2-\theta^2)} - \frac{3\beta}{8}\right)$

$r'_{23}(0) = B^3\left(\frac{3\beta}{4} - \frac{\alpha^2}{(\varphi^2-\theta^2)}\right), r_{24} = \left(\frac{\alpha^2\alpha_1^3 B}{8(\varphi^2-\theta^2)} - \frac{3\alpha_1^3\beta}{16}\right)$

$r_{24}(0) = B^3\left(\frac{\beta}{16} - \frac{\alpha^2}{8(\varphi^2-4\theta^2)}\right), r'_{24}(0) = B^3\left(\frac{3\alpha^2}{8(\varphi^2-\theta^2)} - \frac{3\beta}{16}\right), r_{25} = \frac{\alpha\alpha_1\alpha_5}{2} = r_{27},\ \ r_{25}(0) = r_{27}(0) =$
$B^2\alpha^2 S_0,\ \ r'_{25}(0) = r'_{27}(0) = \alpha S_0 B^3$

$r_{26} = \frac{\alpha\alpha_1\beta_5}{2} = r_{28}\,; r_{26}(0) = r_{28}(0) = 0, r'_{26}(0) = r'_{28}(0) = 0$

$$\therefore U_{3m}^{(30)}(\hat{t},\tau) = \alpha_{11}(\tau)cos\Omega\hat{t} + \beta_{11}(\tau)sin\Omega\hat{t} + \frac{r_{22}cos\theta\hat{t}}{\Omega^2-\theta^2} + \frac{r_{23}cos2\theta\hat{t}}{\Omega^2-4\theta^2} + \frac{r_{24}cos3\theta\hat{t}}{\Omega^2-9\theta^2} + \left\{\frac{r_{25}\cos(\varphi+\theta)\,\hat{t} + r_{26}\sin(\varphi+\theta)\,\hat{t}}{\Omega^2-(\varphi+\theta)^2}\right\} + \left\{\frac{r_{27}\cos(\varphi-\theta)\,\hat{t} + r_{28}\sin(\varphi-\theta)\,\hat{t}}{\Omega^2-(\varphi-\theta)^2}\right\} \qquad (4.52b)$$

$$\alpha_{11}(0) = -\left[\frac{r_{22}}{\Omega^2-\theta^2} + \frac{r_{23}}{\Omega^2-4\theta^2} + \frac{r_{24}}{\Omega^2-9\theta^2} + \frac{r_{25}}{\Omega^2-(\varphi+\theta)^2} + \frac{r_{27}}{\Omega^2-(\varphi+\theta)^2}\right]\Bigg|\tau = 0 \qquad (4.52c)$$

$$\beta_{11}(0) = \frac{-1}{\Omega}\left[\frac{r_{26}\,(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} + \frac{r_{28}(\varphi-\theta)}{\Omega^2-(\varphi+\theta)^2}\right]\Bigg|\tau = 0 \qquad (4.53)$$

So far, it follows that

$$U^{(30)} = U_m^{(30)}(1-cos2mx) + U_{2m}^{(30)}(1-cos4mx) + U_{3m}^{(30)}(1-cos6mx)$$

From (3.24),

$$O(\epsilon^3\delta)\ :\ U^{(31)}_{,\hat{t}\hat{t}} + U^{(31)}_{,xxxx} +\ 2\lambda U^{(31)}_{,xx} + U^{(31)}$$
$$= -(\omega_1')^2 U^{(11)}_{,\hat{t}\hat{t}} - 2\left(\omega_1' U^{(21)}_{,\hat{t}\,\tau} + \omega_2' U^{(11)}_{,\hat{t}\,\tau}\right) -\ 2U^{(30)}_{,\hat{t}\,\tau} +\ 2\left(\omega_1' U^{(20)}_{,\hat{t}\hat{t}} + \omega_2' U^{(10)}_{,\hat{t}\hat{t}}\right)$$
$$- \left(\omega_1'' U^{(20)}_{,\hat{t}} + \omega_2''^{+} U^{(10)}_{,\hat{t}}\right) - 2\left\{U^{(30)}_{\hat{t}} +\ \left(\omega_1' U^{(20)}_{,\hat{t}} + \omega_2'^{+} U^{(10)}_{,\hat{t}}\right)\right\}$$
$$- \alpha\left(\mathrm{U}^{(10)}\mathrm{U}^{(21)} +\ \mathrm{U}^{(11)}\mathrm{U}^{(20)}\right)\ + 3\beta\left(\mathrm{U}^{(10)}\right)^2\left(\mathrm{U}^{(11)}\right)$$

Substituting on the RHS of (3.24) gives

$$\mathrm{U}^{(31)}_{,\hat{t}\hat{t}} + \mathrm{U}^{(31)}_{,xxxx} +\ 2\lambda\mathrm{U}^{(31)}_{,xx} + \mathrm{U}^{(31)} = -\Big[(\omega_1')^2 U^{(11)}_{m,\hat{t}}(1-cos2mx) +\ 2\Big\{\omega_1'\Big(U^{(21)}_{m,\hat{t}\tau}(1-cos2mx) + U^{(21)}_{2m,\hat{t}\tau}(1-cos4mx)\Big) + \omega_2' U^{(11)}_{2m,\hat{t}\tau}(1-cos2mx)\Big\} + 2\Big\{U^{(30)}_{m,\hat{t}\tau}(1-cos2mx) + U^{(30)}_{2m,\hat{t}\tau}(1-cos4mx) + U^{(30)}_{3m,\hat{t}\tau}(1-cos6mx)\Big\} - 2\Big\{\omega_1'\Big(U^{(20)}_{m,\hat{t}\hat{t}}(1-cos2mx) + U^{(20)}_{2m,\hat{t}\hat{t}}(1-cos4mx)\Big) + \omega_2' U^{(10)}_{m,\hat{t}\hat{t}}(1-cos2mx)\Big\} + \Big\{\omega_1'' U^{(20)}_{m,\hat{t}}(1-cos2mx) + U^{(20)}_{2m,\hat{t}}(1-cos4mx) + \omega_2'' U^{(10)}_{m,\hat{t}}(1-cos2mx)\Big\} + 2\Big\{\mathrm{U}^{(30)}_{\mathrm{m},\hat{t}}(1-\cos 2mx) + \mathrm{U}^{(30)}_{2\mathrm{m},\hat{t}}(1-\cos 4mx) + \mathrm{U}^{(30)}_{3\mathrm{m},\hat{t}}(1-\cos 6mx) + \omega_1'\Big(\mathrm{U}^{(20)}_{\mathrm{m},\hat{t}}(1-\cos 2mx) +\ \mathrm{U}^{(20)}_{2\mathrm{m}\hat{t}}(1-\cos 4mx) + \omega_2'\mathrm{U}^{(10)}_{\mathrm{m},\hat{t}}(1-\cos 2mx)\Big)\Big\} + \alpha\Big\{\mathrm{U}^{(10)}_{\mathrm{m}}(1-\cos 2mx)\Big(\mathrm{U}^{(21)}_{\mathrm{m}}(1-\cos 2mx) + \mathrm{U}^{(21)}_{2\mathrm{m}}(1-\cos 4mx)\Big) +\ \mathrm{U}^{(11)}_{\mathrm{m}}(1-\cos 2mx)\Big(\mathrm{U}^{(20)}_{\mathrm{m}}(1-\cos 2mx) + \mathrm{U}^{(20)}_{2\mathrm{m}}(1-\cos 4mx)\Big)\Big\} - 3\beta\Big\{\left(\mathrm{U}^{(10)}_{\mathrm{m}}\right)^2\mathrm{U}^{(11)}_{\mathrm{m}}(1-\cos 2mx)^3\Big\}\Big] \tag{4.54}$$

Further simplification of (4.54) yields

$$U^{(31)}_{,\hat{t}\hat{t}} + U^{(31)}_{,xxxx} +\ 2\lambda U^{(31)} + U^{(31)} = -\ \Big[(\omega_1')^2 U^{(21)}_{m,\hat{t}\hat{t}}(1-cos2mx) + 2\Big\{\omega_1'\Big(U^{(21)}_{m,\hat{t}\tau}(1-cos2mx) + U^{(21)}_{2m,\hat{t}\tau}(1-cos4mx)\Big)\omega_2' U^{(11)}_{m,\hat{t}\tau}(1-cos2mx)\Big\} + 2\Big\{U^{(30)}_{m,\hat{t}\tau}(1-cos2mx) + U^{(30)}_{2m,\hat{t}\tau}(1-cos4mx) + U^{(30)}_{3m,\hat{t}\tau}(1-cos6mx)\Big\} - 2\Big\{\omega_1' U^{(20)}_{m,\hat{t}\hat{t}}(1-cos2mx) + U^{(20)}_{2m,\hat{t}\hat{t}}(1-cos4mx) + \omega_2' U^{(10)}_{m,\hat{t}\hat{t}}(1-cos2mx)\Big\} + \Big\{\omega''_1 U^{(20)}_{m,\hat{t}}(1-cos2mx) + U^{(20)}_{2m,\hat{t}}(1-cos4mx) + \omega_2'' U^{(10)}_{m,\hat{t}}(1-cos2mx)\Big\}$$
$$+2\Big\{U^{(30)}_{m,\hat{t}}(1-cos2mx) + U^{(30)}_{2m,\hat{t}}(1-cos4mx) + U^{(30)}_{3m,\hat{t}}(1-cos6mx) + \omega_1'\Big(U^{(20)}_{m,\hat{t}}(1-cos2mx) + U^{(20)}_{2m,\hat{t}}(1-cos4mx)\Big) + \omega_2'\Big(U^{(10)}_{m,\hat{t}}(1-cos2mx)\Big)\Big\} +\ \alpha\Big\{U^{(10)}_m U^{(21)}_m\left(\tfrac{3}{2} - 2cos2mx + \tfrac{1}{2}cos4mx\right) + U^{(10)}_m U^{(21)}_{2m}\left(1 - \tfrac{1}{2}cos2mx - cos4mx + \tfrac{1}{2}cos6mx\right)\Big\} + \alpha\Big\{U^{(11)}_m U^{(20)}_m\left(\tfrac{3}{2} - 2cos2mx + \tfrac{1}{2}cos4mx\right) + U^{(11)}_m U^{(20)}_{2m}\left(1 - \tfrac{1}{2}cos2mx - cos4mx + \tfrac{1}{2}cos6mx\right)\Big\} - 3\beta\left(U^{(10)}_m\right)^2 U^{(11)}_m\left(\tfrac{5}{2} - \tfrac{15}{4}cos2mx - \tfrac{3}{2}cos4mx - \tfrac{1}{4}cos6mx\right)\Big] \tag{4.55}$$

Let

$$U^{(31)}\sum_{n=1}^{\infty} U^{(31)}(1 - cos2nx)$$

The LHS of (4.55) becomes

$$\sum_{n=1}^{\infty}\left[U^{(31)}_{n,\hat{t}\hat{t}}(1-cos2nx) + (-16n^4 + 8\lambda n^2 + 1)U^{(31)}_n cos2nx\right]$$

Multiplying (4.55) through cos2mx and integrating from 0 to π and from n=m, gives

$$-\frac{\pi}{2}\left[U^{(31)}_{m,\hat{t}\hat{t}} + (16m^4 - 8\lambda m^2 + 1)U^{(31)}_m\right] =$$
$$-\Bigg[(\omega_1')^2 U^{(11)}_{m,\hat{t}\hat{t}}\left(-\tfrac{\pi}{2}\right) +\ \ 2\left\{\omega_1' U^{(21)}_{m,\hat{t}\tau}\left(-\tfrac{\pi}{2}\right) + \omega_2' U^{(11)}_{m,\hat{t}\tau}\left(-\tfrac{\pi}{2}\right)\right\} + 2\left\{U^{(30)}_{m,\hat{t}\,\tau}\left(-\tfrac{\pi}{2}\right)\right\} - 2\left\{\begin{matrix}\omega_1' U^{(20)}_{m,\hat{t}\hat{t}} \\ +\omega_2' U^{(10)}_{m,\hat{t}\hat{t}}\left(-\tfrac{\pi}{2}\right)\end{matrix}\right\} + \left\{\omega_1' U^{(20)}_{m,\hat{t}\tau}\left(-\tfrac{\pi}{2}\right) + \omega_2'' U^{(10)}_{m,\hat{t}}\left(-\tfrac{\pi}{2}\right)\right\} + 2\left\{U^{(30)}_{m,\hat{t}}\left(-\tfrac{\pi}{2}\right) + \omega_1' U^{(20)}_{m,\hat{t}\tau}\left(-\tfrac{\pi}{2}\right) + \omega_2' U^{(10)}_{m,\hat{t}}\left(-\tfrac{\pi}{2}\right)\right\} + \alpha\left\{-2U^{(10)}_m U^{(21)}_m\left(-\tfrac{\pi}{2}\right) - U^{(10)}_m U^{(21)}_m\left(-\tfrac{\pi}{2}\right) - 2U^{(11)}_m U^{(21)}_m\left(-\tfrac{\pi}{2}\right) - 2U^{(11)}_m U^{(20)}_m\left(-\tfrac{\pi}{2}\right) - U^{(11)}_m U^{(20)}_{2m}\left(-\tfrac{\pi}{2}\right)\right\} + 3\beta\left(U^{(10)}_m\right)^2 U^{(11)}_m\left(-\tfrac{15}{4}\right)\Bigg] \tag{4.56}$$

A further simplification of (4.56) yields

$$U^{(31)}_{m,\hat t\hat t}+\theta^2U^{(31)}_m=(\omega_1')^2U^{(11)}_{m,\hat t\hat t}-2\left\{\omega_1'U^{(21)}_{m,\hat t\tau}+\omega_2'U^{(11)}_{m,\hat t\tau}\right\}-2\left\{U^{(30)}_{m,\hat t\,\tau}\right\}+2\left\{\omega_1'U^{(20)}_{m,\hat t\hat t}+\omega_2'U^{(10)}_{m,\hat t\hat t}\right\}-\left\{\omega_1'U^{(20)}_{m,\hat t}+\omega_2''U^{(10)}_{m,\hat t}\right\}-2\left\{U^{(30)}_{m,\hat t}+\omega_1'U^{(20)}_{m,\hat t}+\omega_2'U^{(10)}_{m,\hat t}\right\}+\alpha\left\{2U^{(10)}_mU^{(21)}_m+U^{(10)}_mU^{(21)}_m+2U^{(11)}_mU^{(21)}_m+U^{(11)}_mU^{(20)}_{2m}\right\}-\frac{45}{4}\beta\left(U^{(10)}_m\right)^2U^{(11)}_m \qquad (4.57)$$

The initial conditions for (4.57) are

$$U^{(31)}_m(0{,}0)=0;$$
$$U^{(31)}_{m,\hat t}(0{,}0)+\omega_1'(0)U^{(21)}_{m,\hat t}(0{,}0)+\omega_2'(0)U^{(11)}_{m,\hat t}(0{,}0)+U^{(30)}_{m,\tau}(0{,}0)=0$$

Multiplying (4.55) through by cos4mx and integrating from 0 to π and for n=2m gives

$$-\frac{\pi}{2}\left[U^{(31)}_{m,\hat t\hat t}+(256m^4-32\lambda m^2+1)U^{(30)}_{2m}\right]$$
$$=-\Big[2\omega_1'U^{(21)}_{2m,\hat t\tau}\left(-\frac{\pi}{2}\right)+2U^{(30)}_{2m,\hat t\tau}\left(-\frac{\pi}{2}\right)-2\omega_1'U^{(20)}_{2m,\hat t\hat t}\left(-\frac{\pi}{2}\right)+\omega_1''U^{(20)}_{2m,\hat t}\left(-\frac{\pi}{2}\right)+2U^{(30)}_{2m,\hat t}\left(-\frac{\pi}{2}\right)+2\omega_1'U^{(20)}_{2m,\hat t}\left(-\frac{\pi}{2}\right)+\alpha\left\{\frac{1}{2}U^{(10)}_mU^{(21)}_m\left(-\frac{\pi}{2}\right)-U^{(10)}_mU^{(21)}_{2m}\left(-\frac{\pi}{2}\right)+\frac{1}{2}U^{(11)}_mU^{(20)}_m\left(-\frac{\pi}{2}\right)-U^{(11)}_mU^{(20)}_{2m}\left(-\frac{\pi}{2}\right)\right\}-3\beta\left(U^{(10)}_m\right)^2U^{(11)}_m\left(\frac{3}{2}\right)\Big] \qquad (4.58)$$

$$\Rightarrow U^{(31)}_{2m,\hat t\hat t}+\varphi^2U^{(30)}_{2m}=-\Big[2\omega_1'U^{(21)}_{2m,\hat t\tau}+2U^{(30)}_{2m,\hat t\tau}-2\omega_1'U^{(20)}_{2m,\hat t\hat t}+\omega_1''U^{(20)}_{2m,\hat t}+2U^{(30)}_{2m,\hat t}+2\omega_1'U^{(20)}_{2m,\hat t}+\alpha\left\{\frac{1}{2}U^{(10)}_mU^{(21)}_m-U^{(10)}_mU^{(21)}_{2m}+\frac{1}{2}U^{(11)}_mU^{(20)}_m-U^{(11)}_mU^{(20)}_{2m}\right\}-\frac{9}{4}\beta\left(U^{(10)}_m\right)^2U^{(11)}_m\Big] \qquad (4.59)$$

The initial conditions are

$$U^{(31)}_{2m}(0{,}0)=0;\ U^{(31)}_{2m}(0{,}0)+\omega_1'(0)U^{(20)}_{2m,\hat t}(0{,}0)=0$$

Multiplying (4.56) through by cos6mx and integrating from 0 to π and for n=3m, the result is

$$-\frac{\pi}{2}\left[U^{(31)}_{m,\hat t\hat t}+(1296m^4-72\lambda m^2+1)U^{(31)}_{2m}\right]=-\left[2U^{(30)}_{3m,\hat t\tau}\left(-\frac{\pi}{2}\right)+2U^{(30)}_{3m,\hat t}\left(-\frac{\pi}{2}\right)+\alpha\left\{\frac{1}{2}U^{(10)}_mU^{(21)}_{2m}\left(-\frac{\pi}{2}\right)+\frac{1}{2}U^{(11)}_mU^{(20)}_{2m}\left(-\frac{\pi}{2}\right)\right\}-3\beta\left(-\frac{1}{4}\right)\left(U^{(10)}_m\right)^2U^{(11)}_m\right] \qquad (4.60)$$

A further simplification of (4.60) yields

$$U^{(31)}_{3m,\hat t\hat t}+\Omega^2U^{(30)}_{3m}=-\left[2U^{(30)}_{3m,\hat t\tau}+2U^{(30)}_{3m,\hat t}+\alpha\left\{\frac{1}{2}U^{(10)}_mU^{(21)}_{2m}+\frac{1}{2}U^{(11)}_mU^{(20)}_{2m}\right\}+\frac{3}{4}\beta\left(U^{(10)}_m\right)^2U^{(11)}_m\right] \qquad (4.61)$$

The initial conditions for (4.61) are

$$U^{(31)}_{3m}(0{,}0)=0;\ \ U^{(31)}_{3m,\hat t}(0{,}0)=0$$

Further simplification of terms in (4.57) yields

$$U^{(31)}_{m,\hat t\hat t}+\theta^2U^{(31)}_m=(\omega_1')^2\theta^2\beta_2\sin\theta\hat t-2\left[\omega_1'(-\theta\alpha_6'\sin\theta\hat t+\theta\beta_6'\cos\theta\hat t)-\left(\frac{-2\theta r_3'\sin2\theta\hat t+2\theta r_4'\cos2\theta\hat t}{3\theta^2}\right)\right]-2\omega_2'\beta_2'\theta\cos\theta\hat t-2\Big[-\alpha_9'\theta\sin\theta\hat t+\beta_9'\theta\cos\theta\hat t+\frac{2r_{11}'\sin2\theta\hat t}{3\theta}+\frac{3r_{12}'\sin3\theta\hat t}{8\theta}-\frac{1}{\varphi(2\theta+\varphi)}\{-r_{13}'(\varphi+\theta)\sin(\varphi+\theta)\hat t+r_{14}'(\varphi+\theta)\cos(\varphi+\theta)\hat t\}+\frac{1}{\varphi(2\theta-\varphi)}\{-r_{15}'(\varphi-\theta)\sin(\varphi-\theta)\hat t+r_{16}'(\varphi-\theta)\cos(\varphi-\theta)\hat t\}\Big]+2\omega_1'\left\{-\theta^2\alpha_4\cos\theta\hat t+\frac{4r_1\cos2\theta\hat t}{3}\right\}+2\omega_2'(-\alpha_1\theta^2\cos\theta\hat t)-\omega_1'\left\{-\theta\alpha_4\sin\theta\hat t+\frac{2r_1\sin2\theta\hat t}{3\theta}\right\}-\omega_2''(-\alpha_1\theta\sin\theta\hat t)-2\Big\{-\alpha_9\sin\theta\hat t+\beta_9\cos\theta\hat t+\frac{2r_{11}\sin2\theta\hat t}{3\theta}+\frac{3r_{12}\sin3\theta\hat t}{8\theta}-\frac{1}{\varphi(2\theta+\varphi)}\{-(\varphi+\theta)r_{13}\sin(\varphi+\theta)\hat t+r_{14}(\varphi+\theta)\cos(\varphi+\theta)\hat t\}+\frac{1}{\varphi(2\theta-\varphi)}\{-(\varphi+\theta)r_{15}\sin(\varphi-\theta)\hat t+(\varphi-\theta)r_{16}\cos(\varphi-\theta)\hat t\}\Big\}-2\omega_1'\left\{-\theta\alpha_4\sin\theta\hat t+\frac{2r_1\sin2\theta\hat t}{3\theta}\right\}-2\omega_2'(-\theta\alpha_1\sin\theta\hat t)+2\alpha\Big\{\left(\frac{\alpha_1\alpha_6}{2}-\frac{Br_2}{\theta^2}\right)+\left(\frac{\alpha_1r_2}{2}-\frac{Br_3}{6\theta^2}+B\alpha_6\right)\cos\theta\hat t+\left(\frac{B\beta_6-\alpha_1\alpha_4}{6\theta^2}\right)\sin\theta\hat t+\left(\frac{\alpha_1\alpha_6}{2}-\frac{Br_3}{3\theta^2}\right)\cos2\theta\hat t+\left(\frac{\alpha_1\beta_6}{2}-\frac{Br_4}{3\theta^2}\right)\sin\theta\hat t-\frac{\alpha_1r_3}{6\theta^2}\cos3\theta\hat t-\frac{\alpha_1r_4}{6\theta^2}\sin3\theta\hat t\Big\}+\alpha\Big\{\left(\frac{\alpha_1r_6}{2(\varphi^2-4\theta^2)}+\frac{Br_5}{\varphi^2-\theta^2}\right)\sin\theta\hat t+\left(\frac{\alpha_1r_5}{2(\varphi^2-4\theta^2)}-\frac{Br_6}{\varphi^2-4\theta^2}\right)\sin2\theta\hat t+\frac{\alpha_1r_6\sin3\theta\hat t}{2(\varphi^2-4\theta^2)}+B\alpha_7\cos\varphi\hat t+B\beta_7\sin\varphi\hat t+$$

$\frac{\alpha_1\alpha_7}{2}cos(\varphi+\theta)\hat{t}+\frac{\alpha_1\beta_7}{2}sin(\varphi+\theta)\hat{t}+\frac{\alpha_1\alpha_7}{2}cos(\varphi-\theta)\hat{t}+\frac{\alpha_1\beta_7}{2}sin(\varphi-\theta)\hat{t}\Big\}+2\alpha\Big\{\frac{\beta_2\alpha_4}{2}sin2\theta\hat{t}+$
$\frac{\beta_2 r_0}{\theta^2}sin\theta\hat{t}-\frac{\beta_2 r_1}{6\theta^2}(sin3\theta\hat{t}-sin\theta\hat{t})\Big\}$ (4.62)

To ensure a uniformly valid solution in $\hat{t}$, demands equating to zero the coefficients of $\sin\theta\,\hat{t}$ and $\cos\theta\,\hat{t}$ in (4.62) as further expanded. The coefficient of $\sin\theta\,\hat{t}$ leads to

$$\alpha_9'+\alpha_9=h_1(\tau) \quad (4.63a)$$

$$h_1(\tau)=-\frac{1}{2\theta}\Big[2\omega_1'\theta\alpha_6'+\omega_2''\alpha_1\theta+\omega_1'\theta\alpha_4+2\omega_1'\theta\alpha_4+2\omega_2'\theta\alpha_1+2\alpha\Big(\frac{B\beta_6-\alpha_1 r_4}{6\theta^2}\Big)\Big] \quad (4.63)$$

$$\therefore\ \alpha_9=e^{-\tau}\Big[\int e^s h_1(s)ds+\alpha_9(0)\Big] \quad (4.64)$$

The coefficient of $\cos\theta\,\hat{t}$ yields

$$\beta_9'+\beta_9=h_2(\tau) \quad (4.65)$$

$$h_2(\tau)=-\frac{1}{2\theta}\Big[2\omega_1'\theta\beta_6'+2\omega_2'\beta_2'\theta+2\omega_1'\theta^2\alpha_4+2\omega_2'\alpha_1\theta^2-2\alpha\Big(\frac{\alpha_1 r_2}{\theta^2}-\frac{\alpha_1 r_3}{6\theta^2}+B\alpha_6\Big)\Big] \quad (4.66)$$

$$\therefore\ \beta_9=e^{-\tau}\Big[\int e^s h_2(s)ds+\beta_9(0)\Big] \quad (6.67)$$

The remaining equation in (4.63)

$$U_{m,\hat{t}\hat{t}}^{(31)}+\theta^2U_m^{(31)}=r_{29}+r_{30}\sin2\theta\hat{t}+r_{31}\cos2\theta\hat{t}+r_{32}\cos3\theta\hat{t}+r_{33}\cos3\theta\hat{t}+r_{34}\cos\varphi\hat{t}+r_{35}\sin\varphi\hat{t}$$
$$+r_{36}\cos(\varphi+\theta)\,\hat{t}+r_{37}\sin(\varphi+\theta)\,\hat{t}+r_{38}\cos(\varphi-\theta)\,\hat{t}$$
$$+r_{39}\sin(\varphi-\theta)\,\hat{t} \quad (4.68)$$

The initial conditions are

$$U_m^{(31)}(0,0)=0;\ U_{m,\hat{t}}^{(31)}(0,0)+\omega_1'(0)U_{m,\hat{t}}^{(21)}(0,0)+\omega_2'(0)U_{m,\hat{t}}^{(11)}(0,0)+U_{m,\tau}^{(30)}(0,0)=0$$

where,

$$r_{29}=2\alpha\Big(\frac{\alpha_1\alpha_6}{2}+\frac{r_2B}{\theta^2}\Big);\ \ r_{29}(0)=0$$

$$r_{30}=\Big[\frac{-4r_3'}{3\theta}-\frac{4r_1'}{3\theta}-\frac{4r_{11}'}{3\theta}-\frac{2\omega_1'r_1}{3\theta}-\frac{4r_{11}}{3\theta}-\frac{4\omega_1'r_1}{3\theta}+2\alpha\Big(\frac{\alpha_1\beta_6}{2}-\frac{r_4B}{3\theta^2}\Big)+\alpha\Big(\frac{\alpha_1 r_5}{2(\varphi^2-\theta^2)}+\frac{Br_6}{\varphi^2-4\theta^2}\Big)$$
$$+\alpha\beta_2\alpha_4+\frac{\alpha^2\alpha_1B\beta_2}{2(\varphi^2-\theta^2)}-\frac{45}{4}\beta B\alpha_1\beta_2\Big]$$

$$r_{30}(0)=B^3\Big(\frac{-8\alpha}{3\theta B}-\frac{4S_{21}}{3\theta}+\frac{2\alpha}{3\theta^3}-\frac{4S_4}{3\theta}+\frac{4\alpha}{3\theta^3}-\frac{2\alpha^2}{3\theta^3}+\frac{\alpha^2}{2\theta(\varphi^2-\theta^2)}+\frac{\alpha^2}{2\theta B(\varphi^2-\theta^2)}-\frac{45}{4\theta}\Big)=B^3S_{25},$$

$$S_{25}=\Big(-\frac{8\alpha}{3B\theta}-\frac{4S_{21}}{2\theta}+\frac{2\alpha}{\theta^3}-\frac{4S_4}{3\theta}-\frac{2\alpha^2}{\theta^3}+\frac{\alpha^2}{2\theta(\varphi^2-\theta^2)}+\frac{S_1}{(\varphi^2-\theta^2)}+\frac{\alpha^2}{2\theta B(\varphi^2-\theta^2)}-\frac{45\beta}{4\theta}\Big)$$

$$r_{31}=\Big[\frac{8\omega_1'r_1}{3}+2\alpha\Big(\frac{\alpha_1\alpha_6}{2}-\frac{Br_3}{3\theta^2}\Big)\Big],\ r_{31}(0)=\frac{-8\alpha B^3}{3\theta^2}$$

$$r_{32}=\Big[-\frac{\alpha\alpha_1 r_3}{3\theta^2}\Big],\ r_{32}(0)=0$$

$$r_{33}=\left[\begin{matrix}\frac{-3r_{12}'}{4\theta}-\frac{3r_{12}}{4\theta}-\frac{\alpha\alpha_1 r_4}{3\theta^2}+\frac{\alpha\alpha_1 r_6}{2(\varphi^2-4\theta^2)}-\frac{\alpha\beta_2 r_1}{3\theta^2}+\frac{\alpha^2\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)}\\ -\frac{45}{16}\beta B\alpha_1\beta_2\end{matrix}\right]r_{33}(0)$$
$$=B^3\Big(-\frac{3S_{61}}{4\theta}-\frac{3S_5}{3\theta}+\frac{\alpha^2}{3\theta^3}-\frac{\alpha S_1}{2\theta(\varphi^2-\theta^2)}\Big)=B^3S_{26}$$
$$S_{26}=\Big(-\frac{3S_{16}}{3\theta}-\frac{3S_5}{3\theta}+\frac{\alpha^2}{3\theta^3}-\frac{\alpha S_1}{2(\varphi^2-4\theta^2)}\Big)$$

$$r_{34}=[\alpha B\alpha_7]\,,\ \ r_{34}(0)=0\,,$$

$$r_{35}=[\alpha B\beta_7]\,,\ \ r_{35}(0)=\frac{\alpha^2}{3\varphi^3}+\frac{\alpha^2S_0}{\varphi}+\frac{\alpha^2}{2\varphi(\varphi^2-4\theta^2)}-\frac{\alpha^2}{\varphi(\varphi^2-4\theta^2)}$$

$$r_{36}=\Big[\frac{2r_{14}'(\varphi+\theta)}{\varphi(2\theta+\varphi)}-\frac{2r_{14}(\varphi+\theta)}{\varphi(2\theta+\varphi)}+\frac{\alpha\alpha_1\alpha_7}{2}-\frac{\alpha\beta_2\beta_5}{2}\Big],\ \ r_{36}(0)=0$$

$$r_{37}=\Big[-\frac{2r_{13}'(\varphi+\theta)}{\varphi(2\theta+\varphi)}+\frac{2r_{13}(\varphi+\theta)}{\varphi(2\theta+\varphi)}+\frac{\alpha\alpha_1\beta_7}{2}+\alpha\Big(\frac{\beta_2\alpha_5}{2}\Big)\Big]$$

$$r_{37}(0)=B^3\Big(\frac{6\alpha(\varphi+\theta)S_0}{\varphi(2\theta+\varphi)}+\frac{\alpha S_{23}}{2}-\frac{S_0\alpha}{2\theta}\Big)=B^3S_{27}$$

$$S_{27}=\Big(\frac{6\alpha(\varphi+\theta)S_0}{\varphi(2\theta+\varphi)}-\frac{\alpha S_{43}}{2}-\frac{\alpha S_0}{2\theta}\Big)$$

$$r_{38} = \left[\frac{-2r'_{16}(\varphi-\theta)}{\varphi(2\theta-\varphi)} + \frac{2r_{16}(\varphi-\theta)}{\varphi(2\theta-\varphi)} + \frac{\alpha\alpha_1\alpha_7}{2} + \frac{\alpha\beta_2\beta_5}{2}\right],\ r_{38}(0) = 0$$

$$r_{39} = \left[\frac{2r'_{15}(\varphi-\theta)}{\varphi(2\theta-\varphi)} + \frac{2r_{15}(\varphi-\theta)}{\varphi(2\theta-\varphi)} + \frac{\alpha\alpha_1\beta_7}{2} - \alpha\left(\frac{\beta_2\alpha_5}{2}\right)\right]$$

$$r_{39}(0) = B^3\left(\frac{4\alpha(\varphi-\theta)S_0}{\varphi(2\theta+\varphi)} - \frac{2\alpha(\varphi-\theta)S_0}{\varphi(2\theta-\varphi)} - \frac{\alpha^2S_3}{2\varphi} + \frac{\alpha S_0}{2\theta}\right) = B^3S_{29}$$

$$S_{29} = \left(-\frac{3\alpha}{\theta^2(\varphi^2-\theta^2)} - \frac{2\alpha}{(\varphi^2-\theta^2)}\right)$$

Solving (4.69), the following is obtained

$$U_m^{(31)} = \alpha_{12}\cos\theta\hat{t} + \beta_{12}\sin\theta\hat{t} + \frac{r_{29}}{\theta^2} + \frac{r_{30}\sin 2\theta\hat{t} + r_{31}\cos 2\theta\hat{t}}{\theta^2-4\theta^2} + \frac{r_{32}\cos 3\theta\hat{t} + r_{33}\sin 3\theta\hat{t}}{\theta^2-9\theta^2} + \frac{r_{34}\cos\varphi\hat{t} + r_{35}\sin\varphi\hat{t}}{\theta^2-\varphi^2} + \frac{r_{36}\cos(\varphi+\theta)\hat{t} + r_{37}\sin(\varphi+\theta)\hat{t}}{\varphi(2\theta-\varphi)} + \frac{r_{38}\cos(\varphi-\theta)\hat{t} + r_{39}\sin(\varphi-\theta)\hat{t}}{\varphi(2\theta-\varphi)} \quad (4.69)$$

$$\alpha_{12}(0) = -\left[\frac{r_{29}}{\theta^2} + \frac{r_{31}}{\theta^2-4\theta^2} + \frac{r_{32}}{\theta^2-9\theta^2} + \frac{r_{34}}{\theta^2-\varphi^2} - \frac{r_{36}}{\varphi(2\theta-\varphi)} + \frac{r_{38}}{\varphi(2\theta-\varphi)} - \frac{2\alpha B^3}{3\theta^4} + \frac{1}{2\theta^2}\left(\frac{B^2}{\theta^4} + \frac{16\alpha B^3}{3\theta^2} - \frac{10\alpha^2B^3}{\theta^2} + \frac{3\alpha^2B^3}{2\varphi^2} + \frac{\alpha^2B^3}{4(\varphi^2-4\theta^2)} + \frac{225\beta B^3}{16}\right) + \left(\alpha'_9 + \frac{r'_{10}}{\theta^2} - \frac{r'_{11}}{3\theta^2} - \frac{r'_{12}}{8\theta^2} - \frac{r'_{13}}{\varphi(2\theta+\varphi)} + \frac{r'_{15}}{\varphi(2\theta-\varphi)}\right)\right]\Bigg|\ \tau = 0 \quad (4.70a)$$

$$\beta_{12}(0) = \frac{-1}{\theta}\left[\frac{2\theta r_{30}}{\theta^2-4\theta^2} + \frac{3\theta r_{33}}{\theta^2-9\theta^2} + \frac{\varphi r_{35}}{\theta^2-\varphi^2} + \frac{(\varphi+\theta)r_{37}}{\varphi(2\theta-\varphi)} + \frac{(\varphi-\theta)r_{39}}{\varphi(2\theta-\varphi)}\right] at\ \tau = 0 \quad (4.70b)$$

Substituting in (4.59) gives

$$U_{2m,\hat{t}\hat{t}}^{(31)} + \varphi^2U_{2m}^{(31)} = -\Bigg[2\omega'_1\left\{-\varphi\alpha'_7\sin\varphi\hat{t} + \varphi\beta'_7\cos\varphi\hat{t} + \frac{\theta r'_5\cos\theta\hat{t}}{\varphi^2-\theta^2} + \frac{2\theta r'_6\cos 2\theta\hat{t}}{\varphi^2-4\theta^2}\right\} + 2\Big\{-\varphi\alpha'_{10}\sin\varphi\hat{t} + \varphi\beta'_{10}\cos\varphi\hat{t} - \frac{\theta r'_{18}\sin\theta\hat{t}}{\varphi^2-\theta^2} - \frac{2\theta r'_{19}\sin 2\theta\hat{t}}{\varphi^2-4\theta^2} - \frac{3\theta r'_{20}\sin 3\theta\hat{t}}{\varphi^2-9\theta^2}\Big\} - \alpha\omega'_1\left\{\frac{2B\alpha_1\theta^2\cos\theta\hat{t}}{\varphi^2-\theta^2} - \frac{2\theta\alpha_1^2\cos 2\theta\hat{t}}{\varphi^2-4\theta^2}\right\} + \alpha(\omega''_1 + 2\omega'_1)\left\{\frac{B\alpha_1\theta\sin\theta\hat{t}}{\varphi^2-\theta^2} - \frac{\theta\alpha_1^2\sin 2\theta\hat{t}}{\varphi^2-4\theta^2}\right\} + 2\left\{-\varphi\alpha_{10}\sin\varphi\hat{t} + \varphi\beta_{10}\cos\varphi\hat{t} - \frac{\theta r_{18}\sin\theta\hat{t}}{\varphi^2-\theta^2} - \frac{2\theta r_{19}\sin 2\theta\hat{t}}{\varphi^2-4\theta^2} - \frac{3\theta r_{20}\sin 3\theta\hat{t}}{\varphi^2-9\theta^2}\right\} + \frac{\alpha}{2}\Big\{\left(\frac{\alpha_1\alpha_6}{2} + \frac{r_2B}{\theta^2}\right) + \left(\frac{\alpha_1 r_2}{\theta^2} - \frac{\alpha_1 r_3}{6\theta^2} + B\alpha_6\right)\cos\theta\hat{t} + \left(B\beta_6 - \frac{\alpha_1 r_4}{6\theta^2}\right)\sin\theta\hat{t} + \left(\frac{\alpha_1\beta_6}{2} - \frac{r_4B}{3\theta^2}\right)\sin 2\theta\hat{t} - \frac{\alpha_1 r_3}{6\theta^2}\cos 3\theta\hat{t} - \frac{\alpha_1 r_4}{6\theta^2}\sin 3\theta\hat{t}\Big\} - \alpha\Big\{\left(\frac{\alpha_1 r_6}{2(\varphi^2-4\theta^2)} + \frac{Br_5}{\varphi^2-\theta^2}\right)\sin\theta\hat{t} + \frac{\alpha_1\alpha_7}{2}\cos(\varphi+\theta)\hat{t} + \frac{\alpha_1\beta_7}{2}\sin(\varphi+\theta)\hat{t} + \frac{\alpha_1\alpha_7}{2}\cos(\varphi-\theta)\hat{t} + \frac{\alpha_1\beta_7}{2}\sin(\varphi-\theta)\hat{t} + \left(\frac{\alpha_1 r_5}{2(\varphi^2-\theta^2)} + \frac{Br_6}{\varphi^2-4\theta^2}\right)\sin 2\theta\hat{t} + \frac{\alpha_1 r_6}{2(\varphi^2-4\theta^2)}\sin 3\theta\hat{t} + B\alpha_7\cos\varphi\hat{t} + B\beta_7\sin\varphi\hat{t}\Big\} + \frac{\alpha}{2}\left\{\frac{\beta_2\alpha_4\sin 2\theta\hat{t}}{2} + \frac{\beta_2 r_0\sin\theta\hat{t}}{\theta^2} - \frac{\beta_2 r_1}{6\theta^2}(\sin 3\theta\hat{t} - \sin\theta\hat{t})\right\} - \alpha\left\{\frac{\alpha\beta_2}{2}\left(\frac{\frac{\alpha_1^2}{2}+B^2}{\varphi^2}\right) - \frac{\alpha\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)}\sin\theta\hat{t} + \frac{\alpha\alpha_1B\beta_2}{2(\varphi^2-\theta^2)}\sin 2\theta\hat{t} + \frac{\alpha\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)}\sin 3\theta\hat{t}\right\} + \frac{9\beta}{4}\left[\left\{\beta_2\left(B^2 + \frac{\alpha_1^2}{2}\right) - \frac{\beta_2\alpha_1^2}{4}\right\}\sin\varphi\hat{t} + \beta_2B\alpha_1\sin 2\theta\hat{t} + \frac{\beta_2\alpha_1^2}{4}\sin 3\theta\hat{t}\right]\Bigg] \quad (4.71)$$

To ensure uniformly valid solution in $\hat{t}$needs equating the coefficients of $\cos\varphi\hat{t}$ and $\sin\varphi\hat{t}$ to zero.
Equating the coefficient of $\cos\varphi\hat{t}$ yields

$$-2\omega'_1\varphi\beta'_7 - 2\varphi\beta'_{10} - 2\beta_{10}\varphi + B\alpha_1\alpha_7 = 0$$

$$\therefore \beta'_{10} + \beta_{10} = \frac{1}{2\varphi}[-2\omega'_1\varphi\beta'_7 + B\alpha_1\alpha_7] \quad (4.72a)$$

$$\therefore \beta'_{10} + \beta_{10} = h_3(\tau) \quad (4.72b)$$

where,

$$h_3(\tau) = \frac{1}{2\varphi}[-2\omega'_1\varphi\beta'_7 + B\alpha_1\alpha_7] \quad (4.72c)$$

It therefore follows that,

$$\beta_{10} = e^{-\tau}\left[\int h_3(s)e^s ds + \beta_{10}(0)\right] \quad (4.72d)$$

The coefficient of $\sin\varphi\hat{t}$ leads to

$$2\omega'_1\varphi\alpha'_7 + 2\varphi\alpha'_{10} + 2\alpha_{10}\varphi + B\beta_7\alpha = 0$$

$$\alpha'_{10} + \alpha_{10} = h_4(\tau) \quad (4.72e)$$

$$h_4(\tau) = -\frac{1}{2\varphi}[2\omega'_1\varphi\alpha'_7 + B\beta_7\alpha] \quad (4.72f)$$

$$\therefore \alpha_{10} = e^{-\tau}\left[\int h_4(s)e^s ds + \alpha_{10}(0)\right] \quad (4.72g)$$

The remaining equation in (4.71) is

$U^{(31)}_{2m,\hat{t}\hat{t}} + \varphi^2 U^{(31)}_{2m} = r_{40}\cos\theta\hat{t} + r_{41}\sin\theta\hat{t} + r_{42}\cos2\theta\hat{t} + r_{43}\sin2\theta\hat{t} + r_{44}\cos3\theta\hat{t} + r_{45}\sin3\theta\hat{t} + r_{46}\cos(\varphi+\theta)\hat{t} + r_{47}\sin(\varphi+\theta)\hat{t} + r_{48}\cos(\varphi-\theta)\hat{t} + r_{49}\sin(\varphi-\theta)\hat{t}$

(4.73)

The initial conditions are

$$U^{(31)}_{2m}(0,0) = 0;\; U^{(31)}_{2m,\hat{t}}(0,0) + \omega_1'(0)U^{(21)}_{2m,\hat{t}}(0,0) + U^{(30)}_{2m,\tau}(0,0) = 0$$

where,

$$r_{40} = \frac{-2\theta r_5^1\omega_1'}{\varphi^2-\theta^2} + \frac{2\alpha\omega_1' B\alpha_1\theta^2}{\varphi^2-\theta^2} - \frac{\alpha}{2}\left(\left(\frac{\alpha_1\alpha_6}{2} + \frac{r_2 B}{\theta^2}\right) + \left(\frac{\alpha_1 r_2}{\theta^2} - \frac{\alpha_1 r_3}{6\theta^2}\right) + B\alpha_6\right)$$

$$r_{40}(0) = B^3\left(\frac{-3\alpha}{\theta^2(\varphi^2-\theta^2)} - \frac{2\alpha}{(\varphi^2-\theta^2)}\right)$$

$$r_{41} = \left[\frac{2\theta r_{18}^1}{\varphi^2-\theta^2} + \frac{(\omega_1''+2\omega_1')\alpha B\alpha_1\theta}{\varphi^2-\theta^2} + \frac{2\theta r_{18}}{\varphi^2-\theta^2} - \frac{\alpha}{2}\left(B\beta_6 - \frac{\alpha_1 r_4}{6\theta^2}\right) + \alpha\left(\frac{\alpha_1 r_6}{2(\varphi^2-4\theta^2)} + \frac{B r_5}{\varphi^2-\theta^2}\right) - \frac{\alpha\beta_2 r_0}{2\theta^2} - \frac{\alpha\beta_2 r_1}{12\theta^2} + \left\{\frac{\alpha^2\beta_2}{2\varphi^2}\left(\frac{\alpha_1^2}{2} + B^2\right) - \frac{\alpha^2\alpha_1 B\beta_2}{8(\varphi^2-4\theta^2)} + \frac{9\beta}{4}\left(\beta_2\left(B^2 + \frac{\alpha_1^2}{2}\right)\right)\right\}\right]$$

$$r_{41}(0) = B^3\left(\frac{2\theta S_{24}}{(\varphi^2-\theta^2)} - \frac{2\alpha}{\theta(\varphi^2-\theta^2)} + \frac{2\theta S_7}{\varphi^2-\theta^2} - \frac{2\alpha^2}{9\theta^4} - \frac{\alpha S_1}{2(\varphi^2-4\theta^2)} + \frac{\alpha^2}{\theta(\varphi^2-\theta^2)} - \frac{17\alpha^2}{12\theta^3} - \frac{3\alpha^2}{(4\theta-\varphi^2)} - \frac{\alpha^2}{8\theta(\varphi^2-4\theta^2)} - \frac{45\beta}{16\theta}\right)$$

$r_{42} = \left[\frac{4\theta\omega_1' r_6}{(\varphi^2-4\theta^2)} - \frac{2\theta\alpha\omega_1'\alpha_1^2}{(\varphi^2-4\theta^2)}\right]$, $r_{42}(0) = B^3\left(\frac{2\alpha}{\theta(\varphi^2-4\theta^2)} - \frac{4S_{14}}{\theta(\varphi^2-4\theta^2)}\right)$

$$r_{43} = \left[\frac{4\theta\omega_1' r_{19}}{(\varphi^2-4\theta^2)} + \frac{(\omega_1''+2\omega_1')\alpha\theta\alpha_1^2}{\varphi^2-4\theta^2} + \frac{4\theta r_{19}}{(\varphi^2-4\theta^2)} - \frac{\alpha}{2}\left(\frac{\alpha_1\beta_6}{2} - \frac{B r_4}{3\theta^2}\right) + \alpha\left(\frac{\alpha_1 r_5}{2(\varphi^2-\theta^2)} + \frac{B r_6}{\varphi^2-4\theta^2}\right) + \frac{\alpha^2\alpha_1 B\beta_2}{2(\varphi^2-\theta^2)} + \frac{9}{4}\beta\beta_2 B\alpha_1\right]$$

$$r_{43}(0) = B^3\left(\frac{4\theta S_{10}}{(\varphi^2-4\theta^2)} + \frac{2\alpha}{\theta(\varphi^2-\theta^2)} + \frac{4\theta S_8}{\varphi^2-4\theta^2} + \frac{\alpha^2}{6\theta^3} + \frac{2\alpha^2}{2\theta(\varphi^2-\theta^2)} + \frac{\alpha S_1}{(\varphi^2-\theta^2)} + \frac{9\beta}{4\theta}\right)$$

$r_{44} = \left[\frac{\alpha}{2}\left(\frac{\alpha_2 r_3}{6\theta^2}\right)\right]$, $r_{44}(0) = 0$

$$r_{45} = \left[\frac{6\theta r_{20}'}{(\varphi^2-9\theta^2)} + \frac{6\theta r_{20}}{(\varphi^2-9\theta^2)} + \frac{\alpha}{2}\left(\frac{\alpha_1 r_4}{6\theta^2}\right) + \frac{\alpha\alpha_1 r_6}{2(\varphi^2-4\theta^2)} + \frac{\alpha\beta_2 r_1}{12} + \frac{\alpha^2\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)} + \frac{9}{16}\beta\beta_2\alpha_1^2 B\right]$$

$$r_{45}(0) = B^3\left(\frac{6\theta S_{34}}{(\varphi^2-4\theta^2)} + \frac{6\theta S_9}{(\varphi^2-\theta^2)} - \frac{\alpha^2}{12\theta^3} - \frac{\alpha S_1}{2(\varphi^2-\theta^2)} - \frac{\alpha^2}{12\theta} - \frac{9\beta}{16\theta}\right)$$

$r_{46} = \left[\frac{\alpha_1\alpha_7}{2}\right]$, $r_{46}(0) = 0$, $r_{47} = \left[\frac{\alpha\alpha_1\beta_7}{2}\right]$, $r_{47}(0) = -\alpha B^3 S_{43}$

$r_{48} = \left[\frac{\alpha\alpha_1\alpha_7}{2}\right]$, $r_{48}(0) = 0$, $r_{49} = \left[\frac{\alpha\alpha_1\beta_7}{2}\right]$, $r_{49}(0) = -\alpha B^3 S_{43}$

$$S_{43} = \frac{\alpha S_0}{\varphi} + \frac{\alpha}{2\varphi^3} + \frac{\alpha}{2\alpha(\varphi^2-4\theta^2)} - \frac{\alpha}{\alpha(\varphi^2-\theta^2)} - \frac{2\theta\alpha S_1}{\varphi(\varphi^2-4\theta^2)}$$

$$\therefore U^{(31)}_{2m} = \alpha_{13}\cos\varphi\hat{t} + \beta_{13}\sin\varphi\hat{t} + \frac{r_{42}\cos\theta\hat{t} + r_{43}\sin\theta\hat{t}}{\varphi^2-\theta^2} + \frac{r_{44}\cos2\theta\hat{t} + r_{45}\sin2\theta\hat{t}}{\varphi^2-4\theta^2} + \frac{r_{46}\cos3\theta\hat{t} + r_{47}\sin3\theta\hat{t}}{\varphi^2-9\theta^2} - \frac{r_{48}\cos(\varphi+\theta)\hat{t} + r_{49}\sin(\varphi+\theta)\hat{t}}{\theta(2\varphi+\theta)} + \frac{r_{50}\cos(\varphi-\theta)\hat{t} + r_{39}\sin(\varphi-\theta)\hat{t}}{\theta(2\varphi-\theta)} \quad (4.74)$$

where, from the first initial condition

$$\alpha_{13}(0) = -\left[\frac{r_{42}}{\varphi^2-\theta^2} + \frac{r_{44}}{\varphi^2-4\theta^2} + \frac{r_{46}}{\varphi^2-9\theta^2} - \frac{r_{48}}{\theta(2\varphi+\theta)} + \frac{r_{50}}{\theta(2\varphi-\theta)}\right] \; at\; \tau = 0 (4.75)$$

and from the second initial condition, it follows that

$$\left[\beta_{13}(0)\varphi + \frac{\theta\, r_{43}}{\varphi^2-\theta^2} + \frac{2\theta r_{45}}{\varphi^2-4\theta^2} + \frac{3\theta r_{47}}{\varphi^2-9\theta^2} - \frac{(\theta+\varphi)r_{49}}{\theta(2\varphi+\theta)} + \frac{(\theta-\varphi)r_{51}}{\theta(2\varphi-\theta)} + \alpha_{10}'(0) + \frac{r_{17}'}{\varphi^2} + \frac{r_{18}'}{\varphi^2-\theta^2} + \frac{r_{19}'}{\varphi^2-4\theta^2} + \frac{r_{20}'}{\varphi^2-9\theta^2}\right] = 0$$

$$\therefore \beta_{13}(0) = -\frac{1}{\varphi}\left[\frac{\theta r_{43}}{\varphi^2-\theta^2} + \frac{2\theta r_{45}}{\varphi^2-4\theta^2} + \frac{3\theta r_{47}}{\varphi^2-9\theta^2} - \frac{(\theta+\varphi)r_{49}}{\theta(2\varphi+\theta)} + \frac{(\theta-\varphi)r_{51}}{\theta(2\varphi-\theta)} + \alpha'_{10}(0) + \frac{r'_{17}}{\varphi^2} + \frac{r'_{18}}{\varphi^2-\theta^2} + \frac{r'_{19}}{\varphi^2-4\theta^2} + \frac{r'_{20}}{\varphi^2-9\theta^2}\right] \quad (4.75b)$$

Substituting in (4.61)

$$U^{(31)}_{3m,\hat{t}\hat{t}} + \Omega^2 U^{(31)}_{3m} = -\left[2U^{(30)}_{3m,\hat{t}\tau} + 2U^{(30)}_{3m,\hat{t}} + \alpha\left\{\frac{1}{2}U^{(10)}_{m}U^{(21)}_{2m} + \frac{1}{2}U^{(11)}_{m}U^{(20)}_{2m}\right\} + \frac{3}{4}\beta\left(U^{(10)}_{m}\right)^2 U^{(11)}_{m}\right] \quad (4.75c)$$

Further simplification of (4.75c)yields

$$U^{(31)}_{3m,\hat{t}\hat{t}} + \Omega^2 U^{(31)}_{3m} = -\Bigg[2\Big\{-\Omega\alpha'_{11}\sin\Omega\hat{t} + \Omega\beta'_{11}\cos\Omega\hat{t} - \frac{\theta r'_{22}\sin\theta\hat{t}}{\Omega^2-\theta^2} - \frac{2\theta r'_{23}\sin 2\theta\hat{t}}{\Omega^2-4\theta^2} - \frac{3\theta r'_{24}\sin 3\theta\hat{t}}{\Omega^2-9\theta^2} - \frac{(\varphi+\theta)r'_{25}\sin(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2} + \frac{(\varphi+\theta)r'_{26}\cos(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2} - \frac{(\varphi-\theta)r'_{27}\sin(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2} + \frac{(\varphi-\theta)r'_{28}\cos(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2}\Big\} + 2\Big\{-\Omega\alpha_{11}\sin\Omega\hat{t} + \Omega\beta_{11}\cos\Omega\hat{t} - \frac{\theta r_{22}\sin\theta\hat{t}}{\Omega^2-\theta^2} - \frac{2\theta r_{23}\sin 2\theta\hat{t}}{\Omega^2-4\theta^2} - \frac{3\theta r_{24}\sin 3\theta\hat{t}}{\Omega^2-9\theta^2} - \frac{(\varphi+\theta)r_{25}\sin(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2} + \frac{(\varphi+\theta)r_{26}\cos(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2} - \frac{(\varphi-\theta)r_{27}\sin(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2} + \frac{(\varphi-\theta)r_{28}\cos(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2}\Big\} + \frac{\alpha}{2}\Big\{\left(\frac{\alpha_1 r_6}{2(\varphi^2-4\theta^2)} + \frac{Br_5}{\varphi^2-\theta^2}\right)\sin\theta\hat{t} + \frac{\alpha_1\alpha_7}{2}\cos(\varphi+\theta)\hat{t} + \frac{\alpha_1\beta_7}{2}\sin(\varphi+\theta)\hat{t} + \frac{\alpha_1\alpha_7}{2}\cos(\varphi-\theta)\hat{t} + \frac{\alpha_1\beta_7}{2}\sin(\varphi-\theta)\hat{t} + \left(\frac{\alpha_1 r_5}{2(\varphi^2-\theta^2)} + \frac{Br_6}{\varphi^2-4\theta^2}\right)\sin 2\theta\hat{t} + \frac{\alpha_1 r_6}{2(\varphi^2-4\theta^2)}\sin 3\theta\hat{t} + B\alpha_7\sin\varphi\hat{t} + B\beta_7\sin\varphi\hat{t}\Big\} + \frac{\alpha}{2}\left\{\frac{\beta_2\alpha}{2}\left(\frac{B^2+\frac{\alpha_1^2}{2}}{\varphi^2}\right) - \frac{\alpha\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)}\sin\theta\hat{t} + \frac{\alpha\alpha_1 B\beta_2}{2(\varphi^2-\theta^2)}\sin 2\theta\hat{t} + \frac{\alpha\alpha_1^2\beta_2}{8(\varphi^2-4\theta^2)}\sin 3\theta\hat{t} - \frac{\beta_2\beta_5}{2}\cos(\varphi+\theta)\hat{t}\right\} + \frac{3\beta}{4}\left\{\left(\beta_2\left(B^2+\frac{\alpha_1^2}{2}\right) - \frac{\beta_2\alpha_1^2}{4}\right)\sin\theta\hat{t} + \beta_2 B\alpha_1\sin 2\theta\hat{t} + \frac{\beta_2\alpha_1^2}{4}\sin 3\theta\hat{t}\right\}\Bigg] \quad (4.76)$$

To ensure uniformly valid solution in $\hat{t}$, needs equating the coefficients of $\cos\Omega\hat{t}$ and $\sin\Omega\hat{t}$ to zero. The coefficients of $\cos\Omega\hat{t}$ yields

$$-2\Omega\beta'_{11} - 2\Omega\beta_{11} - \frac{\alpha B\beta_7}{2} = 0 \quad (4.77a)$$

$$\therefore \beta'_{11} + \beta_{11} = -\frac{\alpha B\alpha_7}{2\Omega} = h_5(\tau) \quad (4.77b)$$

where

$$h_5(\tau) = -\frac{\alpha B\alpha_7}{2\Omega} \quad (4.77c)$$

$$\therefore \beta_{11} = e^{-\tau}\left[\int h_5(\tau)e^{s}ds + \beta_{11}(0)\right] \quad (4.77d)$$

The coefficients of $\sin\Omega\hat{t}$ yields

$$-2\Omega\alpha'_{11} - 2\Omega\alpha_{11} - \frac{\alpha B\beta_7}{2} = 0 \quad (4.77e)$$

where

$$h_6(\tau) = \frac{\alpha B\beta_7}{4\Omega} \quad (4.77f)$$

$$\therefore \alpha_{11} = e^{-\tau}\left[\int h_6(\tau)e^{s}ds + \alpha_{11}(0)\right] \quad (4.77g)$$

The remaining equation (4.76) is:

$$U^{(31)}_{3m,\hat{t}\hat{t}} + \Omega^2 U^{(31)}_{2m} = r_{50}\sin\theta\hat{t} + r_{51}\sin 2\theta\hat{t} + r_{52}\sin 3\theta\hat{t} + r_{53}\cos(\varphi+\theta)\hat{t} + r_{54}\sin(\varphi+\theta)\hat{t} + r_{55}\cos(\varphi-\theta)\hat{t} + r_{56}\sin(\varphi-\theta)\hat{t} \quad (4.78)$$

The initial conditions are

$$U^{(31)}_{3m}(0,0) = 0;\ U^{(31)}_{3m,\hat{t}}(0,0) + U^{(30)}_{3m,\tau}(0,0) = 0$$

$$r_{50} = \frac{2\theta r^1_{22}}{\Omega^2-\theta^2} + \frac{2\theta r_{22}}{\Omega^2-\theta^2} - \frac{\alpha\alpha_1 r_6}{4(\varphi^2-4\theta^2)} - \frac{B\alpha r_5}{2(\varphi^2-4\theta^2)} - \frac{\alpha^2\beta_2}{4}\left(\frac{\frac{\alpha_1^2}{2}+B^2}{\varphi^2}\right) + \frac{\alpha^2\alpha_1\beta_2}{16(\varphi^2-4\theta^2)} - \frac{3\alpha\beta\beta_2}{8}\left(B^2+\frac{\alpha_1^2}{2}\right) + \frac{3\alpha\beta\alpha_1^2}{32}$$

$$r_{50}(0) = B^3\left(\frac{2\theta S_{17}}{(\Omega^2-\theta^2)} + \frac{2\theta S_{11}}{(\Omega^2-\theta^2)} + \frac{\alpha S_1}{4(\varphi^2-4\theta^2)} + \frac{\alpha^2}{2\theta(\varphi^2-4\theta^2)} + \frac{3\alpha^2}{8\theta\varphi^2} + \frac{\alpha^2}{16B(\varphi^2-4\theta^2)} + \frac{9\alpha\beta}{16\theta} + \frac{3\alpha\beta}{32B}\right)$$

$$r_{51} = \frac{4\theta r^1_{23}}{\Omega^2-4\theta^2} + \frac{4\theta r_{23}}{\Omega^2-4\theta^2} - \frac{\alpha\alpha_1 r_5}{4(\varphi^2-\theta^2)} - \frac{B\alpha r_6}{2(\varphi^2-4\theta^2)} - \frac{\alpha^2\alpha_1 B\beta_2}{4(\varphi^2-4\theta^2)} - \frac{\alpha\alpha_1\beta_2 B}{2}$$

$$r_{51}(0) = B^3\left(\frac{4\theta S_{18}}{(\Omega^2 - 4\theta^2)} + \frac{4\theta S_{12}}{(\Omega^2 - 4\theta^2)} + \frac{\alpha^2}{2\theta(\varphi^2 - \theta^2)} - \frac{\alpha S_1}{2(\varphi^2 - 4\theta^2)} - \frac{\alpha}{2\theta}\right)$$

$$r_{52} = \frac{6\theta r_{24}^1}{\Omega^2 - 9\theta^2} + \frac{6\theta r_{24}}{\varphi^2 - 9\theta^2} - \frac{\alpha\alpha_1 r_6}{4(\varphi^2 - 4\theta^2)} - \frac{\alpha^2\beta_2\alpha_1^2}{16(\varphi^2 - 4\theta^2)} + \frac{\alpha\alpha_1^2\beta_2}{8}$$

$$r_{52}(0) = B^3\left(\frac{6\theta S_{19}}{(\Omega^2 - 9\theta^2)} + \frac{6\theta S_{13}}{(\Omega^2 - 9\theta^2)} + \frac{\alpha S_1}{4(\varphi^2 - 4\theta^2)} - \frac{\alpha^2}{16\theta(\varphi^2 - 4\theta^2)} + \frac{\alpha}{8\theta}\right)$$

$r_{53} = -\frac{2r'_{26}(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} - \frac{2r_{26}(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} - \frac{\alpha\alpha_1\alpha_7}{4}$, $r_{53}(0) = 0$

$r_{54} = \frac{2r'_{25}(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} + \frac{2r_{25}(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} - \frac{\alpha\alpha_1\beta_7}{4}$, $r_{54}(0) = B^3\left(\frac{6\alpha S_0\,(\varphi+\theta)}{\Omega^2-(\varphi+\theta)^2} + \frac{\alpha S_{43}}{4}\right)$

$r_{55} = \frac{-2r'_{28}(\varphi-\theta)}{\Omega^2-(\varphi-\theta)^2} - \frac{2r_{28}(\varphi-\theta)}{\Omega^2-(\varphi-\theta)^2} - \frac{\alpha\alpha_1\alpha_7}{4}$, $r_{55}(0) = \frac{-4\alpha S_0 B^3}{\Omega^2-(\varphi-\theta)^2}$

$r_{56} = \frac{2r'_{27}(\varphi-\theta)}{\Omega^2-(\varphi-\theta)^2} + \frac{2r_{27}(\varphi-\theta)}{\Omega^2-(\varphi-\theta)^2} + \frac{\alpha\alpha_1\beta_7}{4}$, $r_{56}(0) = B^3\left(\frac{6\alpha S_0\,(\varphi-\theta)}{\Omega^2-(\varphi-\theta)^2} + \frac{\alpha S_{43}}{4}\right)$

Therefore;

$$U_{3m}^{(31)} = \alpha_{14}\cos\Omega\hat{t} + \beta_{14}\sin\Omega\hat{t} + \frac{r_{50}\sin\theta\hat{t}}{\Omega^2-\theta^2} + \frac{r_{51}\sin2\theta\hat{t}}{\Omega^2-4\theta^2} + \frac{r_{52}\sin3\theta\hat{t}}{\Omega^2-9\theta^2} + \left(\frac{r_{53}\cos(\varphi+\theta)\hat{t}+r_{54}\sin(\varphi+\theta)\hat{t}}{\Omega^2-(\varphi+\theta)^2}\right) + \left(\frac{r_{55}\cos(\varphi-\theta)\hat{t}+r_{56}\sin(\varphi-\theta)\hat{t}}{\Omega^2-(\varphi-\theta)^2}\right) \quad (4.79)$$

Therefore,

$$\alpha_{14}(0) = -\left[\frac{r_{53}}{\Omega^2-(\varphi+\theta)^2} + \frac{r_{55}}{\Omega^2-(\varphi-\theta)^2}\right]\Big|\tau = 0 \quad (4.80a)$$

$$\Omega\beta_{14}(0) = -\frac{\theta r_{50}}{\Omega^2-\theta^2} - \frac{2\theta r_{51}}{\Omega^2-4\theta^2} - \frac{3\theta r_{52}}{\Omega^2-9\theta^2} - \frac{(\varphi+\theta)r_{54}}{\Omega^2-(\varphi+\theta)^2} - \frac{(\varphi-\theta)r_{56}}{\Omega^2-(\varphi+\theta)^2} - \alpha'_{11} - \frac{r'_{22}}{\Omega^2-\theta^2} - \frac{r'_{23}}{\Omega^2-4\theta^2} - \frac{r'_{24}}{\Omega^2-9\theta^2} - \frac{r'_{25}}{\Omega^2-(\varphi+\theta)^2} - \frac{r'_{27}}{\Omega^2-(\varphi-\theta)^2}$$

Therefore;

$$\beta_{14}(0) = \frac{-1}{\Omega}\left[\frac{(\theta r_{50}+r'_{22})}{\varphi^2-\theta^2} + \frac{(2\theta r_{51}+r'_{23})}{\varphi^2-4\theta^2} + \frac{(3\theta r_{52}+r'_{24})}{\varphi^2-9\theta^2} - \frac{((\theta+\varphi)r_{54}+r'_{24})}{\Omega^2-(\varphi+\theta)^2} + \frac{(\theta-\varphi)r_{56}+r'_{27}}{\Omega^2-(\varphi-\theta)^2} + \alpha'_{11}(0)\right] \quad (4.80b)$$

So far, it follows that

$$U^{(31)} = U_m^{(31)}(1 - cos2mx) + U_{2m}^{(31)}(1 - cos4mx) + U_{3m}^{(31)}(1 - cos6mx) \quad (4.81)$$

The summary of the solution so far is,

$$U(x,t,\tau) = \left(U^{(10)} + \delta U^{(11)} + \delta^2 U^{(12)} + \cdots\right) + \epsilon^2\left(U^{(20)} + \delta U^{(21)} + \delta^2 U^{(22)} + \cdots\right) + \epsilon^3\left(U^{(30)} + \delta U^{(31)} + \delta^2 U^{(32)} + \cdots\right) + \cdots \quad (4.82)$$

4.2 Maximum Displacement of the Column

The dynamic buckling load is obtained from the maximization $\frac{d\lambda}{dU_a} = 0$, where $U_a$is the maximum displacement and λ is the load parameter. The conditions for maximum displacement are,

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial w}{\partial t} = 0 \quad (4.83a)$$

But from (3.12), it follows that

$$\frac{\partial w}{\partial t} = U_{,\hat{t}} + (\omega'_1\epsilon + \omega'_2\epsilon^2 + \cdots)U_{,\hat{t}} + \delta U_{,\tau} = 0 \quad (4.83b)$$

The aim is to determine the maximum displacement;

$$U_a = U(x_a\,, \hat{t}_a\,, t_a)$$

where $x_a\,, t_a\,, \tau_a\ and\ \hat{t}_a$ are the values of $x, t, \tau, and\ \hat{t}$ respectively at maximum displacement and are to be next determined before finally determining the maximum displacement.

From the first condition of maximization, $\frac{\partial U}{\partial x} = 0$, this means

$$\epsilon\left[\frac{\partial U^{(10)}}{\partial x} + \delta\frac{\partial U^{(11)}}{\partial x} + \cdots\right] + \epsilon^2\left[\frac{\partial U^{(20)}}{\partial x} + \delta\frac{\partial U^{(20)}}{\partial x} + \cdots\right] + \epsilon^3\left[\frac{\partial U^{(30)}}{\partial x} + \delta\frac{\partial U^{(30)}}{\partial x} + \cdots\right] = 0 \quad (4.84)$$

i.e,

$$
\begin{aligned}
2m\epsilon\big[U_m^{(10)}sin2mx + \delta U_m^{(11)}sin2mx + \cdots\big]&\\
+ \epsilon^2 \Big[2m\, U_m^{(20)} sin2mx + 4m\, U_{2m}^{(20)} sin4mx&\\
+ \cdots \delta\Big\{2m\, U_m^{(21)} sin2mx + 4m\, U_{2m}^{(21)} sin4mx + \cdots\Big\}\Big]&\\
+ \epsilon^3 \Big[2m\, U_m^{(30)} sin2mx + 4m\, U_{2m}^{(30)} sin4mx + 6m\, U_{3m}^{(30)} sin6mx + \ldots\ldots&\\
+ \delta\Big\{2m\, U_m^{(31)} sin2mx + 4m\, U_{2m}^{(31)} sin4mx + 6m\, U_{3m}^{(31)} sin6mx + \cdots\Big\} + \cdots\Big]&\\
= 0 \qquad (4.85)&
\end{aligned}
$$

The equation (4.85) is satisfied if $sin2mx_a = 0$, 5where x_a is the value of x at maximum displacement. This means, $2mx_a = \pi n,\ n = 0,1,2,3 \ldots.,\ set\ n = 1,\ \ x_a = \frac{\pi}{2m}$

Substituting, $x_a = \frac{\pi}{2m}\ in\ U(x, \hat{t}, \tau)$, gives

$$U(x_a, \hat{t}\,, \tau) = 2\epsilon\big[U_m^{(10)} + \delta U_m^{(11)} + \cdots\big] + 2\epsilon^2\big[\, U_m^{(20)} + \delta\, U_{2m}^{(21)} + \cdots\big] + 2\epsilon^3\big[\big(\, U_m^{(30)} +\, U_{3m}^{(30)}\big) + \delta\big(\, U_m^{(31)} +\, U_{3m}^{(31)}\big) + \cdots\big] \qquad (4.86)$$

Let $\hat{t}_a$, t_a and τ_a be the values of $\hat{t}$, t and τ respectively at maximum displacement and let them be expanded asymptotically as

$$\hat{t}_a = \hat{t}_0 + \delta\hat{t}_{01} + \delta^2\hat{t}_{02} + \epsilon(\hat{t}_{10} + \delta\hat{t}_{11} + \delta^2\hat{t}_{12} + \cdots) + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \cdots) \qquad (4.87a)$$

$$
\begin{aligned}
t_a = t_0 + \delta t_{01} + \delta^2 t_{02} + \cdots \ldots + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \cdots)&\\
+ \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \cdots) \qquad (4.87b)&
\end{aligned}
$$

$$\tau_a = \delta[t_0 + \delta t_{01} + \delta^2 t_{02} + \cdots \ldots + \epsilon(t_{10} + \delta t_{11} + \delta^2 t_{12} + \cdots) + \epsilon^2(t_{20} + \delta t_{21} + \delta^2 t_{22} + \cdots)] \qquad (4.87c)$$

Evaluating (4.87c) at the máximum values and simplifying, the following are obtained:

$\hat{t}_0 \ = \frac{\pi}{\theta},\quad t_0 = \frac{\pi}{\theta},\quad t_{10} = -\frac{t_0 B}{\theta^2},\qquad t_{20} = \hat{t}_{20} - \hat{t}_{10}\,\omega_1'(0) - t_0\omega_2'(0)$ and

$$\hat{t}_{20} \ = \frac{B^2\alpha S_0 \sin\varphi \hat{t}_0}{\theta^2}\left[\frac{(\varphi-\theta)}{\Omega^2 - (\varphi-\theta)^2} - \frac{(\varphi+\theta)}{\Omega^2 - (\varphi+\theta)^2} - \frac{(\varphi+\theta)}{\varphi(2\theta+\varphi)} + \frac{(\varphi-\theta)}{\varphi(2\theta-\varphi)}\right]$$

Let U_a be the maximum displacement. We now substitute for x_a;

$$
\begin{aligned}
U\left(\frac{\pi}{2m}, \hat{t}, \tau\right) = \epsilon\big[2U_m^{10} + 2\delta U_m^{(11)} \ldots\big] + \epsilon^2\big[2U_m^{20} + 2\delta U_{2m}^{(21)} \ldots\big]&\\
+ \epsilon^3\big[\big(2U_m^{30} + 2U_{3m}^{(30)} \ldots\big) + \delta\big(2U_m^{31} + 2\delta U_{3m}^{(31)} \ldots\big)\big] \qquad (4.88)&
\end{aligned}
$$

Expanding each of the terms in (4.88) and evaluating (4.88) at maximum values and noting that all $U_m^{(ij)}$ are evaluated at $(\hat{t}_0, 0)$, the following are obtained

Therefore,

$$
\begin{aligned}
U_a = 2\epsilon\Big[U_m^{(10)} + \delta\Big\{\hat{t}_0 U_{m,\hat{t}}^{(10)} + t_0 U_{m,\tau}^{(10)} + U_m^{(11)}\Big\} + \cdots\Big]&\\
+ 2\epsilon^2\Big[\hat{t}_{10} U_{m,\hat{t}}^{(10)} + U_m^{(20)}&\\
+ \delta\Big\{\hat{t}_{11} U_{m,\hat{t}}^{(10)} + t_{10} U_{m,\tau}^{(10)} + \hat{t}_{01}\hat{t}_{10} U_{m,\hat{t}\hat{t}}^{(10)} + \hat{t}_{10} t_0 U_{m,\hat{t}\tau}^{(10)} + \hat{t}_{10} U_{m,\hat{t}}^{(11)} + \cdots\Big\} + \cdots\Big]&\\
+ 2\epsilon^3\Big[\hat{t}_{20} U_{m,\hat{t}}^{(10)} + \frac{(\hat{t}_{10})^2}{2} U_{m,\hat{t}\hat{t}}^{(10)} + \hat{t}_{10} U_{m,\hat{t}}^{(20)} + \big(U_m^{(30)} + U_{3m}^{(30)}\big)&\\
+ \delta\Big\{t_{21} U_{m,\hat{t}}^{(10)} + \hat{t}_{20} U_{m,\tau}^{(10)} + \hat{t}_{10}\hat{t}_{11} U_{m,\hat{t}\hat{t}}^{(10)} + \hat{t}_{20} t_0 U_{m,\hat{t}\tau}^{(10)} + \hat{t}_{10} t_{10} U_{m,\hat{t}\tau}^{(10)} + \hat{t}_{20} U_{m,\hat{t}}^{(11)} \ldots\Big\}&\\
+ \frac{1}{2}(t_{10})^2 U_{m,\hat{t}\hat{t}}^{(11)} + \hat{t}_{11} U_{m,\hat{t}}^{(20)} + t_{10} U_{m,\tau}^{(20)} + \hat{t}_{10} t_0 U_{m,\hat{t}\tau}^{(20)} + \hat{t}_{10} U_{m,\hat{t}}^{(21)}&\\
+ \hat{t}_{01}\big(U_m^{(30)} + U_{3m}^{(30)}\big)_{\hat{t}} + t_0\big(U_m^{(30)} + U_{3m}^{(30)}\big)_{,\tau} + \cdots\Big]\ at\ \tau = 0 \qquad (4.89)&
\end{aligned}
$$

Therefore,

$$U_a = 2\epsilon\big[U_m^{(10)} + \delta t_0 U_{m,\tau}^{(10)} + \cdots\big] + 2\epsilon^2\big[U_m^{(20)} + \delta t_{10} U_{m,\tau}^{(10)} + \cdots\big] + 2\epsilon^3\Big[\big(U_m^{(30)} + U_{3m}^{(30)}\big) + \delta t_{20} U_{m,\tau}^{(10)} + \delta\hat{t}_{20} U_{m,\hat{t}}^{(11)} + \delta t_{10} U_{m,\tau}^{(20)} + \delta t_0\big(U_m^{(30)} + U_{3m}^{(30)}\big)_{,\tau} + \cdots\Big]\ \text{at}\ \tau = 0 \qquad (4.90)$$

In what follows, simplifications of the terms in (4.89)-(4.90) are carried out to obtain the following

$$U_m^{(10)}(\hat{t}_0,0)=2BU_{m,\tau}^{(10)}(\hat{t}_0,0)=-B \qquad (4.91)$$

$$U_m^{(20)}(\hat{t}_0,0)=-\alpha_4(0)+\frac{r_0(0)}{\theta^2}+\frac{r_1(0)}{3\theta^2}=\frac{-r_1(0)}{3\theta^2}+\frac{r_0(0)}{\theta^2}+\frac{r_0(0)}{\theta^2}-\frac{r_1(0)}{\theta^2}=2\left[\frac{-r_1(0)}{3\theta^2}+\frac{r_0(0)}{\theta^2}\right]=2\left[\frac{\alpha B^2}{3\theta^2}-\frac{3\alpha B^2}{\theta^2}\right]=\frac{-16\alpha B^2}{3\theta^2} \qquad (4.92)$$

$$U_m^{(30)}\left(\frac{\pi}{\theta},0\right)=\frac{135B^3\beta}{8\theta^2}\left[1+\frac{8\theta^2}{135}\left\{\left(\frac{\alpha}{\beta}\right)S_0\left(\frac{1}{\varphi(2\theta+\varphi)}-\frac{1}{\varphi(2\theta-\varphi)}\right)\cos\left(\frac{\varphi\pi}{\theta}\right)\right\}+\frac{2}{3\theta^2}\left(\frac{\alpha^2}{\beta}\right).\frac{8\theta^2}{135\beta}(3k_3-k_4)\right]=\frac{135B^3\beta}{8\theta^2}(1+A_{31}) \qquad (4.93)$$

where,

$$A_{31}=\left[1+\frac{8\theta^2}{135}\left\{\left(\frac{\alpha}{\beta}\right)S_0\left(\frac{1}{\varphi(2\theta+\varphi)}-\frac{1}{\varphi(2\theta-\varphi)}\right)\cos\left(\frac{\varphi\pi}{\theta}\right)\right\}+\frac{2}{3\theta^2}\left(\frac{\alpha^2}{\beta}\right).\frac{8\theta^2}{135\beta}(3k_3-k_4)\right]$$

$$S_0=\left(\frac{\alpha}{\varphi^2-\theta^2}-\frac{\alpha}{4(\varphi^2-4\theta^2)}-\frac{3\alpha}{4\varphi^2}\right),k_3=\left(\frac{10}{3\theta^2}-\frac{1}{(\varphi^2-\theta^2)}\right)$$

$$k_4=\left(\frac{2}{3\theta^2}-\frac{1}{(\varphi^2-\theta^2)}+\frac{8}{3\theta\alpha}\right),k_5=\left(\frac{1}{3\theta^2}+\frac{1}{4(\varphi^2-\theta^2)}\right)$$

Similarly,

$$U_{3m}^{(30)}=-B^3\beta\left(A_{32}+\left(\frac{\alpha}{\beta}\right)S_0A_{33}\right) \qquad (4.94)$$

where,

$$A_{32}=\left[\frac{\frac{15}{16}\left(1-\frac{16\frac{\alpha^2}{\beta}k_{11}}{15}\right)(1+\cos\Omega\hat{t}_0)}{\Omega^2-\theta^2}+\frac{\frac{3}{8}(1-k_{12})(1-\cos\Omega\hat{t}_0)}{\Omega^2-4\theta^2}+\frac{(1-k_{13})(1+\cos\Omega\hat{t}_0)}{16(\Omega^2-9\theta^2)}\right]$$

and

$$A_{33}=\left[\frac{1+\cos\Omega\hat{t}_0}{\Omega^2-(\varphi+\theta)^2}-\frac{1+\cos\Omega\hat{t}_0}{\Omega^2-(\varphi-\theta)^2}\right],k_{12}=\left[-\frac{4}{3}\left(\frac{\alpha^2}{\beta}\right)\left(\frac{1}{\varphi^2-\theta^2}\right)\right],\ k_{13}=\left[2\left(\frac{\alpha^2}{\beta}\right)\left(\frac{1}{\varphi^2-4\theta^2}\right)\right]$$

$$U_{m,\tau}^{(20)}\left(\frac{\pi}{\theta},0\right)=-\alpha_4'(0)+\frac{r_0'(0)}{\theta^2}-\frac{r_1'(0)}{3\theta^2} \qquad (4.95)$$

From (4.24h),

$$\alpha_4'(0)=-\alpha_1(0)+\frac{1}{2\theta}\left[\alpha B\beta_2(0)-2\theta^2\omega_1'(0)\beta_2(0)-\omega_1''(0)\theta\alpha_1(0)-2\omega_1'(0)\alpha_1\theta\right]\alpha_1(0)=\frac{-13\alpha B^2}{3\theta^2}+\frac{4B^2}{\theta} \qquad (4.96)$$

$$U_{m,\tau}^{(20)}\left(\frac{\pi}{\theta},0\right)=\left(\frac{13\alpha B^2}{3\theta^2}-\frac{4B^2}{\theta}\right)+\frac{r_0'(0)}{\theta^2}-\frac{r_1'(0)}{3\theta^2}=\left(\frac{13\alpha B^2}{3\theta^2}-\frac{4B^2}{\theta}\right)+\frac{2\alpha B^2}{\theta^2}-\frac{2\alpha B^2}{3\theta^2}=\frac{17\alpha B^2}{3\theta^2}-\frac{4B^2}{\theta}=B^2\left(\frac{17\alpha}{3\theta^2}-\frac{4}{\theta}\right) \qquad (4.97)$$

Also,

$$\omega_2''=-\frac{1}{2\theta^2}\left[\frac{(\omega_1')^2\theta^2}{\alpha_1}+\frac{2\omega_1'\theta^2}{\alpha_1}-2\alpha\left(\frac{r_0}{\theta^2}-\frac{r_1}{6\theta^2}+3\left(\frac{\alpha_4}{\alpha_1}\right)\right)-\left\{\frac{\left(\frac{\alpha_1^2}{2}+B^2\right)}{\varphi^2}+\frac{\alpha_1^2\alpha^2}{4(\varphi^2-4\theta^2)}\right\}-\frac{45\beta}{4}\left(\frac{\alpha_1^2}{4}+B^2\right)\right] \qquad (4.98)$$

$$\therefore\ \omega_2''(0)=-\frac{1}{2\theta^2}\left[\frac{\theta^2\left\{\alpha_1'(0)\left(\omega_1'(0)\right)^2-2\alpha_1\omega_1''(0)\omega_1'(0)\right\}}{\alpha_1^2(0)}+2\theta^2\left\{\alpha_1'(0)\left(\omega_1'(0)\alpha_4(0)\right)-\alpha_1(0)\left(\omega_1''(0)\alpha_4(0)+\omega_1'(0)\alpha_4'(0)\right)\right\}-2\alpha\left(\frac{r_0(0)}{\theta^2}-\frac{r_1'(0)}{6\theta^2}+B\left(\frac{\alpha_1(0)\alpha_4'(0)-\alpha_4(0)\alpha_1'(0)}{\alpha_1^2(0)}\right)\right)-\left\{\frac{\alpha_1'(0)\alpha_1(0)\alpha^2}{\varphi^2}+\frac{\alpha^2\alpha_1(0)\alpha_1'(0)}{4(\varphi^2-4\theta^2)}-\frac{45\beta}{4}\left(\frac{\alpha_1(0)\alpha_1'(0)}{2}\right)\right\}\right]$$

$$= -\frac{1}{2\theta^2}\left[\theta^2\left\{\frac{B^3}{\theta^4 B^2}\right\} + 2\theta^2\left\{\frac{B^2}{\theta^2}.\frac{8\alpha B^2}{3\theta^2} + B\left(\frac{B}{\theta^2}.B^2 S_{51}\right)\right\}\right.$$
$$- 2\alpha\left\{\frac{2\alpha B^2}{\theta^2} - \frac{2\alpha B^2}{6\theta^2} + B\left(\frac{-B.B^2 S_{51} - \frac{8\alpha B^3}{3\theta^2}}{B^2}\right)\right\} - \left\{\alpha^2\left(\frac{-B^2}{\varphi^2}\right) + \frac{\alpha^2 B(-B)}{2(\varphi^2 - 4\theta^2)}\right\}$$
$$\left. - \frac{45\beta(-B^2)}{8}\right]$$
$$= -\frac{1}{2\theta^2}\left[\frac{B}{\theta^2} + 2\left\{\frac{B^4\alpha}{3\theta^2} + B^4 S_{51}\right\} - 2\alpha^2\left\{\frac{5B^2}{3\theta^2} - B^2\left(\frac{S_{51}}{\alpha} + \frac{8}{3\theta^2}\right)\right\} + \alpha^2 B^2\left(\frac{1}{\varphi^2} - \frac{1}{2(\varphi^2 - 4\theta^2)}\right)\right.$$
$$\left. + \frac{45\beta B^2}{8}\right] \qquad (4.99)$$

$$\Rightarrow \omega_2''(0) = -\frac{1}{2\theta^2}\left[\frac{B}{\theta^2} - 2\alpha^2 B^2\left\{\frac{5}{3\theta^2} - \left(\frac{S_{51}}{\alpha} + \frac{8}{3\theta^2}\right)\right\} + \alpha^2 B^2\left(\frac{1}{\varphi^2} - \frac{1}{2(\varphi^2 - 4\theta^2)}\right) + \frac{45\beta B^2}{8} + 2B^4\alpha\left(\frac{1}{3\theta^2} + \frac{S_{51}}{\alpha}\right)\right] \qquad (4.100)$$

Similarly,

$$U_{m,\tau}^{(30)}\left(\frac{\pi}{\theta}, 0\right) = -B^3 S_{65} + \frac{B^3 S_{20}}{\theta^2} + \frac{B^3 S_{20}}{\theta^2} + \frac{B^3 S_{21}}{3\theta^2} + \frac{2\alpha B^3 S_0}{\theta^2} cos\left(\frac{\varphi\pi}{\theta}\right)\left[\frac{1}{(2\theta - \varphi)} - \frac{1}{(2\theta + \varphi)}\right]$$

$$\Rightarrow U_{m,\tau}^{(30)}\left(\frac{\pi}{\theta}, 0\right) = B^3 S_{65} + \frac{B^3 S_{20}}{\theta^2} + \frac{B^3 S_{20}}{\theta^2} + \frac{B^3 S_{21}}{3\theta^2}$$
$$+ \frac{2\alpha B^3 S_0}{\theta^2} cos\left(\frac{\varphi\pi}{\theta}\right)\left[\frac{1}{(2\theta - \varphi)} - \frac{1}{(2\theta + \varphi)}\right] \qquad (4.101)$$

where,

$$S_{65} = -S_{64} + \frac{S_{20}}{\theta^2} + \frac{S_{21}}{3\theta^2} + \frac{2\alpha S_0}{\theta^2} cos\left(\frac{\varphi\pi}{\theta}\right)\left[\frac{1}{(2\theta - \varphi)} - \frac{1}{(2\theta + \varphi)}\right]$$

where,

$$S_{64} = S_{62} - S_{63}, \quad S_{62} = -\frac{1}{2\theta^2}\left[\frac{6\alpha^2}{\theta^3} - 2\theta S_{49} + \frac{\alpha^2}{3\theta^3} - \frac{\theta\omega_2''(0)}{B^2}\right], h_1(0) = B^3 S_{62},$$
$$S_{63} = \left(\frac{-S_3}{\theta^2} + \frac{S_4}{3\theta^2} + \frac{S_5}{8\theta^2} + \frac{\alpha S_0}{(\varphi(2\theta - \varphi)}\right)$$

Also,

$$U_{3m,\tau}^{(30)} = \alpha_{11}'(0)cos\Omega\hat{t}_0 + \beta_{11}'(0)sin\Omega\hat{t}_0 + \frac{r_{22}'(0)cos\theta\hat{t}_0}{\Omega^2 - \theta^2} + \frac{r_{23}'(0)cos2\theta\hat{t}_0}{\Omega^2 - 4\theta^2} + \frac{r_{24}'(0)cos3\theta\hat{t}_0}{\Omega^2 - 9\theta^2}$$
$$+ \frac{r_{25}'(0)cos(\varphi + \theta)\hat{t}_0 + r_{26}'(0)sin(\varphi + \theta)\hat{t}_0}{\Omega^2 - (\varphi + \theta)^2}$$
$$+ \frac{r_{27}'(0)cos(\varphi - \theta)\hat{t}_0 + r_{28}'(0)sin(\varphi - \theta)\hat{t}_0}{\Omega^2 - (\varphi - \theta)^2} \qquad (4.102)$$

$$\alpha_{11}'(0) = h_6(0) - \alpha_{11}(0) = \frac{\alpha B^3 S_{43}}{4\Omega} - B^3 S_{48} = B^3 S_{66}, \qquad S_{66} = \frac{\alpha S_{43}}{4\Omega} - S_{48}$$

Similarly, $\beta_{11}'(0) = h_5(0) - \beta_{11}(0) = -\beta_{11}(0) = 0 \; since \; h_5(0) = 0$

$$\therefore U_{3m,\tau}^{(30)}\left(\frac{\pi}{\theta}, 0\right) = B^3 S_{66} cos\Omega\left(\frac{\pi}{\theta}\right) - \frac{B^3 S_{17}}{\Omega^2 - \theta^2} + \frac{B^3 S_{18}}{\Omega^2 - 4\theta^2} - \frac{B^3 S_{19}}{\Omega^2 - 9\theta^2} - \frac{2\alpha B^3 S_0 cos\left(\frac{\varphi\pi}{\theta}\right)}{\Omega^2 - (\varphi + \theta)^2}$$
$$- \frac{2\alpha B^3 S_0 cos\left(\frac{\varphi\pi}{\theta}\right)}{\Omega^2 - (\varphi - \theta)^2}$$

i.e,

$$U_{3m,\tau}^{(30)}\left(\frac{\pi}{\theta}, 0\right) = B^3 S_{67} \qquad (4.103)$$

where,

$$S_{67} = S_{66} cos\Omega\left(\frac{\pi}{\theta}\right) - \frac{S_{17}}{\Omega^2 - \theta^2} + \frac{S_{18}}{\Omega^2 - 4\theta^2} - \frac{S_{19}}{\Omega^2 - 9\theta^2} - 2\alpha S_0 cos\left(\frac{\varphi\pi}{\theta}\right)\left[\frac{1}{\Omega^2 - (\varphi+\theta)^2} - \frac{1}{\Omega^2 - (\varphi-\theta)^2}\right]$$

Therefore, the maximum displacement is

$$U_a\left(\frac{\pi}{\theta}, 0\right) = 2\epsilon[2B - t_0 B\delta + \cdots] + 2\epsilon^2\left[\frac{-16\alpha B^2}{3\theta^2} - \frac{t_0 B(-B)\delta}{\theta^2} + \cdots\right] + 2\epsilon^3\left[\frac{135B^3\beta(1+A_{31})}{8\theta^2} - B^3\beta\left(A_{32} + \frac{\alpha}{\beta}S_0 A_{33}\right)\right] + \delta\left[-t_{20}B - \hat{t}_{20}B + t_{10}B^2\left(\frac{17\alpha}{3\theta^2} - \frac{4}{\theta^2}\right) + t_0 B^2(S_{65} + S_{67}) + \cdots\right] \quad (4.104)$$

i.e,

$$U_a\left(\frac{\pi}{\theta}, 0\right) = \left[4B\epsilon\left(1 - \frac{t_0\delta}{2}\right) - \frac{32\alpha B^2\epsilon^2}{3\theta^2}\left(1 - \frac{3\delta t_0}{16\alpha} + \cdots\right) + \frac{135\beta(1+A_{31})B^3\epsilon^3}{4\theta^2}\left\{1 - \frac{8\theta^2\left(A_{32} + \frac{\alpha}{\beta}S_0 A_{33}\right)}{135(1+A_{31})}\right\} + \frac{8\delta\theta^2}{135\beta(1+A_{31})}\left\{\frac{-t_{20}}{B^2} - \frac{\hat{t}_{20}}{B^2} + \frac{t_{10}}{B}\left(\frac{17\alpha}{3\theta^2} - \frac{4}{\theta}\right) + t_0(S_{65} + S_{67})\right\}\right] \quad (4.105)$$

A further simplification of (4.105) yields

$$U_a\left(\frac{\pi}{\theta}, 0\right) \equiv U_a = 4B\epsilon D_1 - \frac{32\alpha B^2 D_2\epsilon^2}{3\theta^2} + \frac{135\beta(1+A_{31})B^3\epsilon^3}{4\theta^2}[D_3 + D_4] + \cdots \quad (4.106)$$

where ,

$$D_1 = 1 - \frac{t_0\delta}{2}, \qquad D_2 = 1 - \frac{3t_0\delta}{16\alpha}, \qquad D_3 = 1 - \frac{8\theta^2\left(A_{32} + \frac{\alpha}{\beta}S_0 A_{33}\right)}{135(1+A_{31})}$$

$$D_4 = \frac{8\delta\theta^2}{135\beta(1+A_{31})}\left\{\frac{-t_{20}}{B^2} - \frac{\hat{t}_{20}}{B^2} + \frac{t_{10}}{B}\left(\frac{17\alpha}{3\theta^2} - \frac{4}{\theta}\right) + t_0(S_{65} + S_{67})\right\}$$

Equation (4.106) can be rewritten as

$$U_a = 4B\epsilon D_1 - \frac{32\alpha B^2 D_2\epsilon^2}{3\theta^2} + \frac{135\beta(1+A_{31})B^3 D_3\epsilon^3}{4\theta^2}\left[1 + \frac{D_4}{D_3}\right] + \cdots \quad (4.107)$$

Equation (4.107) can further be rewritten as,

$$U_a = \epsilon c_1 + \epsilon^2 c_2 + \epsilon^2 c_3 + \cdots \quad (4.108a)$$

where,

$$c_1 = 4BD_1, \; c_2 = -\frac{32\alpha B^2 D_2}{3\theta^2}, \; c_3 = \frac{135\beta(1+A_{31})B^3 D_3}{4\theta^2}\left(1 + \frac{D_4}{D_3}\right) = \frac{135\beta(1+A_{31})B^3 D_3(1+D_5)}{4\theta^2}$$

where, $D_5 = \left(\frac{D_4}{D_3}\right)$

To reverse the series (4.108a) as in Ette (2007), we have

$$\epsilon = d_1 U_a + d_2 U_a^2 + d_3 U_a^3 + \cdots \quad (4.108b)$$

By substituting for U_a in (4.108b) and equating the coefficients of powers of ϵ, (4.108b) becomes

$$\epsilon = d_1(\epsilon c_1 + \epsilon^2 c_2 + \epsilon^2 c_3 + \cdots) + d_2(\epsilon c_1 + \epsilon^2 c_2 + \epsilon^3 c_3 + \cdots)^2 + d_3(\epsilon c_1 + \epsilon^2 c_2 + \epsilon^3 c_3 + \cdots)^3 \quad (4.109a)$$

$$O(\epsilon): 1 = d_1 c_1$$

$$\therefore\; d_1 = \frac{1}{c_1}$$

$$O(\epsilon^2): 0 = d_1 c_1 + d_2 c_1^2$$

$$\therefore\; d_2 = \frac{d_1 c_2}{c_1^2} = -\frac{c_2}{c_1^3}$$

$$O(\epsilon^3): 0 = d_1 c_3 + 2d_2 c_1 c_2 + d_3 c_1^3$$

$$\therefore\; d_3 = \frac{-(d_1 c_3 + 2d_2 c_1 c_2)}{c_1^3} = \frac{2c_2^2 - c_1 c_3}{c_1^5}$$

4.3 The Dynamic Buckling Load, λ_D of the Column

As in (3.1), the dynamic buckling load λ_D is now obtained from the maximization, $\frac{d\lambda}{dU_a} = 0$. This is easily done from (4.108a) to yield,

$$\frac{d\epsilon}{dU_a} = \left(\frac{d\epsilon}{d\lambda}\cdot\frac{d\lambda}{dU_{aD}}\right) = 0$$

$$\therefore\; d_1 + 2U_{aD} d_2 + 3d_3 U_{aD}^2 = 0 \quad (4.110)$$

Where, U_{aD} is the value of U_a at buckling and solving (4.110) yields,

$$U_{aD} = \frac{1}{3d_3}\left\{-d_2 \pm (d_2^2 - 3d_1d_3)^{\frac{1}{2}}\right\} \qquad (4.111)$$

The negative root sign in (4.111) is considered because the positive root sign is of no physical significance. Therefore,

$$U_{aD} = \frac{1}{3d_3}\left\{-d_2 - (d_2^2 - 3d_1d_3)^{\frac{1}{2}}\right\} \qquad (4.112)$$

Further simplification of (4.112) yields

$$U_{aD} = \frac{1}{\frac{-c_3}{c_1^4}\left(1 - \frac{2c_2^2}{c_1c_3}\right)}\left[-\sqrt{\frac{3c_3}{c_1^5}}\left(1 - \frac{5c_2^2}{3c_1c_3}\right)\left\{\left\{1 - \frac{c_2}{\sqrt{3c_1c_3}\left(1 - \frac{5c_2^2}{3c_1c_3}\right)^{\frac{1}{2}}}\right\}\right\}\right] \qquad (4.113)$$

i.e,

$$U_{aD} = \sqrt{\frac{c_1^3}{3c_3}}\left[\sqrt{\left(1 - \frac{5c_2^2}{3c_1c_3}\right)}\left\{\frac{1 - \frac{c_2}{\sqrt{3c_1c_3}\left(1 - \frac{5c_2^2}{3c_1c_3}\right)^{\frac{1}{2}}}}{\frac{2c_2^2}{c_1c_3}}\right\}\right] \qquad (4.114)$$

But,

$$\sqrt{\frac{c_1^3}{3c_3}} = \frac{1}{\sqrt{3}}\left\{\frac{\{4BD_1\}^3}{\frac{3\{135\beta(1+A_{31})B^3D_3(1+D_5)\}}{4\theta^2}}\right\}^{\frac{1}{2}} = \frac{1}{\sqrt{3}}\left\{\frac{64B^3D_1^3.4\theta^2}{405\beta B^3D_3(1+D_5)(1+A_{31})}\right\}^{\frac{1}{2}} = \frac{16\theta D_1^{\frac{3}{2}}}{9\sqrt{15\beta D_3\,(1+D_5)(1+A_{31})}} =$$

$$\frac{16\theta}{9\sqrt{15\beta}\,(1+D_5)(1+A_{31})}\left(\frac{D_1^{\frac{3}{2}}}{D_3^{\frac{1}{2}}}\right) = \frac{16\theta D_6}{9\sqrt{15\beta}} = \frac{16\theta D_6}{9\sqrt{15}\beta^{\frac{1}{2}}} \qquad (1.115)$$

where, $D_6 = \frac{\left(\frac{D_1^3}{D_3}\right)^{\frac{1}{2}}}{\sqrt{(1+D_5)(1+A_{31})}}$

Further simplification of terms in (4.114) yields

$$U_{aD} = \frac{16\theta D_6}{9\sqrt{15}\beta^{\frac{1}{2}}}\left[D_7^{\frac{1}{2}}\left\{\frac{1-D_8}{D_9}\right\}\right] = \frac{16\theta D_6 D_{10}}{9\sqrt{15}\beta^{\frac{1}{2}}} \qquad (4.116)$$

where,

$$D_7 = 1 - \frac{5c_2^2}{3c_1c_3} = \left[1 + \frac{1024\left(\frac{\alpha^2}{\beta}\right)D_2^2}{729\theta^2 D_1D_3(1+D_5)(1+A_{31})}\right]$$

$$D_8 = 1 - \frac{c_2}{\sqrt{3c_1c_3}\left(1 - \frac{5c_2^2}{3c_1c_3}\right)^{\frac{1}{2}}} = 1 + \frac{32\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)D_2}{27\sqrt{5}\theta\sqrt{D_1D_3D_7(1+D_5)(1+A_{31})}}$$

$$D_9 = \left(1 - \frac{2c_2^2}{c_1c_3}\right) = 1 - \frac{2048D_2^2\left(\frac{\alpha^2}{\beta}\right)}{1215\theta^2 D_1D_3(1+A_{31})(1+D_5)}$$

Writing, $D_{10} = \left[D_7^{\frac{1}{2}}\left\{\frac{1-D_8}{D_9}\right\}\right]$, (4.116) becomes, $U_{aD} = \frac{16\theta D_6 D_{10}}{9\sqrt{15}\beta^{\frac{1}{2}}}$

To determine the dynamic buckling load, λ_D, (4.108a) is evaluated at buckling to get,

$$\epsilon = d_1U_{aD} + d_2U_{aD}^2 + d_3U_{aD}^3 + \cdots \qquad (4.117)$$

Multiplying equation (4.117) by 3, the following is obtained

$$3\epsilon = 3d_1U_{aD} + 3d_2U_{aD}^2 + 3d_3U_{aD}^3 + \cdots = 3(d_1U_{aD} + d_2U_{aD}^2) + U_{aD}(3d_3U_{aD}^2) + \cdots \qquad (4.118)$$

But from (4.110),

$$3d_3U_{aD}^2 = -d_1 - 2d_2U_{aD} \qquad (4.118)$$

Substituting (4.118) for $3d_3U_{aD}^2$ in (4.117) yields,

$$3\epsilon = 3(d_1U_{aD} + d_2U_{aD}^2 + U_{aD} + \cdots = 2d_1U_{aD} + d_2U_{aD}^2 = 2d_1U_{aD}\left(1 + \frac{d_2U_{aD}}{2d_1}\right) \qquad (4.119)$$

On substituting for d_1, d_2 in equation (4.119), the following is obtained

$$3\epsilon = \frac{2}{c_1} U_{aD}\left(1 - \frac{c_2 U_{aD}}{2c_1^2}\right) \qquad (4.120)$$

On substituting for c_1, c_2 *and* U_{aD} in equation (4.120), the following is obtained

$$3\epsilon = \frac{2\left(\frac{16\theta D_6 D_{10}}{9\sqrt{15}\beta^{\frac{1}{2}}}\right)}{4BD_1}\left[1 - \frac{\left(\frac{-32\alpha B^2 D_2}{3\theta^2}\right)}{2(4BD_1)^2}\left\{\frac{16\theta D_6 D_{10}}{9\sqrt{15}\beta^{\frac{1}{2}}}\right\}\right] = \frac{8\theta D_6 D_{10}}{9\sqrt{15}D_1\beta^{\frac{1}{2}}B}\left[1 + \left(\frac{\alpha D_2}{(D_1\theta)^2}\right)\left(\frac{16\theta D_6 D_{10}}{9\sqrt{15}\beta^{\frac{1}{2}}}\right)\right] = \frac{8\theta D_6 D_{10}}{9\sqrt{15}D_1\beta^{\frac{1}{2}}B}\left[1 + \frac{16\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)D_2 D_6 D_{10}}{27\sqrt{15}D_1^2\theta}\right] \qquad (4.121)$$

$$\Rightarrow\ 3\epsilon = \frac{8(16m^4 - 8\lambda_D m^2 + 1)^{\frac{1}{2}} D_6 D_{10}(16m^4 - 8\lambda_D m^2 + 1)}{9\sqrt{15}D_1\beta^{\frac{1}{2}}.\,8\lambda_D m^2 \overline{a_m}}\left[1 + \frac{16\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)D_2 D_6 D_{10}}{27\sqrt{15}D_1^2\theta}\right]$$

i.e,

$$3\epsilon = \frac{(16m^4 - 8\lambda_D m^2 + 1)^{\frac{3}{2}} D_6 D_{10}}{9\sqrt{15}D_1\beta^{\frac{1}{2}}(\lambda_D m^2 \overline{a_m})}\left[1 + \frac{16\left(\frac{\alpha}{\beta^{\frac{1}{2}}}\right)D_2 D_6 D_{10}}{27\sqrt{15}D_1^2\theta(\lambda_D)}\right]$$

$$\therefore\ (16m^4 - 8\lambda_D m^2 + 1)^{\frac{3}{2}} = 27\sqrt{15}D_1(\lambda_D\epsilon)\, m^2\overline{a_m}\left[1 + \frac{16\left(\frac{\alpha}{\beta^{\left(\frac{1}{2}\right)}}\right)D_2 D_6 D_{10}}{27\sqrt{15}D_1^2\theta(\lambda_D)}\right]^{-1} \qquad (4.122)$$

A simple computer programme, written on Q-basic, gives the values of the dynamic buckling loads, λ_D, at different values of ϵ and δ using equation (4.122).

Table 1: Relationship between the Dynamic Buckling Load and the Imperfection Parameters for different values of damping factors, using equation (4.122).

$\bar{a}_1\epsilon$	λ_D for δ = 0	λ_D for δ = 0.01	λ_D for δ = 0.03
0.01	1.87913	1.87789	1.87548
0.02	1.81858	1.81736	1.81496
0.03	1.77694	1.77571	1.77332
0.04	1.74427	1.74306	1.74068
0.05	1.71702	1.71582	1.71345
0.06	1.69344	1.69225	1.68989
0.07	1.67257	1.67138	1.66903
0.08	1.65376	1.65257	1.65023

0.09	1.63659	1.63541	1.63307
0.1	1.62076	1.61959	1.61726

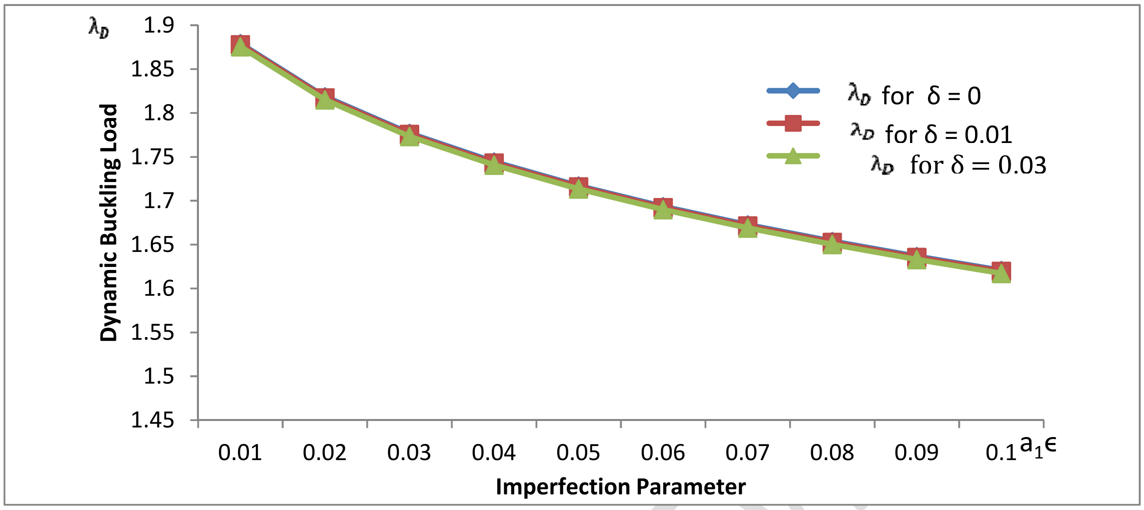

Figure 1: Relationship between the Dynamic Buckling Load and the Imperfection Parameters for different values of damping factors, using equation (4.122).

Table 2: Relationship between the Dynamic Buckling Load and the damping factors for different values of Imperfection Parameters, using equation (4.122).

δ	$\lambda_D for\ a_1\epsilon = 0.01$	$\lambda_D for\ a_1\epsilon = 0.03$	$\lambda_D for\ a_1\epsilon = 0.05$
0.01	1.87789	1.87789	1.87789
0.02	1.87667	1.87667	1.87667
0.03	1.87548	1.87548	1.87548
0.04	1.87431	1.87431	1.87431
0.05	1.87316	1.87316	1.87316
0.06	1.87204	1.87204	1.87204
0.07	1.87093	1.87093	1.87093
0.08	1.86985	1.86985	1.86985
0.09	1.86878	1.86878	1.86878
0.1	1.86773	1.86773	1.86773

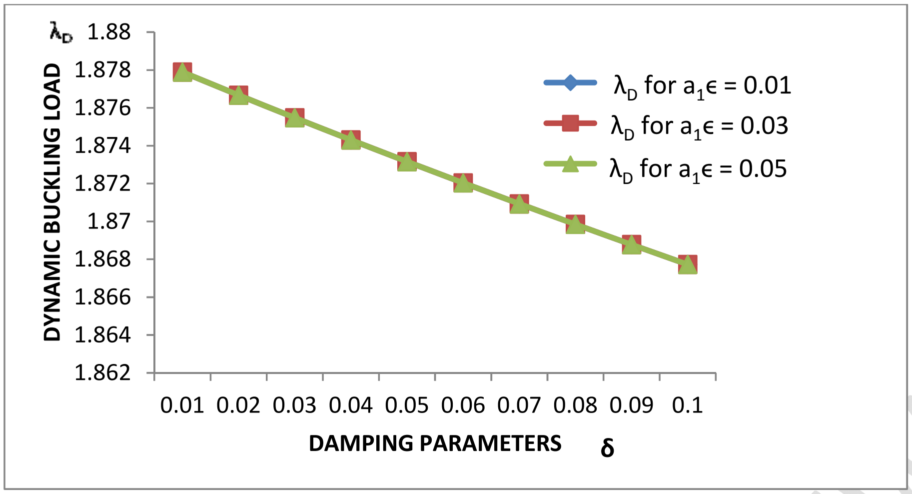

Figure 2: Relationship between the Dynamic Buckling Load and the damping factors for different values of Imperfection Parameters, using equation (4.122).

4.4 Analysis of the Result

The analysis of the result of the simple elastic model column structure tapped by a step load and lying on a quadratic-cubic foundation is hereby presented. The dynamic buckling load decreases with increased imperfection amplitude and vice-versa. This is equivalent to saying that, the nearer the structure is to a perfect nature, the more stable it is for a step load. Besides, we clearly observe that, within the limit of accuracy retained in this work, there is no marked difference in the values of λ_D for the different cases of $\delta = 0.01 \; and \; 0.03$.

5.0 Conclusion

The perturbation and asymptotic techniques applied in this work made it possible to change ordinary differential equations to partial differential equations. These techniques helped us to analyze this problem asymptotically, which could not have been possible if we had solved the ordinary differential equation using the traditional means of solving ordinary differential equations. We were able to establish that dynamic buckling load decreases as imperfection increases. Though we have limited our analysis to an elastic model structure with quadratic-cubic nonlinearity, we can, in principle, extend this analysis to any other elastic model structure while taking care of whatever nonlinearities inherent in such problems. We expect this to be another phase of development in subsequent investigations.

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