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Original Research Article

On A Generalized Pentanacci Sequence

Abstract. The well known Pentanacci sequence is a fifth order recurrence sequence. In this paper, we define other generalized Pentanacci sequence and establish some properties of this sequence using matrix methods.

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1. Introduction and Preliminaries

Pentanacci sequence $\{P_n\}_{n\geq 0}$ and Pentanacci sequence $\{Q_n\}_{n\geq 0}$ are defined by the fifth-order recurrence relations

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, \qquad P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4 \tag{1.1}$$

and

$$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4} + Q_{n-5}, \qquad Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15 \tag{1.2}$$

respectively. P_n is the sequence $A001591$ in [5] and Q_n is the sequence $A074048$ in [5]. Pentanacci sequence has been studied by many authors, see for example [2], [3], [4].

The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -P_{-(n-1)} - P_{-(n-2)} - P_{-(n-3)} - P_{-(n-4)} + P_{-(n-5)}$$

and

$$Q_{-n} = -Q_{-(n-1)} - Q_{-(n-2)} - Q_{-(n-3)} - Q_{-(n-4)} + Q_{-(n-5)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1) and (1.2) hold for all integer n.

Next, we present the first few values of the Pentanacci and Pentanacci-Lucas numbers with positive and negative subscripts in the following Table 1.

Table 1. A few Pentanacci and Pentanacci-Lucas Numbers

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
P_n	2	0	0	0	-1	1	0	0	0	0	1	1	2	4	8	16	31	61	120
Q_n	-1	-1	-1	-7	9	-1	-1	-1	-1	5	1	3	7	15	31	57	113	223	439

For all integers n, usual Pentanacci and Pentanacci-Lucas numbers can be expressed using Binet's formulas

$$\begin{aligned} P_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)} \end{aligned}$$

(see Theorem 2.2 in [7]) or

$$P_n = \frac{\alpha-1}{6\alpha-10}\alpha^{n-1} + \frac{\beta-1}{6\beta-10}\beta^{n-1} + \frac{\gamma-1}{6\gamma-10}\gamma^{n-1} + \frac{\delta-1}{6\delta-10}\delta^{n-1} + \frac{\lambda-1}{6\lambda-10}\lambda^{n-1} \tag{1.3}$$

(see for example [1])

and

$$Q_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n$$

respectively, where $\alpha, \beta, \gamma, \delta$ and λ are the roots of the equation

$$x^5 - x^4 - x^3 - x^2 - x - 1 = 0. \tag{1.4}$$

Moreover, the approximate value of $\alpha, \beta, \gamma, \delta$ and λ are given by

$$\begin{aligned} \alpha &= 1.9659 \\ \beta &= -0.67835 + 0.45854i \\ \gamma &= -0.67835 - 0.45854i \\ \delta &= 0.19538 + 0.84885i \\ \lambda &= 0.19538 - 0.84885i. \end{aligned}$$

In fact, there are no solutions of the characteristic equation (1.4) in terms of radicals, see [8].

The generating functions for the Pentanacci sequence $\{P_n\}_{n\geq 0}$ and Pentanacci-Lucas sequence $\{Q_n\}_{n\geq 0}$ are

$$f_{P_n}(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-x-x^2-x^3-x^4-x^5} \quad \text{and} \quad f_{Q_n}(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{5-4x-3x^2-2x^3-x^4}{1-x-x^2-x^3-x^4-x^5},$$

respectively, (see [7]).

2. Main Results

We consider the generalized Tribonacci sequence defined by

$$E_n = E_{n-1} + E_{n-2} + E_{n-3} + E_{n-4} + E_{n-5}, \qquad E_0 = 5, E_1 = 1, E_2 = 2, E_3 = 0, E_4 = 4. \tag{2.1}$$

The following Table 2 presents the first few values of the generalized Pentanacci numbers E_n with positive and negative subscripts:

Table 2. Generalized Pentanacci numbers E_n with non-negative and negative indices

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
E_n	-4	12	-4	-13	9	0	4	-4	-4	5	1	2	0	4	12	19	37	72	144

Obviously, $x^5 - x^4 - x^3 - x^2 - x - 1 = 0$ is also the characteristic equation of the sequence (2.1) and it produces four roots as $\alpha, \beta, \gamma, \delta$ and λ which are given above. The following Theorem presents the generating function of generalized Pentanacci numbers E_n.

THEOREM 2.1. *The generating function of generalized Pentanacci numbers E_n is given as*

$$f_{E_n}(x) = \frac{5 - 4x - 4x^2 - 8x^3 - 4x^4}{1 - x - x^2 - x^3 - x^4 - x^5}. \tag{2.2}$$

Proof. Let

$$f_{E_n}(x) = \sum_{n=0}^{\infty} E_n x^n$$

be generating function of generalized Pentanacci numbers. Using (2.1) and some calculation, we obtain

$$\begin{aligned} & f_{E_n}(x) - xf_{E_n}(x) - x^2 f_{E_n}(x) - x^3 f_{E_n}(x) - x^4 f_{E_n}(x) - x^5 f_{E_n}(x) \\ = \; & E_0 + (E_1 - E_0)x + (E_2 - E_1 - E_0)x^2 + (E_3 - E_2 - E_1 - E_0)x^3 + (E_4 - E_3 - E_2 - E_1 - E_0)x^4 \end{aligned}$$

which gives (2.2).

We define the square matrix E of order 5 as:

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det E = 1$. E is called the generating matrix for the sequence (2.1).

THEOREM 2.2.

(a): *For $n \geq 1$, we have*

$$\begin{pmatrix} E_{n+4} \\ E_{n+3} \\ E_{n+2} \\ E_{n+1} \\ E_{n} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} E_{n+4} \\ E_{n+2} \\ E_{n+1} \\ E_{n} \\ E_{n-1} \end{pmatrix}. \tag{2.3}$$

(b): *For $n \geq 0$, we have*

$$\begin{pmatrix} E_{n+4} \\ E_{n+3} \\ E_{n+2} \\ E_{n+1} \\ E_{n} \end{pmatrix} = E^{n} \begin{pmatrix} E_{4} \\ E_{3} \\ E_{2} \\ E_{1} \\ E_{0} \end{pmatrix}. \tag{2.4}$$

Proof. (a) and (b) can be proved by using induction on n.

Next we present Binet formula for the generalized Pentanacci sequence $\{E_n\}$.

THEOREM 2.3 (Binet Formula for the Generalized Tetranacci Sequence).

$$\begin{aligned} E_n &= 4\Big(\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\Big) \\ &\quad + 8\Big(\frac{\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\Big) \\ &\quad + 3\Big(\frac{\alpha^{n-2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n-2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^{n-2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n-2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n-2}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\Big) \\ &\quad + 2\Big(\frac{\alpha^{n-3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n-3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^{n-3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n-3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n-3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\Big) \\ &= 4P_{n-3} + 8P_{n-4} + 3P_{n-5} + 2P_{n-6}. \end{aligned}$$

Proof. The general form of the generalized Tribonacci sequence can be expressed in the following form

$$E_n = A\alpha^n + B\beta^n + C\gamma^n + D\delta^n + E\lambda^n \tag{2.5}$$

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where A, B, C and D are constants that can be determined by the initial conditions. Thus putting the values $n = 0$, $n = 1$, $n = 2$, $n = 3$ and $n = 4$ in (2.5), we obtain

$$\begin{aligned} A + B + C + D + E &= 5 \\ A\alpha + B\beta + C\gamma + D\delta + E\lambda &= 1 \\ A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 + E\lambda^{n2} &= 2 \\ A\alpha^3 + B\beta^3 + C\gamma^3 + D\delta^3 + E\lambda^3 &= 0 \\ A\alpha^4 + B\beta^4 + C\gamma^4 + D\delta^4 + E\lambda^4 &= 4. \end{aligned}$$

We can write above system in a matrix form as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta & \lambda \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \lambda^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 & \lambda^3 \\ \alpha^4 & \beta^4 & \gamma^4 & \delta^4 & \lambda^4 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}$$

or

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta & \lambda \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \lambda^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 & \lambda^3 \\ \alpha^4 & \beta^4 & \gamma^4 & \delta^4 & \lambda^4 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}.$$

Solving the above matrix system of equations for A, B, C and D, we get

$$\begin{aligned} A &= \frac{2(\beta\lambda + \beta\gamma + \lambda\gamma + \beta\delta + \lambda\delta + \gamma\delta) + 4 - (\beta\lambda\gamma + \beta\lambda\delta + \beta\gamma\delta + \lambda\gamma\delta) + 5\beta\lambda\gamma\delta}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} \\ B &= \frac{2(\alpha\lambda + \alpha\gamma + \alpha\delta + \lambda\gamma + \lambda\delta + \gamma\delta) + 4 - (\alpha\lambda\gamma + \alpha\lambda\delta + \alpha\gamma\delta + \lambda\gamma\delta) + 5\alpha\lambda\gamma\delta}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ C &= \frac{2(\alpha\beta + \alpha\lambda + \beta\lambda + \alpha\delta + \beta\delta + \lambda\delta) + 4 - (\alpha\beta\lambda + \alpha\beta\delta + \alpha\lambda\delta + \beta\lambda\delta) + 5\alpha\beta\lambda\delta}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ D &= \frac{2(\alpha\beta + \alpha\lambda + \alpha\gamma + \beta\lambda + \beta\gamma + \lambda\gamma) + 4 - (\alpha\beta\lambda + \alpha\beta\gamma + \alpha\lambda\gamma + \beta\lambda\gamma) + 5\alpha\beta\lambda\gamma}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ E &= \frac{2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) + 4 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) + 5\alpha\beta\gamma\delta}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}. \end{aligned}$$

Note that we have the following identities:

$$\begin{aligned}
\alpha+\beta+\gamma+\delta+\lambda &= 1,\\
\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta &= -1,\\
\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta &= 1,\\
\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta &= -1\\
\alpha\beta\gamma\delta\lambda &= 1.
\end{aligned}$$

It now follows that

$$\begin{aligned}
& 2(\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)+4-(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)+5\beta\lambda\gamma\delta\\
= & \frac{1}{\alpha^3}(4\alpha^3+8\alpha^2+3\alpha+2)=(4+8\alpha^{-1}+3\alpha^{-2}+2\alpha^{-3})
\end{aligned}$$

Similarly we have

$$\begin{aligned}
2(\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)+4-(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)+5\alpha\lambda\gamma\delta &= (4+8\beta^{-1}+3\beta^{-2}+2\beta^{-3})\\
2(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)+4-(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)+5\alpha\beta\lambda\delta &= (4+8\gamma^{-1}+3\gamma^{-2}+2\gamma^{-3})\\
2(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)+4-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)+5\alpha\beta\lambda\gamma &= (4+8\delta^{-1}+3\delta^{-2}+2\delta^{-3})\\
2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)+4-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)+5\alpha\beta\gamma\delta &= (4+8\lambda^{-1}+3\lambda^{-2}+2\lambda^{-3}).
\end{aligned}$$

Hence we get

$$\begin{aligned}
E_n &= A\alpha^n+B\beta^n+C\gamma^n+D\delta^n+E\lambda^n\\
&= 4(\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad+\frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)})\\
&\quad+8(\frac{\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad+\frac{\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)})\\
&\quad+3(\frac{\alpha^{n-2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^{n-2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad+\frac{\gamma^{n-2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^{n-2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^{n-2}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)})\\
&\quad+2(\frac{\alpha^{n-3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^{n-3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad+\frac{\gamma^{n-3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^{n-3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^{n-3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)})\\
&= 4P_{n-3}+8P_{n-4}+3P_{n-5}+2P_{n-6}.
\end{aligned}$$

Identities which is given in the following Lemma can be established using by matrix methods.

LEMMA 2.4.

(a): $P_n = \frac{1}{16\,857}(1761E_{n+4} - 801E_{n+3} - 1317E_{n+2} - 105E_{n+1} - 861E_n)$,

(b): $E_n = 4P_{n+4} - 8P_{n+3} - 4P_{n+2} + 9P_{n+1}$,

(c): $E_n = \frac{1}{9584}(-296Q_{n+4} + 6192Q_{n+3} - 8552Q_{n+2} - 7568Q_{n+1} + 8448Q_n)$,

(d): $Q_n = \frac{1}{16\,857}(39E_{n+4} - 1176E_{n+3} + 10\,041E_{n+2} + 4602E_{n+1} + 11\,889E_n)$.

We now present a matrix formula for E_n which is called Simson formula.

THEOREM 2.5 (Simson formula). *For $n \geq 0$ we have*

$$\begin{vmatrix} E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \end{vmatrix} = 16\,857$$

Proof. The proof follows from

$$\begin{vmatrix} E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \end{vmatrix} = \begin{vmatrix} E_4 & E_3 & E_2 & E_1 & E_0 \\ E_3 & E_2 & E_1 & E_0 & E_{-1} \\ E_2 & E_1 & E_0 & E_{-1} & E_{-2} \\ E_1 & E_0 & E_{-1} & E_{-2} & E_{-3} \\ E_0 & E_{-1} & E_{-2} & E_{-3} & E_{-4} \end{vmatrix}. \tag{2.6}$$

The formula (2.6) is given in Soykan [6].

We now obtain the result of Theorem 2.3 (Binet formula for the generalized Tribonacci sequence $\{E_n\}$) using matrix method.

Second Proof of Theorem 2.3 using matrix method (diagonalization).

The characteristic equation of the generating matrix E is

$$0 = |E - xI_4| = \begin{vmatrix} 1-x & 1 & 1 & 1 & 1 \\ 1 & -x & 0 & 0 & 0 \\ 0 & 1 & -x & 0 & 0 \\ 0 & 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 1 & -x \end{vmatrix} = -(x^5 - x^4 - x^3 - x^2 - x - 1)$$

where x is the eigenvalue of E and I_5 is the 5×5 unite matrix. Note that $\alpha, \beta, \gamma, \delta$ and λ are the roots of the characteristic equation $x^5 - x^4 - x^3 - x^2 - x - 1 = 0$ and also $\alpha, \beta, \gamma, \delta$ and λ are the five eigenvalues of the square matrix E. Next we find the eigenvectors corresponding to the eigenvalues $\alpha, \beta, \gamma, \delta$ and λ. We can find the eigenvector by solving the following system of linear equations:

$$(E - xI_5)u_x = 0$$

where u_x is the column vector of order 4×1. First we find the eigenvector corresponding to the eigenvalue α. Then from

$$(E - \alpha I_5)u_\alpha = \begin{pmatrix} 1-\alpha & 1 & 1 & 1 & 1 \\ 1 & -\alpha & 0 & 0 & 0 \\ 0 & 1 & -\alpha & 0 & 0 \\ 0 & 0 & 1 & -\alpha & 0 \\ 0 & 0 & 0 & 1 & -\alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = 0$$

we have the system

$$\begin{aligned} u_2 + u_3 + u_4 + u_5 - u_1(\alpha - 1) &= 0 \\ u_1 - \alpha u_2 &= 0 \\ u_2 - \alpha u_3 &= 0 \\ u_3 - \alpha u_4 &= 0 \\ u_4 - \alpha u_5 &= 0. \end{aligned}$$

If we take $u_5 = c$ in above system we obtain $u_1 = c\alpha^4, u_2 = c\alpha^3, u_3 = c\alpha^2, u_4 = c\alpha$. Thus the eigenvectors corresponding to α are of the form $\begin{pmatrix} c\alpha^4 \\ c\alpha^3 \\ c\alpha^2 \\ c\alpha \\ c \end{pmatrix}$ and in particular if we take $c = 1$ then the eigenvectors corresponding to α is $\begin{pmatrix} \alpha^4 \\ \alpha^3 \\ \alpha^2 \\ \alpha \\ 1 \end{pmatrix}$. Similarly, using the same technique, we see that the eigenvectors corresponding to β, γ, δ and λ are $\begin{pmatrix} \beta^4 \\ \beta^3 \\ \beta^2 \\ \beta \\ 1 \end{pmatrix}$, $\begin{pmatrix} \gamma^4 \\ \gamma^3 \\ \gamma^2 \\ \gamma \\ 1 \end{pmatrix}$, $\begin{pmatrix} \delta^4 \\ \delta^3 \\ \delta^2 \\ \delta \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \lambda^4 \\ \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{pmatrix}$, respectively. Let

$$P = \begin{pmatrix} \alpha^4 & \beta^4 & \gamma^4 & \delta^4 & \lambda^4 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 & \lambda^3 \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \lambda^2 \\ \alpha & \beta & \gamma & \delta & \lambda \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Now let

$$D = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

i.e., D is the diagonal matrix in which the eigenvalues of E are on the main diagonal. Then using the diagonalization of the generating matrix E we obtain $E = PDP^{-1}$. So we get

$$E^n = (PDP^{-1})^n = PD^nP^{-1}.$$

Using the above last equality and (2.4) and comparing the fourth row entries of the matrices we obtain desired result.

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