Original research papers:

FIXED POINTS OF CONTRACTIVE TYPE MAPS IN CMS OVER BANACH ALGEBRA

ABSTRACT

Our goal of this paper is to vouch some fixed point theorems for contractive type maps in a CMS over Banach algebra, which unify, extend and generalize most of the existing relevant fixed point theorems from Shaoyuan Xu and Stojan Radenovic [20].

MSC: 46B20; 46B40; 46J10; 54A05; 47H10

KEYWORDS: CMS over Banach algebra; c-sequence; expansive mapping; fixed point

1. INTRODUCTION

As a generalization of metric spaces, cone metric spaces were scrutinized by Huang and Zhang in 2007(see [1]). In CMS (cone metric space) X, $d(u,v)$ for $u,v \in X$ is a vector in an ordered Banach space E, quite apart from that which is a non-negative real number in general metric spaces. They presented the version of the Banach contraction principle and other fundamental theorems in the setting of cone metric spaces. Afterwards, by omitting the assumption of normality in the theorems of [1], Rezapour and Hamlbarani [2] established some fixed point theorems, as the generalizations and extensions of the analogous results in [1]. Besides, they gave a number of examples to vouch the existence of non-normal cones, which proves that such generalizations are significant. For more details, we refer the reader to [2-14].

Newly, Liu and Xu [15] familiarized the idea of CMS over Banach algebras (which were called CMS over Banach algebras in [15]), replacing Banach spaces by Banach algebras as the underlying spaces of CMS. They replaced the Banach space E by a Banach algebra $\mathcal{A}$ and familiarized the idea of CMS over Banach algebras. In this manner, they vouched some fixed point theorems of generalized Lipschitz mappings with natural and weaker conditions on generalized Lipschitz constant h by means of spectral radius.

2. PRELIMINARIES

For the sake of reciter, we shall recollect some fundamental concepts and lemmas. We begin with the following definition as a recall from [15].

Let $\mathcal{A}$ always is a real Banach algebra. Then $\forall\, u, v, w \in \mathcal{A}, \alpha \in \mathbb{R}$, we have

1. $(uv)w = u(vw)$;
2. $u(v+w) = uv + uw$ and $(u+v)w = uw + vw$;
3. $\alpha(uv) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leq \|u\|\|v\|$.

We shall assume that a Banach algebra has a multiplicative identity e such that $eu = ue = u, \forall\, u \in \mathcal{A}$. An element $u \in \mathcal{A}$ is said to be invertible if there is an inverse element $v \in \mathcal{A}$ such that $uv = vu = e$. The inverse of u is denoted by u^{-1}. For more details, we refer the reader to [16].

The following proposition is given in [16].

Proposition 2.1 Let A be a Banach algebra with a unit e, and $u \in \mathcal{A}$. If the spectral radius $\rho(u)$ of u is less than 1, i.e.,

$$\rho(u) = \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}} = \inf \|u^n\|^{\frac{1}{n}} < 1,$$

then $e - u$ is invertible. Actually,

$$(e-u)^{-1}=\sum_{i=0}^{\infty}u_i.$$

Remark 2.2 From [16] we see that the spectral radius $\rho(u)$ of u satisfies $\rho(u)\leq\|u\|,\forall\, u\in\mathcal{A}$ for all where $\mathcal{A}$ is a Banach algebra with a unit e.

Remark 2.3 (see [16]) In Proposition 2.1, if the condition $'\rho(u)<1'$ is replaced by $'\|u\|<1'$, then the conclusion remains true.

A subset P of $\mathcal{A}$ is called a cone of A if

1. P is nonempty closed and $\{\theta,e\}\subset P$;
2. $\delta P+\mu P\subset P$ for all nonnegative real numbers δ,μ;
3. $P^2=PP\subset P$;
4. $P\cap(-P)=\{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P\subset\mathcal{A}$, we can define a partial ordering $\preccurlyeq$ with respect to P by $u\preccurlyeq v$ if and only if $v-u\in P$. $u\prec v$ will stand for $u\preccurlyeq v$ and $u=v$, while $u\ll v$ will stand for $v-u\in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P. If $intP\neq\emptyset$, then P is called a solid cone.

The cone P is called normal if there is a number $M>0$ such that, $\forall\, u,v\in\mathcal{A}$, $\theta\preccurlyeq u\preccurlyeq v\Rightarrow\|u\|\leq M\|v\|$. The least positive number satisfying the above is called the normal constant of P [1].

In the following we always assume that A is a Banach algebra with a unit e, P is a solid cone in A and $\preccurlyeq$ is the partial ordering with respect to P.

Definition 2.4 ([1, 15, 17]) Let X be a nonempty set. Suppose that the mapping $d_c\colon X\times X\to\mathcal{A}$ satisfies

1. $\theta\prec d_c(u,v),\forall\, u,v\in X$ and $d_c(u,v)=\theta\Leftrightarrow u=v$;
2. $d_c(u,v)=d_c(v,u),\forall\, u,v\in X$;
3. $d_c(u,v)\preccurlyeq d_c(u,w)+d_c(w,v),\forall\, u,v,w\in X$.

Then d_c is called a cone metric on X, and (X, d_c) is called a cone metric space (CMS) over Banach algebra $\mathcal{A}$.

Definition 2.5 (See [1, 15, 17]) Let (X, d_c) be a CMS over a Banach algebra $\mathcal{A}$, $x \in X$ and let $\{u_n\}_{n=0}^{\infty} \subset X$ be a sequence. Then:

(1). $\{u_n\}_{n=0}^{\infty}$ converges to u whenever for each $c \in \mathcal{A}$ with $c \gg \theta$ there is a natural number N such that $d_c(u_n, u) \ll c$ for all $n \geq N$. We write $\lim_{n\to\infty} u_n = u$ or $u_n \to u\ (n \to \infty)$.

(2). $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence whenever for each $c \in \mathcal{A}$ with $c \gg \theta$ there is a natural number N such that $d_c(u_n, u_m) \ll c$ for all $n, m \geq N$.

(3). (X, d_c) is a complete CMS if every Cauchy sequence is convergent.

(4). Now, we shall appeal to the following lemmas in the sequel.

Lemma 2.6 (See [18]) If E is a real Banach space with a cone P and if $b \preccurlyeq \mu b$ with $b \in P$ and $1 \leq b < 1$, then $= \theta$.

Lemma 2.7 (See [9]) If E is a real Banach space with a solid cone P and if $\theta \preccurlyeq x \ll c$ for each $\theta \ll c$, then $x = \theta$.

Lemma 2.8 (See [9]) If E is a real Banach space with a solid cone P and $\{x_n\} \subset P$ is a sequence with $\|x_n\| \to 0\ (n \to \infty)$ then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$, i.e. x_n is a c-sequence

Finally, let us recall the concept of generalized Lipschitz mapping defining on the cone metric spaces over Banach algebras, which is introduced in [15].

Definition 2.9 (See [15]) Let (X, d_c) be a CMS over a Banach algebra $\mathcal{A}$. A mapping $T: X \to X$ is called a generalized Lipschitz mapping if there exists a vector $h \in P$ with $\rho(h) < 1$ and for all $u, v \in X$, one has $d_c(Tu, Tv) \preccurlyeq h d_c(u, v)$

Remark 2.10 In Definition 2.9, we only suppose the spectral radius of h is less than 1, while $\|h\| < 1$ is not assumed. Generally speaking, it is meaningful since by Remark 2.2, the condition $\rho(h) < 1$ is weaker than that $\|h\| < 1$.

Remark 2.11 (see [16]) If $\rho(u) < 1$, then $\|u\|^n \to 0\ (n \to \infty)$.

Lemma 2.12 ([16]) Let $\mathcal{A}$ be a Banach algebra with a unit $e, h, k \in \mathcal{A}$. If h commutes with k, then

$$\rho(h+k) \leq \rho(h) + \rho(k),$$

$$\rho(hk) \leq \rho(h)\rho(k).$$

Lemma 2.13 ([16]) If E is a real Banach space with a solid cone P

(1). If $a_1, a_2, a_3 \in E$ and $a_1 \preccurlyeq a_2 \ll a_3$, then $a_1 \ll a_3$.

(2). If $a_1 \in P$ and $a_1 \ll a_3$ for each $a_3 \gg \theta$, then $a_1 = \theta$.

Lemma 2.14 ([20]) Let P be a solid cone in a Banach algebra $\mathcal{A}$. Suppose that $h \in P$ and $\{x_n\} \subset P$ is a c-sequence. Then $\{hx_n\}$ is a c-sequence.

Proposition 2.15 (See [20]) Let P be a solid cone in a Banach space $\mathcal{A}$ and let $\{u_n\}, \{v_n\} \subset X$ be sequences. If $\{u_n\}$ and $\{v_n\}$ are c-sequences and $\gamma, \delta > 0$ then $\{\gamma u_n + \delta u_n\}$ is a c-sequence.

Proposition 2.16 (See [20]) Let P be a solid cone in a Banach algebra $\mathcal{A}$. and let $\{u_n\} \subset P$ is a sequence.. Then the following conditions are equivalent:

(1). $\{u_n\}$ is a c-sequence.

(2). For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \prec c$ for $n \geq n_0$.

(3). For each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $u_n \preccurlyeq c$ for $n \geq n_0$.

Lemma 2.17 ([16]) Let A be a Banach algebra with a unit $e, h \in A$, then $\lim_{n\to\infty} \|h^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(h)$ satisfies

$$\rho(h) = \lim_{n\to\infty} \|h^n\|^{\frac{1}{n}} = \inf \|h^n\|^{\frac{1}{n}}$$

If $\rho(h) < |\omega|$, then $(\omega e - h)$ is invertible in A; moreover,

$$(\omega e - h)^{-1} = \sum_{i=0}^{\infty} \frac{h^i}{\omega^{i+1}}$$

where ω is a complex constant.

Lemma 2.18 ([16]) Let A be a Banach algebra with a unit e and $h \in A$. If ω is a complex constant and $\rho(h) < |\omega|$, then

$$\rho((\omega e - h)^{-1}) \leq \frac{1}{|\omega| - \rho(h)}$$

Lemma 2.19 ([16]) Let $\mathcal{A}$ be a Banach algebra with a unit e and P be a solid cone in $\mathcal{A}$. Let $h \in \mathcal{A}$ and $u_n = h^n$. If $\rho(h) < 1$, then $\{u_n\}$ is a c-sequence.

Lemma 2.20 ([22]). Let T and S be weakly compatible self-maps of a set X. If T and S have a unique point of coincidence $w = Tu = Su$, then w is the unique common fixed point of T and S.

3. MAIN RESULTS

Theorem 3.1 Let (X, d_c) be a cone metric space over Banach algebra A and P be a solid cone in A. Let $h_i \in P$ $(i = 1, \ldots, 5)$ be generalized Lipschitz constants with $\rho(h_1) + \rho(h_2 + h_3 + h_4 + h_5) < 1$. Suppose that h_1 commutes with $h_2 + h_3 + h_4 + h_5$ and the mappings $T, S : X \to X$ satisfy that

$$d_c(Tu, Tv) \preccurlyeq h_1 d_c(Su, Sv) + h_2 d_c(Tu, Su) + h_3 d_c(Tv, Sv) + h_4 d_c(Su, Tv) + h_5 d_c(Tu, Sv) \quad (3.1)$$

for all $u, v \in X$. If the range of S contains the range of T and $S(X)$ is a complete subspace, then T and S have a unique point of coincidence in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point.

Proof. Suppose $u_0 \in X$ be an arbitrary point. Since $T(X) \subset S(X)$, $\exists\, u_1 \in X$ such that $Tu_0 = Su_1$. By induction, a sequence $\{Tu_n\}$ can be chosen such that $Tu_n = Su_{n+1}$ $(n = 0,1,2, \ldots)$. Thus, by (3.1), for any natural number n, on the one hand, we obtain

$$d_c(Su_{n+1}, Su_n) = d_c(Tu_n, Tu_{n-1})$$

$$\preccurlyeq h_1 d_c(Su_n, Su_{n-1}) + h_2 d_c(Tu_n, Su_n) + h_3 d_c(Tu_{n-1}, Su_{n-1})$$

$$+ h_4 d_c(Su_n, Tu_{n-1}) + h_5 d_c(Tu_n, Su_{n-1})$$

$$= h_1 d_c(Su_n, Su_{n-1}) + h_2 d_c(Su_{n+1}, Su_n) + h_3 d_c(Su_n, Su_{n-1})$$

$$+h_4 d_c(Su_n, Su_n) + h_5 d_c(Su_{n+1}, Su_{n-1})$$

$$\preccurlyeq (h_1 + h_3 + h_5) d_c(Su_n, Su_{n-1}) + (h_2 + h_5) d_c(Su_{n+1}, Su_n)$$

This $\Longrightarrow$

$$(e - h_2 - h_5) d_c(Su_{n+1}, Su_n) \preccurlyeq (h_1 + h_3 + h_5) d_c(Su_n, Su_{n-1}) \quad (3.2)$$

For another thing,

$$d_c(Su_n, Su_{n+1}) = d_c(Tu_{n-1}, Tu_n)$$

$$\preccurlyeq h_1 d_c(Su_{n-1}, Su_n) + h_2 d_c(Tu_{n-1}, Su_{n-1}) + h_3 d_c(Tu_n, Su_n)$$

$$+h_4 d_c(Su_{n-1}, Tu_n) + h_5 d_c(Tu_{n-1}, Su_n)$$

$$\preccurlyeq h_1 d_c(Su_{n-1}, Su_n) + h_2 d_c(Su_n, Su_{n-1}) + h_3 d_c(Su_{n+1}, Su_n)$$

$$+h_4 d_c(Su_{n-1}, Su_{n+1}) + h_5 d_c(Su_n, Su_n)$$

$$\preccurlyeq (h_1 + h_2 + h_4) d_c(Su_{n-1}, Su_n) + (h_3 + h_4) d_c(Su_n, Su_{n+1})$$

This $\Longrightarrow$

$$(e - h_3 - h_4) d_c(Su_n, Su_{n+1}) \preccurlyeq (h_1 + h_2 + h_4) d_c(Su_{n-1}, Su_n) \quad (3.3)$$

Add up (3.2) and (3.3) produces that

$$(2e - h_2 - h_3 - h_4 - h_5) d_c(Su_n, Su_{n+1}) \preccurlyeq (2h_1 + h_2 + h_3 + h_4 + h_5) d_c(Su_{n-1}, Su_n) \quad (3.4)$$

Taking $k = h_2 + h_3 + h_4 + h_5$, (3.4) yields that

$$(2e - h) d_c(Su_n, Su_{n+1}) \preccurlyeq (2h_1 + h) d_c(Su_{n-1}, Su_n) \quad (3.5)$$

Because

$$\rho(h) \leq \rho(h_1) + \rho(h) < 1$$

leads to $\rho(h) < 1 < 2$, then from Lemma 2.17, it concludes that $2e - h$ is invertible. Furthermore,

$$(2e - h\,)^{-1} = \sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}}$$

In both sides of (2.5), multiplying by $(2e - h\,)^{-1}$, we attain at

$$d_c(Su_n, Su_{n+1}) \preccurlyeq (2e - h\,)^{-1}(2h_1 + h)d_c(Su_{n-1}, Su_n) \tag{3.6}$$

Taking $(2e - h\,)^{-1}(2h_1 + h) = k$, by (3.6), we reach

$$d_c(Su_n, Su_{n+1}) \preccurlyeq k d_c(Su_{n-1}, Su_n) \preccurlyeq \ldots\ldots \preccurlyeq k^n d_c(Su_0, Su_1) = k^n d_c(Su_0, Tu_0) \tag{3.7}$$

Because h_1 commutes with h, it follows that

$$(2e - h\,)^{-1}(2h_1 + h) = \sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}}(2h_1 + h)$$

$$= 2\left(\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}}\right) h_1 + \left(\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}}\right) h$$

$$= 2h_1\left(\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}}\right) + \sum_{i=0}^{\infty} \frac{h^{i+1}}{2^{i+1}}$$

$$= 2h_1\left(\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}}\right) + h\left(\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}}\right)$$

$$= (2h_1 + h)\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}}$$

$$= (2h_1 + h)(2e - h\,)^{-1}$$

To say that $(2e - h\,)^{-1}$ commutes with $(2h_1 + h)$. By Lemma 2.12 and Lemma 2.18, we avail

$$\rho(k) = \rho\big((2e - h\,)^{-1}(2h_1 + h)\big)$$

$$\leq \rho((2e - h\,)^{-1})\rho(2h_1 + h)$$

$$\leq \frac{1}{2-\rho(h)}[2\rho(h_1) + \rho(h)] < 1$$

which vouches that $e - k$ is invertible and $\|k^m\| \to 0$ $(m \to \infty)$. Hence, for any $m \geq 1$; $p \geq 1$ and $k \in P$ with $\rho(k) < 1$, we obtain that

$$d_c(Su_m, Su_{m+p}) \preccurlyeq d_c(Su_m, Su_{m+1}) + d(Su_{m+1}, Su_{m+p})$$

$$\preccurlyeq d_c(Su_m, Su_{m+1}) + d_c(Su_{m+1}, Su_{m+2}) + d_c(Su_{m+2}, Su_{m+p})$$

$$\preccurlyeq d_c(Su_m, Su_{m+1}) + d_c(Su_{m+1}, Su_{m+2}) + \ldots + d_c(Su_{m+p-1}, Su_{m+p})$$

$$\preccurlyeq k^m d_c(Su_0, Tu_0) + k^{m+1} d_c(Su_0, Tu_0) + \ldots + k^{m+p-1} d_c(Su_0, Tu_0)$$

$$= k^m[e + k + \ldots + k^{p-1}] d_c(Su_0, Tu_0)$$

$$\preccurlyeq k^m (e-k)^{-1} d_c(Su_0, Tu_0) \qquad (3.8)$$

With advantage of Lemma 2.19 and Lemma 2.14, we obtain $\{Su_n\}$ is a Cauchy sequence. Since $S(X)$ is complete, $\exists$ $z \in S(X)$ such that $Su_n \to z$ $(n \to \infty)$. Thus $\exists$ $w \in X$ such that $Sw = z$. We shall certify $Tw = z$. In order to finish this, for one thing,

$$d_c(Su_n, Tw) = d_c(Tu_{n-1}, Tw)$$

$$\preccurlyeq h_1 d_c(Su_{n-1}, Sw) + h_2 d_c(Tu_{n-1}, Su_{n-1}) + h_3 d_c(Tw, Sw)$$

$$+ h_4 d_c(Su_{n-1}, Tw) + h_5 d_c(Tu_{n-1}, Sw)$$

$$= h_1 d_c(Su_{n-1}, z) + h_2 d_c(Su_n, Su_{n-1}) + h_3 d_c(Tw, z)$$

$$+ h_4 d_c(Su_{n-1}, Tw) + h_5 d_c(Su_n, z)$$

$$\preccurlyeq h_1 d_c(Su_{n-1}, z) + h_2[d_c(Su_n, z) + d_c(z, Su_{n-1})]$$

$$+ h_3[d_c(Tw, Su_n) + d_c(Su_n, z)]$$

$$+ h_4[d_c(Su_{n-1}, z) + d_c(z, Su_n) + d_c(Su_n, Tw)]$$

$$+h_5 d_c(Su_n, z)$$

This $\Longrightarrow$

$$(e - h_3 - h_4)d_c(Su_n, Tw) \preccurlyeq (h_1 + h_2 + h_4)d_c(Su_{n-1}, z)$$

$$+(h_2 + h_3 + h_4 + h_5)d_c(Su_n, z) \quad (3.9)$$

On the other hand, we obtain

$$d_c(Su_n, Tw) = d_c(Tu_{n-1}, Tw) = d_c(Tw, Tu_{n-1})$$

$$\preccurlyeq h_1 d_c(Sw, Su_{n-1}) + h_2 d_c(Tw, Sw) + h_3 d_c(Tu_{n-1}, Su_{n-1})$$

$$+h_4 d_c(Sw, Tu_{n-1}) + h_5 d_c(Tw, Su_{n-1})$$

$$= h_1 d_c(z, Su_{n-1}) + h_2 d_c(Tw, z) + h_3 d_c(Su_n, Su_{n-1})$$

$$+h_4 d_c(z, Su_n) + h_5 d_c(Tw, Su_{n-1})$$

$$\preccurlyeq h_1 d_c(z, Su_{n-1}) + h_2[d_c(Tw, Su_n) + d_c(Su_n, z)]$$

$$+h_3[d_c(Su_n, z) + d_c(z, Su_{n-1})] + h_4 d_c(z, Su_n)$$

$$+h_5[d_c(Tw, Su_n) + d_c(Su_n, z) + d_c(z, Su_{n-1})]$$

This $\Longrightarrow$

$$(e - h_2 - h_5)d_c(Su_n, Tw) \preccurlyeq (h_1 + h_3 + h_5)d_c(Su_{n-1}, z)$$

$$+(h_2 + h_3 + h_4 + h_5)d_c(Su_n, z) \quad (3.10)$$

Combine (3.9) and (3.10), it follows that

$$(2e - h_2 - h_3 - h_4 - h_5)d_c(Su_n, Tw) \preccurlyeq (2h_1 + h_2 + h_3 + h_4 + h_5)d_c(Su_{n-1}, z)$$

$$+2(h_2 + h_3 + h_4 + h_5)d_c(Su_n, z)$$

$$\Longrightarrow (2e - h)d_c(Su_n, Tw) \preccurlyeq (2h_1 + h)d_c(Su_{n-1}, z) + 2h d_c(Su_n, z)$$

Because

$$\rho(h) \leq \rho(h_1) + \rho(h) < 1 \tag{3.11}$$

thus by Lemma 2.17, it concludes that $2e - h$ is invertible. As a result, it follows immediately from (2.9) that

$$d_c(Su_n, Tw) \preccurlyeq \left((2e - h)\right)^{-1}[(2h_1 + h)d_c(Su_{n-1}, z) + 2hd_c(Su_n, z)]$$

Since $\{d_c(Su_{n-1}, z)\}$ and $\{d_c(Su_{n-1}, z)\}$ are c-sequences, then by Lemma 2.14, we acquire that $\{d_c(Su_n, Tw)\}$ is a c-sequence, thus $Su_n \to Tw$ $(n \to \infty)$. Hence $Tw = Sw = z$. In the following we shall show T and S have a unique point of coincidence.

If $\exists\, w' \neq w$ such that $Tw' = Sw'$. Then we obtain

$$\begin{aligned} d_c(Sw', Sw) &= d_c(Tw', Tw) \\ &\preccurlyeq h_1 d_c(Sw', Sw) + h_2 d_c(Tw', Sw') + h_3 d_c(Tw, Sw) \\ &\quad + h_4 d_c(Sw', Tw) + h_5 d_c(Tw', Sw) \\ &= (h_1 + h_4 + h_5) d_c(Sw', Sw) \end{aligned}$$

Set $\gamma = h_1 + h_4 + h_5,$ then it follows that

$$d_c(Sw', Sw) \preccurlyeq \gamma d_c(Sw', Sw) \preccurlyeq \ldots\ldots \preccurlyeq \gamma^n d_c(Sw', Sw) \tag{3.12}$$

Because of

$$\rho(h) \leq \rho(h_1) + \rho(h) < 1$$

it follows that $\rho(h_1) + \rho(h) < 1$. Since h_1 commutes with h, then by Lemma 2.12,

$$\rho(h_1 + h) \leq \rho(h_1) + \rho(h) < 1$$

Accordingly, by Lemma 2.19, we speculate that $\{(h_1 + h)^n\}$ is a c-sequence. Noticing that $\gamma \preccurlyeq h_1 + h$ leads to $\gamma^n \preccurlyeq (h_1 + h)^n$, we claim that $\{\gamma^n\}$ is a c-sequence. Consequently, in view of (3.12), it is easy to see $d_c(Sw', Sw) = \theta$ that is, $Sw' = Sw$.

Finally, if (T, S) is weakly compatible, then by using Lemma 2.20, we claim that T and S have a unique common fixed point.

Corollary 3.2 Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and let P be the underlying solid cone with $h \in P$ where $\rho(k) < 1$. Suppose he mappings $T, S : X \to X$ satisfy generalized Lipschitz condition:

$$d_c(Tu, Tv) \preccurlyeq h d_c(Su, Sv) \tag{3.13}$$

for all $u, v \in X$. If the range of S contains the range of T and $S(X)$ is a complete subspace, then $T$and S have a unique point of coincidence in X. Moreover, if $T$and S are weakly compatible, then $T$and S have a unique common fixed point.

Proof. Choose $h_1 = h$ and $h_2 = h_3 = h_4 = h_5 = 0$ in Theorem 3.1, we complete the proof.

Corollary 3.3 Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and let P be the underlying solid cone with $h \in P$ where $\rho(k) < 1$. Suppose he mappings $T, S : X \to X$ satisfy generalized Lipschitz condition:

$$d_c(Tu, Tv) \preccurlyeq h\,[d_c(Tu, Sv) + d_c(Tv, Su)] \tag{3.14}$$

for all $u, v \in X$. If the range of S contains the range of T and $S(X)$ is a complete subspace, then $T$and S have a unique point of coincidence in X. Moreover, if $T$and S are weakly compatible, then $T$and S have a unique common fixed point.

Proof. Choose $h_4 = h_5 = h$ and $h_1 = h_2 = h_3 = 0$ in Theorem 3.1, the proof is valid.

Corollary 3.4 Let (X, d_c) be a cone metric space over Banach algebra $\mathcal{A}$ and let P be the underlying solid cone with $h \in P$ where $\rho(k) < 1$. Suppose he mappings $T, S : X \to X$ satisfy generalized Lipschitz condition:

$$d_c(Tu, Tv) \preccurlyeq h\,[d_c(Tu, Su) + d_c(Tv, Sv)] \tag{3.15}$$

for all $u, v \in X$. If the range of S contains the range of T and $S(X)$ is a complete subspace, then $T$and S have a unique point of coincidence in X. Moreover, if $T$and S are weakly compatible, then $T$and S have a unique common fixed point.

Proof. Choose $h_2 = h_3 = h$ and $h_1 = h_4 = h_5 = 0$ in Theorem 3.1, the claim holds.

Corollary 3.5 Let (X, d_c) be a complete cone metric space over Banach algebra $\mathcal{A}$ and P be a solid cone in $\mathcal{A}$. Let $h_i \in P$ $(i = 1, \ldots, 5)$ be generalized Lipschitz constants with $\rho(h_1) + \rho(h_2 + h_3 + h_4 + h_5) < 1$. Suppose that h_1 commutes with $h_2 + h_3 + h_4 + h_5$ and the mapping $T: X \to X$ satisfies that

$$d_c(Tu, Tv) \preccurlyeq h_1 d_c(u, v) + h_2 d_c(Tu, u) + h_3 d_c(Tv, v) + h_4 d_c(u, Tv) + h_5 d_c(Tu, v) \quad (3.16)$$

for all $u, v \in X$. then T has a unique fixed point in X.

Proof. Taking $S = I_X$ (Identity mapping) in Theorem 3.1, the proof is valid.

Remark 3.6

1. If we take $S = I_X$ and choose $h_1 = h$ and $h_2 = h_3 = h_4 = h_5 = 0$ in Theorem 3.1, we get Theorem 3.1 of Shaoyuan Xu and Stojan Radenovic [20].
2. If we take $S = I_X$ and choose $h_4 = h_5 = h$ and $h_1 = h_2 = h_3 = 0$ in Theorem 3.1, we have Theorem 3.2 of Shaoyuan Xu and Stojan Radenovic [20].
3. If we take $S = I_X$ and choose $h_1 = h_4 = h_5 = 0$ and $h_2 = h_3 = h$ in Theorem 3.1, we obtain Theorem 3.3 of Shaoyuan Xu and Stojan Radenovic [20].
4. If we take $S = I_X$ in Corollary 3.2, we obtain Theorem 3.1 of Shaoyuan Xu and Stojan Radenovic [20].
5. If we take $S = I_X$ in Corollary 3.3, we get Theorem 3.2 of Shaoyuan Xu and Stojan Radenovic [20].
6. If we take $S = I_X$ in Corollary 3.4, we get Theorem 3.3 of Shaoyuan Xu and Stojan Radenovic [20].
7. Choose $h_1 = h$ and $h_2 = h_3 = h_4 = h_5 = 0$ in Corollary 3.5, we have Theorem 3.1 of Shaoyuan Xu and Stojan Radenovic [20].
8. Choose $h_4 = h_5 = h$ and $h_1 = h_2 = h_3 = 0$ in Corollary 3.5, we get Theorem 3.2 of Shaoyuan Xu and Stojan Radenovic [20].

9. Choose $h_1 = h_4 = h_5 = 0$ and $h_2 = h_3 = h$ in Corollary 3.5, we obtain Theorem 3.3 of Shaoyuan Xu and Stojan Radenovic [20].

Example 3.7 Let $X = [0,1]$ and $\mathcal{A}$ be the set of all real valued functions on X which also have continuous derivatives on X with the norm$\|u\| = \|u\|_\infty + \|u'\|_\infty$ and the usual multiplication. Let $P = \{u \in \mathcal{A}, u(t) \geq 0, t \in X\}$. It is clear that P is a nonnormal cone and $\mathcal{A}$ is a Banach algebra with a unit $e = 1$. Define a mapping $d_c: X \times X \to \mathcal{A}$ by $d_c(u, v) = |u - v|e^t$. We make a conclusion that (X; d) is a complete cone b-metric space over Banach algebra $\mathcal{A}$. Now define the mappings $T, S: X \to X$ by $Tu = \frac{u}{8}$ and $Su = \frac{u}{2}$. Choose $h_1 = \frac{1+t}{8}, h_2 = \frac{1+t}{12}, h_3 = \frac{1+t}{16}, h_4 = h_5 = 0$. Simple calculations show that all conditions of Theorem 2.9 are satisfied. Therefore, 0 is the unique common fixed point of T and S.

Competing interests

The authors declare that they have no competing interests.

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