Approximate Solution Technique for Singular Fredholm Integral Equations of the First Kind with Oscillatory Kernels

ABSTRACT

An efficient quadrature formula has been developed for evaluating numerically certain singular Fredholm integral equations of the first kind with oscillatory trigonometric kernels. The method is based on the Lagrange interpolation formula and the orthogonal polynomial considered are the Legendre polynomials whose zeros served as interpolation nodes. A test example has been provided for the verification and validation of the rule developed.

Keywords: Singular kernel, Oscillatory kernel, Lagrange interpolation, orthogonal polynomial, Legendre polynomial

1. INTRODUCTION

The Fredholm integral equations of the first kind with oscillatory kernels

$$\int_{-1}^{1} \frac{k(x,t)}{t-x} u(t)dt = f(x), \qquad \upsilon \geq 0, \qquad i^2 = -1, \qquad -1 < x < 1, \qquad (1)$$

where f is a given continuous function, have wide applications in mathematics, physics, engineering and other applied and computational sciences. Many efficient methods have been developed for the evaluation of oscillatory integrals. The earliest numerical methods for evaluating rapidly oscillatory functions are based on the piecewise approximation by second-degree polynomials over an even number of subintervals and then integrated exactly. Such a method is due to Filon [6]. Improvement on the Filon's method was done by Flinn [7], whose approximation used fifth-degree polynomials. Stetter [16] used the idea of approximating the transformed function by polynomials in $\frac{1}{t}$. Miklosko [10] proposed to use an interpolating quadrature formula with the Chebyshev's nodes. Piessens and Poleunis [13] approximated the function by a sum of Chebyshev polynomials. Ting and Luke [17] approximated integrals whose integrands are oscillatory and contain singularities at the endpoints of the interval of integration by expanding the function in series of orthogonal polynomials over the interval of integration with respect to the weight function. Different numerical techniques like collocation and Galerkin's methods [4, 8], asymptotic method [9] generalized quadrature rule [5] and modified Clenshaw-Curtis method [18] have also been developed.

In this paper, we are concerned with the application of collocation technique to provide solution of the Fredholm integral equations of the form

$$\int_{-1}^{1} \frac{e^{i\upsilon t}}{t-x} u(t)dt = f(x), \qquad \upsilon \geq 0, \qquad i^2 = -1, \qquad -1 < x < 1, \quad (2)$$

where u is the unknown function and f is a given continuous function. The integral in (2) is oscillatory and has a singularity of Cauchy type. To deal with this pertinent problem, we present a method based on the Lagrange interpolation formula and on properties of orthogonal polynomials. The orthogonal polynomials we will consider are the Legendre polynomials. Suppose q_{n-1} is the Lagrange interpolation polynomial of degree n-1 interpolating to u at the zeros, $t_1, t_2, t_3, \cdots, t_n$, of the Legendre polynomial P_n of degree n. Then by the Lagrange interpolation formula

$$q_{n-1}(t) = \sum_{k=1}^{n} \frac{P_n(t)u(x_k)}{(t-x_k)P_n'(x_k)} + e_n(t)\,, \qquad (3)$$

where

$$e_n(t) = \frac{u^{(n+1)}(\xi_t)}{(n+1)!} \prod_{j=1}^{n} (t-x_j), \quad \xi_t \in (-1,1)$$

is the error due to the interpolation formula.

2. THE APPROXIMATE SOLUTION METHOD

By the substitution of equation (3) in equation (2), we obtain

$$\sum_{k=1}^{n}\frac{u(x_k)}{P_n^{'}(x_k)}\int_{-1}^{1}\frac{P_n(t)}{(t-x_k)(t-x)}dt+E_n(x,v)=f(x), \qquad (4)$$

where $E_n(x,v)=\int_{-1}^{1}\frac{e^{ivt}e_n(t)}{t-x}dt$ is the error due to the quadrature rule. Subsequently, we shall obtain a bound for $E_n(x.v)$. Applying the Christoffel-Darboux formula [1] to equation (4) gives us

$$\sum_{k=1}^{n}\frac{u(t_k)\rho_n\phi_{n+1}}{P_n^{'}(t_k)\phi_n P_{n+1}(t_k)}\sum_{m=0}^{n-1}\frac{P_m(t_k)\mathcal{Z}_m(x,v)}{\rho_m}=-f(x), \qquad (5)$$

where

$$\mathcal{Z}_m(x,v)=\int_{-1}^{1}\frac{e^{ivt}P_m(t)}{t-x}dt \qquad (6)$$

$\rho_n=\int_a^b w(t)P_n^2(t)dt$, $P_n(t)=\phi_n t^n+\cdots+\phi_0$ and ϕ_n is the coefficient of the term t^n in $P_n(t)$. But

$$P_n^{'}(x)=\frac{nxP_n(x)-nP_{n-1}(x)}{x^2-1}, \qquad x\neq\pm 1. \qquad (7)$$

Thus, from equation (5) we have

$$\sum_{k=1}^{n}\frac{u(t_k)(t_k^2-1)}{nP_{n-1}(t_k)(n+1)P_{n+1}(t_k)}\sum_{m=0}^{n-1}(2m+1)P_m(t_k)\mathcal{Z}_m(x,v)=f(x). \qquad (8)$$

The Legendre polynomials $P_n(x)$ satisfy the recurrence relation

$$(1+l)P_{l+1}(x)-(2l+1)xP_l(x)+lP_{l-1}(x)= \qquad (9)$$

and from equations (6) and (9), we can obtain [12]

$$(1+l)\mathcal{Z}_{l+1}(x,v)=(2l+1)x\mathcal{Z}_l(x,v)-l\mathcal{Z}_{l-1}(x,v)+(2l+1)\tilde{\mathcal{Z}}_l(v), \qquad (10)$$

where

$$\tilde{\mathcal{Z}}_l(v)=\int_{-1}^{1}e^{ivt}P_l(t)dt \qquad (11)$$

and from [14], $\tilde{\mathcal{Z}}(v)$ can be defined as

$$\text{Re}\left[\tilde{\mathcal{Z}}(v)\right]=\int_{-1}^{1}cos(vt)P_l(t)dt=2(-1)^k j_{2k}(v),\ l=2k,\ k=0,1,\cdots \qquad (12)$$

$$\text{Im}\left[\tilde{\mathcal{Z}}(v)\right]=\int_{-1}^{1}sin(vt)P_l(t)dt=2(-1)^k j_{2k+1}(v), \qquad l=2k+1, \qquad k=0,1,\cdots$$

$j_k(x)$ are the spherical Bessel functions of the first kind which can be evaluated as in [1 Eq. 10.5]. Let x_j, $j=1,\cdots,n$ defined as

$$x_j=-1+\frac{2}{n+2}(j+1) \qquad (13)$$

be the collocation points. By placing these collocation points in equation (8), we get

$$\sum_{k=1}^{n}\frac{u(t_k)(t_k^2-1)}{nP_{n-1}(t_k)(n+1)P_{n+1}(t_k)}\sum_{m=0}^{n-1}(2m+1)P_m(t_k)\mathcal{Z}_m(x_j,v)$$
$$=f(x_j),\ j=1,2,\cdots,n \qquad (14)$$

which can be written in matrix form as

$$A\,\mathbf{u}=\mathbf{c}, \qquad (15)$$

where

$$A = \frac{u(t_k)(t_k^2-1)}{nP_{n-1}(t_k)(n+1)P_{n+1}(t_k)} \sum_{m=0}^{n-1} (2m+1)P_m(t_k)\mathcal{Z}_m\big(x_j, \upsilon\big),$$

$$\mathbf{c}^T = [f(x_1), \cdots, f(x_n)], \qquad \mathbf{u}^T = [u(x_1), \cdots, u(x_n)]$$

2.1. Techniques in Evaluating $\mathcal{Z}_n(x, \upsilon)$.

According to Abramowitz and Stegun [1], Legendre polynomial $P_l(x)$ satisfy the recurrence relation [12]

$$P_{l+1}(x) = (A_l + B_l x)P_l(x) - C_l P_{l-1}(x), \;\; l = 0,1,\cdots \qquad (15)$$

with $B_l > 0,\; C_l > 0,\; P_0 = 1,\; P_1(x) = A_0 + B_0 x,\; P_{-1} = 0$. By making use of formula (11) and (15), we can write [12]

$$\mathcal{Z}_{l+1}(x,\upsilon) = (A_l + B_l x)\mathcal{Z}_l(x,\upsilon) - C_l\mathcal{Z}_{l-1}(x,\upsilon) + B_l\tilde{\mathcal{Z}}_l(\upsilon). \qquad (16)$$

The starting values are

$$\mathcal{Z}_0(x,\upsilon) = \int_{-1}^{1} \frac{e^{i\upsilon t}}{t-x} dt \qquad (17)$$

and with the help of equation (16) we have

$$\mathcal{Z}_1(x,\upsilon) = (A_0 + xB_0)\mathcal{Z}_0(x,\upsilon) + 2B_0\frac{sin\upsilon}{\upsilon}, \qquad (18)$$

where $A_0,\; B_0$ are gotten from the coefficients in $P_1(x) = A_0 + B_0 x$. From Okecha [12], we have

$$\mathrm{Re}[\mathcal{Z}_0(x,\upsilon)] = \int_{-1}^{1} \frac{cos\upsilon t}{t-x} dx = cos(\upsilon x)Ci(w_1) - sin(\upsilon x)Si(w_1) + sin(\upsilon x)Si(w_2) - cos(\upsilon x)Ci(w_2)$$

$$\mathrm{Im}[\mathcal{Z}_0(x,\upsilon)] = \int_{-1}^{1} \frac{sin\upsilon t}{t-x} dt = cos(\upsilon x)Si(w_1) + sin(\upsilon x)Ci(w_1) - cos(\upsilon x)Si(w_2) - sin(\upsilon x)Ci(w_2) \qquad (19),$$

where

$$w_1 = \upsilon(1-x), \;\; w_2 = -\upsilon(1+x) \qquad (20)$$

and Ci and Si are the cosine and sine integrals respectively. Furthermore, by applying equation (18),

$$\mathrm{Re}[\mathcal{Z}_1(x,\upsilon)] = \frac{2sin\upsilon}{\upsilon} + x\,\mathrm{Re}[\mathcal{Z}_0(x,\upsilon)]$$

$$\mathrm{Im}[\mathcal{Z}_1(x,\upsilon)] = x\,\mathrm{Im}[\mathcal{Z}_0(x,\upsilon)] \qquad (21)$$

2.2. Error Bound Analysis.

We shall give an error bound based on the Lagrange interpolating polynomials but first we consider the following lemma and theorem.

Lemma 1. Given any function $f(x)$ of bounded variation in $[a,b]$, there can be found a polynomial $Q_n(x)$, degree n, such that

$|f(x) - Q_n(x)| < \epsilon$, whenever $n \to \infty, \epsilon \to 0$ (Jackson's Theorem). [11]

Theorem 1. Let f be a function in $\mathcal{C}^{n+1}[-1,1]$, and let p_n be a polynomial of degree $\leq n$ that interpolates the function f at $(n+1)$ distinct points $x_0, x_1, x_2, \cdots, x_n \in [-1,1]$, then for each $x \in [-1,1]$ there exists a point $\xi_x \in [-1,1]$ such that

$$f(x) - p_n(x) = \prod_{i=0}^{n}(x - x_i)\frac{f^{(n+1)}(\xi(x))}{(n+1)!}, \text{ [15]}$$

Let $g(x)$ be the exact solution of equation (2) and $p_n(x)$ be the interpolation polynomial of g. Assume that g is sufficiently smooth, then we can write g as $g = p_n + e_n$, where e_n is the error term and expressed as

$$e_n(x) = \prod_{i=0}^{n}(x - x_i)\frac{g^{(n+1)}(\xi(x))}{(n+1)!}.$$

If $u_n(x)$ is the Lagrange polynomial series solution of equation (2), then $u_n(x)$ satisfies equation (2) on the nodes and so $u_n(x)$ and $p_n(x)$ are the solutions of $Au = c$ and $A\tilde{u} = c + \Delta c$, where

$$\Delta c \;= \int_{-1}^{1}\frac{e^{ivt}e_n(t)}{t - x_i}dt.$$

Theorem 2. Assume that $u(x)$ and $g(x)$ are Lagrange polynomial series solution and the exact solution of equation (2) respectively, and let $p_n(x)$ denote the interpolation polynomial of $g(x)$. If $A, u, \tilde{u}, c$ and Δc are defined as above, and $g(x)$ is sufficiently smooth, then

$$|g(x) - u_n(x)| \le \epsilon \,+\, N\zeta, \qquad (22)$$

where $max_{0<i<n}|u(x_i) - \tilde{u}(x_i)| \le\, N,$ [15]

Proof: By adding and subtracting $p_n(x)$ we have

$$|g(x) - u_n(x)| \le |g(x) - p_n(x)| + |u_n(x) - p_n(x)|$$
$$= |e_n(x)| + |u_n(x) -\, p_n(x)|.$$

By using equation (3) and Lemma 1, we obtain

$$|g(x) - u_n(x)| \le \epsilon + \left|\sum_{i=1}^{n} l_i(x)\,(u_n(t) -_i \tilde{u}_n(t_i))\right|$$
$$\le \epsilon + \left|\sum_{i=1}^{n} l_i(x)\right| |(u_n(t_i) - \tilde{u}_n(t_i))|$$
$$\le \epsilon + N\left|\sum_{i=1}^{n} l_i(x)\right|$$
$$= \;\epsilon \;+\; N\left|\sum_{i=1}^{n} l_i(x)\right|$$
$$\le \epsilon \;+\; N\zeta,$$

where we assume that the upper bound of $|\sum_{i=1}^{n} l_i(x)| = \left|\sum_{i=1}^{n}\frac{P_n(x)}{(x-x_i)P_n'(x_i)}\right|$ is ζ

3. NUMERICAL EXAMPLE

We consider the integral equation

$$\int_{-1}^{1}\frac{\sin(12t)}{t - x}u(t)dt = f(x), \qquad -1 < x < 1\,, \qquad (23)$$

where we have chosen

$$f(x) = \int_{-1}^{1} e^t\,\frac{\sin(12t)}{t - x}dt$$

so that the exact solution will be $u(x) = e^x$. The interpolation points are chosen to be the zeros of the Legendre polynomial, $P_6(x)$ of degree 6

$$t_1 = 0.238619186083197, \quad t_2 = -0.238619186083197$$
$$t_3 = 0.661209386466265, \quad t_4 = -0.661209386466265$$
$$t_5 = 0.932469514203152, \quad t_6 = -0.932469514203152 \qquad (24)$$

Furthermore, the collocation points are chosen to be

$$x_i = -1 + \tfrac{1}{4}(i+1), \quad i = 1, 2, 3, 4, 5, 6 \qquad (25)$$

We use equation (14) and set $n = 6$ to obtain

$$\sum_{k=1}^{6} \frac{u(t_k)(t_k^2 - 1)}{42 P_5(t_k) P_7(t_k)} \sum_{m=0}^{5} (2m+1) P_m(t_k) \mathcal{Z}_m(x_i, 12)$$
$$= \sum_{k=1}^{6} \frac{e^{t_k}(t_k^2 - 1)}{42 P_5(t_k) P_7(t_k)} \sum_{m=0}^{5} (2m+1) P_m(t_k)\, \mathcal{Z}_m(x_i, 12) \qquad (26)$$

$\mathcal{Z}_m(x_i, 12)$ is evaluated at the collocation points defined in relation (25) by using equations (16), (17), and (18). By making use of equation (12) we obtain

$$\tilde{\mathcal{Z}}_1(12) = 2j_1(12), \qquad \tilde{\mathcal{Z}}_3(12) = -2j_3(12), \qquad \tilde{\mathcal{Z}}_5(12) = 2j_5(12)$$

The spherical Bessel functions of the first kind, $j_k(x)$ can be evaluated as follows [1 Eqn. 10.1.11]

$$j_0(z) = \frac{\sin z}{z}, \qquad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$
$$j_2(z) = \left(-\frac{1}{z} + \frac{3}{z^3}\right) \sin z - \frac{3}{z^2} \cos z \qquad (27)$$

The spherical Bessel functions of the first kind satisfy the following recurrence relation [2]

$$j_{n+1}(z) = \frac{2n+1}{z} j_n(z) - j_{n-1}(z), \qquad n \in \mathcal{Z} \qquad (28)$$

With the help of equation (27), the recurrence relation (28), and Matlab software, the different values of spherical Bessel functions are got. The Si and Ci in the evaluation of $\mathcal{Z}_m(x_i, 12)$ are evaluated from a truncated infinite series defined as

$$Si(z) = \sum_{n1}^{50} \frac{(-1)^{n-1} z^{2(n-1)+1}}{(2(n-1)+1)(2(n-1)+1)!}$$
$$Ci(z) = \gamma + \ln|z| + \sum_{n=1}^{50} \frac{(-1)^n z^{2n}}{(2n)(2n)!} \,, \qquad (29)$$

where $\gamma = 0.5772156649$ is the Euler's constant. By solving the equation (26), we obtain the results in Table 1.

Table 1: Approximation for $\int_{-1}^{1}\frac{\sin(12t)}{t-x}u(t)dt=\int_{-1}^{1}e^{t}\frac{\sin(12t)}{t-x}dt$

t_k	**Approx. (u)**	**Exact(u)**	**Abs. Error**
0.238619186083197	1.269495716853893	1.269495003157037	0.0000007136968593
-0.238619186083197	0.787712023380540	0.787714798020595	0.000002774640055
0.661209386466265	1.937132192317572	1.937133661565611	0.000001469248039
-0.661209386466265	0.516219270254926	0.516226639307785	0.000007369052859
0.932469514203152	2.540784657700185	2.540775918748306	0.000008738951879
-0.932469514203152	0.393534762970032	0.393580556483172	0.000032926782852

From the absolute errors shown on Table 1, we can see that the presented method is accurate and efficient and can be improved by increasing n.

4. CONCLUSION

Okecha [12] developed algorithms based on the modified Lagrangian interpolation formula, Legendre polynomial and the Christoffel-Darboux formula to evaluate Cauchy principal value integrals of oscillatory kind. Numerical examples were used to prove the accuracy of the formulae developed. Motivated by this fact, we developed an algorithm to solve singular Fredholm integral equations of the first kind with oscillatory trigonometric kernel and our test example used was derived from example (c) of Okecha [12]. The results we got shows the convergence of the solution and this can be improved by increasing n

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