The cocycle for the non-autonomous stochastic damped wave equations with white noise

Abstract: This paper is devoted to the cocycle of solutions of the non-autonomous stochastic damped wave equations with multiplicative white noise defined on unbounded domains. And we obtain the existence of a pullback absorbing set of the cocycle in a certain parameter region.

Keywords: stochastic damped wave equations, cocycle, pullback absorbing set

1 Introduction

In this paper, we study the asymptotic behavior of solutions for the following non-autonomous stochastic damped wave equation with multiplicative white noise defined on the unbounded domain $\mathbb{R}^n$:

$$du_t + \alpha du + (\beta u + f(u) - \Delta u)dt = g(x,t)dt + \varepsilon u \circ d\omega, \tag{1.1}$$

with initial conditions

$$u(x,\tau) = u_\tau(x), \quad u_t(x,\tau) = u_\tau(x), \tag{1.2}$$

where $x \in \mathbb{R}^n$ with $1 \le n \le 3$, $t > \tau, \tau \in \mathbb{R}, x \in \mathbb{R}^n$, α and β are positive constants, ε are constants, and g is a time-dependent driving force $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, and ω is a two-sided real-valued Wiener process on a probability space. The stochastic equation (1.1) is understood in the sense of Stratonovich's integration.

Stochastic damped wave equations have been used as models to study the phenomena of s-tochastic resonance in physics, where g is a time-dependent input signal and ω is a Wiener process used to test the impact of stochastic fluctuations on g ([1]-[3]). Especially, if $\varepsilon = 0$, Eq. (1.1) is a deterministic wave equations which have been studied by many experts, including global attractors, uniform attractors and pullback attractors, see e.g., [4]-[5] and the references therein. And when the function g does not depend on time, then equation (1.1) becomes an autonomous stochastic wave equation.

The equation (1.1) is a non-autonomous equation that the external term g is time-dependent, and assuming that the external forces term $g(x,t)$ satisfies:

$$\int_{-\infty}^{0} e^{\delta s}\|g(\cdot,\tau+s)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{1.3}$$

It shows that $g(\cdot,t)$ is not bounded in $L^2(\mathbb{R}^n)$ when $t \to \pm\infty$.

In the present paper, we will investigate the long time behavior of non-autonomous stochastic damped wave equation (1.1) with multiplicative white noise. We will use the decomposition technique with the idea of uniform estimates on the tails of solutions to investigate the existence of a random attractor of the stochastic damped wave equation with multiplicative noise defined on $\mathbb{R}^n$.

2 Preliminaries

Let $(\Omega,\mathcal{F},P)$ be a probability space, and $(X,\|\cdot\|_X)$ be a separable Banach space whose Borel σ-algebra is denoted by $\mathscr{B}(X)$.

Defintion 2.1 Let a mapping $\theta_t : \mathbb{R}\times\Omega \to \Omega$ be $(\mathscr{B}(\mathbb{R})\times\mathcal{F},\mathcal{F})$-measurable such that θ_0 is the identity on Ω, $\theta_{t+s} = \theta_t\circ\theta_s$ for all $t,s\in\mathbb{R}$, and $P\theta_t = P$ for all $t\in\mathbb{R}$. A mapping $\Phi : \mathbb{R}^+\times\Omega\times X \to X$ is called a random dynamical system on X over $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$, if for all $\omega\in\Omega$ and $t,s\in\mathbb{R}^+$ the following conditions are satisfied:

(i) $\Phi(t,\omega,\cdot) : \mathbb{R}^+\times\Omega\times X \to X$ is a $(\mathscr{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathscr{B}(X),\mathscr{B}(X))$-measurable mapping;

(ii) $\Phi(0,\omega,\cdot)$ is the identity on X;

(iii) $\Phi(t+s,\omega,\cdot) = \Phi(t,\theta_s\omega,\cdot)\circ\Phi(s,\omega,\cdot)$;

(iv) $\Phi(t,\omega,\cdot) : X \to X$ is continuous.

Defintion 2.2 Let Φ be a random dynamical system on a Banach space X over $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$.

(1) A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $\{\theta_t\}_{t\in\mathbb{R}}$ if for P-a.e. $\omega\in\Omega$,

$$\lim_{t\to\infty} e^{-\zeta t} d(B(\theta_{-t}\omega)) = 0 \quad \text{for all } \ \zeta > 0,$$

where $d(B) = \sup_{x\in B}\|x\|_X$.

(2) Let $\mathcal{D}$ be a collection of random subsets of X. The parametric dynamical system Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X, if for any P-a.e. $\omega\in\Omega$ and any sequences $t_n\to\infty$, $x_n\in B(\theta_{-t_n}\omega)$ with $B = \{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$, the sequence $\{\Phi(t_n,\theta_{-t_n}\omega,x_n)\}$ has a convergent subsequence in X.

(3) Let $\mathcal{D}$ be a collection of random subsets of X and $K = \{K(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$. Then K is called a random absorbing set for Φ in $\mathcal{D}$ if for every $B\in\mathcal{D}$ and P-a.e. $\omega\in\Omega$, there exists $t_B(\omega) > 0$ such

that

$$\Phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\subset K(\omega),\quad \text{for all } t\geq t_B(\omega).$$

In this paper, we will take $\mathcal{D}$ to be the universe of all tempered random subsets of the product Hilbert space $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and prove that the random dynamical system generated by the stochastic wave equation (1.1) on $\mathbb{R}^n$ has a $\mathcal{D}$-pullback absorbing set.

3 The cocycle for the stochastic damped wave equation

In this section, we define a continous cocycle for problem (1.1)-(1.2). Let $\xi=u_t+\delta u$, where δ is a positive number to be determined, then (1.1)-(1.2) can be rewritten as the equivalent system

$$\begin{cases} u_t+\delta u=\xi, \\ \xi_t+(\alpha-\delta)\xi+(\delta^2-\alpha\delta)u-\Delta u+f(u)=g(x,t)+\varepsilon u\circ\frac{d\omega}{dt}, \\ u(x,\tau)=u_0(x),\quad \xi(x,\tau)=\xi_0=u_1(x)+\delta u_0(x). \end{cases} \tag{3.1}$$

There exist a non-negative constant $c_1\geq 0$ such that

$$|f(u_1)-f(u_2)|\leq c_1|u_1-u_2|,\quad f(0)=0,\quad \forall u_1,u_2\mathbb{R}. \tag{3.2}$$

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space as in Section 2. Define $\{\theta_t\}_{t\in\mathbb{R}}$ on Ω by $\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t)$ for all $\omega\in\Omega$ and $t\in\mathbb{R}$, then $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system defined by [6].

To define a cocycle for problem (3.1), we need to convert the system to a deterministic one with random parameters. Now we introduce an Ornstein-Uhlenbeck process given by the Brownian motion. Put

$$z(\theta_t\omega):=-\alpha\int_{-\infty}^0 e^{\alpha s}(\theta_t\omega)(s)ds,\quad \omega\in\Omega,\quad t\in\mathbb{R}, \tag{3.3}$$

and solves the Itô equation

$$dz+\alpha z dt=d\omega(t). \tag{3.4}$$

From [1], it is known that the random variable $|z(\omega)|$ is tempered, and there is a θ_t-invariant set $\widetilde{\Omega}\subseteq\Omega$ of $\mathbb{P}$ measure such that $|z(\theta_t\omega)|$ is continuous in t for every $\omega\in\widetilde{\Omega}$. For convenience, we write $\widetilde{\Omega}$ as Ω.

Let v be a new variable given by $v(x,t)=\xi(x,t)-\varepsilon u(x,t)z(\theta_t\omega)$. By (3.1), we have

$$\begin{cases} u_t=v+\varepsilon uz(\theta_t\omega)-\delta u, \\ v_t+(\alpha-\delta)v+(\delta^2-\alpha\delta+A)u+\varepsilon(v-2\delta u+\varepsilon uz(\theta_t\omega))z(\theta_t\omega)+f(u)=g(x,t), \\ u(x,\tau)=u_0(x),\quad v(x,\tau)=v_0(x), \end{cases} \tag{3.5}$$

where $A = -\Delta, v_0 = u_1 + \delta u_0 - \varepsilon z(\theta_\tau \omega) u_0$.

Let $E = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, endowed with the usual norm

$$\|Y\|_{H^1 \times L^2} = (\|v\|^2 + \|u\|^2 + \|\nabla u\|^2)^{\frac{1}{2}}, \quad for \;\; Y = (u, v)^{\mathcal{T}} \in E, \tag{3.6}$$

where $\|\cdot\|$ denotes the usual norm in $L^2(\mathbb{R}^n)$ and $\mathcal{T}$ stands for the transposition.

The well-posedness of the deterministic problem (3.5) in $E = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ can be established by standard methods as in [6], [7]. One may show that under conditions (3.2), for every $\omega \in \Omega, \tau \in \mathbb{R}$ and $(u_0, v_0) \in E$, problem (3.5) has a unique solution $(u(\cdot, \tau, \omega, u_0), v(\cdot, \tau, \omega, v_0)) \in C([\tau, \infty), E)$ with $(u(\tau, \tau, \omega, u_0), v(\tau, \tau, \omega, v_0)) = (u_0, v_0)$. In addition, for $t \geq \tau$, $(u(t, \tau, \omega, u_0), v(t, \tau, \omega, v_0))$ is $(\mathcal{F}, \mathcal{B}(H^1(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable and continuous in (u_0, v_0) with respect to the norm of E.

Hence, the solution mapping can define a continuous cocycle for (3.1). Let Φ be a mapping, $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ given by

$$\Phi(t, \tau, \omega, (u_0, v_0)) = (u(t + \tau, \tau, \theta_{-\tau}\omega, u_0), v(t + \tau, \tau, \theta_{-\tau}\omega, v_0)) \tag{3.7}$$

for every $(t, \tau, \omega, (u_0, v_0)) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E$, where $v(t + \tau, \tau, \theta_{-\tau}\omega, v_0) = \xi(t + \tau, \tau, \theta_{-\tau}\omega, \xi_0) - \varepsilon z(\theta_t \omega) u(t+\tau, \tau, \theta_{-\tau}\omega, u_0)$ with $v_0 = \xi_0 - \varepsilon z(\omega) u_0$. Then Φ is a continuous cocycle over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ on E. And $\forall t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$, we have

$$\begin{aligned} \Phi(t, \tau - t, \theta_{-t}\omega, (u_0, v_0)) &= (u(\tau, \tau - t, \theta_{-\tau}\omega, u_0), v(\tau, \tau - t, \theta_{-\tau}\omega, v_0)) \\ &= (u(\tau, \tau - t, \theta_{-\tau}\omega, u_0), \xi(\tau, \tau - t, \theta_{-\tau}\omega, \xi_0) - \varepsilon z(\omega) u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)). \end{aligned} \tag{3.8}$$

When deriving uniform estimates on solutions, we need the following condition on g in (1.1):

$$\int_{-\infty}^{0} e^{\delta s} \|g(\cdot, \tau + s)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}, \tag{3.9}$$

and

$$\lim_{k \to \infty} \int_{-\infty}^{0} e^{\delta s} \int_{|x| \geq k} \|g(x, \tau + s)\|^2 dx ds = 0. \tag{3.10}$$

The condition (3.9) shows that $g(\cdot, t)$ is not bounded in $L^2(\mathbb{R})$ when $t \to \pm\infty$.

Let B be a bounded nonempty subset of E, and denote by $\|B\| = \sup_{\varphi \in B} \|\varphi\|_E$. Suppose $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of E satisfying, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{s \to -\infty} e^{\delta s} \|D(\tau + s, \theta_s \omega)\|^2 = 0. \tag{3.11}$$

Denote by $\mathcal{D}$ the collection of all families of bounded nonempty subsets of E,

$$\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies } (3.11)\}. \tag{3.12}$$

It is evident that $\mathcal{D}$ is neighborhood closed. We will obtain the existence of pullback $\mathcal{D}$-attractor in E.

4 Pullback absorbing set

In this section, we derive uniform estimates on the solutions of the stochastic damped wave equations (3.1) defined on $\mathbb{R}^n$ when $t \to \infty$. These estimates are necessary for proving the existence of bounded absorbing sets and the asymptotic compactness of the system. In particular, we will show that the tails of the solutions for large space variables are uniformly small when time is sufficiently large.

We define a new norm $\|\cdot\|_E$ by

$$\|Y\|_E = (\|v\|^2 + (\delta^2 - \alpha\delta)\|u\|^2 + \|\nabla u\|^2)^{\frac{1}{2}}, \tag{4.1}$$

for $Y = (u, v)^{\mathcal{T}} \in E$. It is easy to check that $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H^1 \times L^2}$ in (3.6).

Lemma 4.1 *Assume that $\alpha - 3\delta > 0$, (3.2) and (3.9) hold. Let $\tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau, \omega), \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then there exists $T = T(\tau, \omega, D) > 0$, for all $t \geq T$, the solution of problem (3.5) satisfies*

$$Y(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega)) \leq R(\tau, \omega),$$

and $R(\tau, \omega)$ is given by

$$R(\tau,\omega) = M \int_{-\infty}^{0} \exp\{2\int_0^s [\delta - |\varepsilon||z(\theta_r\omega)| - \beta_1(\frac{1}{2}\varepsilon^2|z(\theta_r\omega)|^2 + \beta_2|\varepsilon||z(\theta_r\omega)|)]dr\}\|g(\cdot, s+\tau)\|^2 ds, \tag{4.2}$$

where M is positive constant independent of τ, ω, D and ε.

Proof. Taking the inner product of the second equation of (3.5) with v in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|^2 &= (\delta - \alpha - \varepsilon z(\theta_t\omega))\|v\|^2 - (\delta^2 - \alpha\delta)(u, v) - (Au, v) \\ &+ \big(\varepsilon z(\theta_t\omega)(2\delta - \varepsilon z(\theta_t\omega))u, v\big) + (g(x,t), v) - (f(u), v). \end{aligned} \tag{4.3}$$

By the first equation of (3.5), we have

$$v = u_t - \varepsilon u z(\theta_t\omega) + \delta u, \tag{4.4}$$

then substituting the above v into the second and third terms on the right-hand side of (4.1), we find that

$$\begin{aligned}(u,v) &= (u, u_t + \delta u - \varepsilon z(\theta_t\omega)u)\\ &= \frac{1}{2}\frac{d}{dt}\|u\|^2 + \delta\|u\|^2 - \varepsilon z(\theta_t\omega)\|u\|^2\\ &\geq \frac{1}{2}\frac{d}{dt}\|u\|^2 + \delta\|u\|^2 - |\varepsilon|\cdot|z(\theta_t\omega)|\cdot\|u\|^2, \end{aligned}\tag{4.5}$$

and

$$\begin{aligned}-(Au,v) &= -(\nabla u, \nabla v)\\ &= -(\nabla u, \nabla u_t + \delta\nabla u - \varepsilon z(\theta_t\omega)\nabla u)\\ &= -\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 - \delta\|\nabla u\|^2 + \varepsilon z(\theta_t\omega)\|\nabla u\|^2\\ &\leq -\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 - \delta\|\nabla u\|^2 + |\varepsilon|\cdot|z(\theta_t\omega)|\cdot\|\nabla u\|^2. \end{aligned}\tag{4.6}$$

Using the Cauchy-Schwartz inequality and the Young inequality, we have

$$\begin{aligned}\big(\varepsilon z(\theta_t\omega)(2\delta - \varepsilon z(\theta_t\omega))u, v\big) &= (2\delta\varepsilon z(\theta_t\omega) - \varepsilon^2 z^2(\theta_t\omega))(u,v)\\ &\leq (2\delta|\varepsilon|\cdot|z(\theta_t\omega)| + \varepsilon^2\cdot|z(\theta_t\omega)|^2)\|u\|\cdot\|v\|\\ &\leq (\delta|\varepsilon|\cdot|z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2\cdot|z(\theta_t\omega)|^2)(\|u\|^2+\|v\|^2), \end{aligned}\tag{4.7}$$

and

$$(g,v) \leq \|g\|\cdot\|v\| \leq \frac{\|g\|^2}{2(\alpha-\delta)} + \frac{\alpha-\delta}{2}\|v\|^2, \tag{4.8}$$

and by (3.2),

$$\begin{aligned}&-(f(u),v) \leq c_1(u, u_t + \delta u - \varepsilon z(\theta_t\omega)u)\\ \leq\ & c_1\frac{d}{dt}\|u\|^2 + c_1\delta|u|^2 + |\varepsilon|\cdot|z(\theta_t\omega)||u|^2. \end{aligned}\tag{4.9}$$

By (4.5)-(4.9), it follows from (4.3) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2 - (\delta - \alpha - \varepsilon z(\theta_t\omega))\|v\|^2 + \frac{1}{2}(c_1+\delta^2-\alpha\delta)\frac{d}{dt}\|u\|^2 + \delta(c_1+\delta^2-\alpha\delta)\|u\|^2\\ &-|\varepsilon||z(\theta_t\omega)|(c_1+\delta^2-\alpha\delta)\|u\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 - (-\delta + |\varepsilon||z(\theta_t\omega)|)\|\nabla u\|^2\\ \leq\ & (\delta|\varepsilon|\cdot|z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2\cdot|z(\theta_t\omega)|^2)(\|u\|^2+\|v\|^2) + \frac{\alpha-\delta}{2}\|v\|^2 + \frac{\|g\|^2}{2(\alpha-\delta)}. \end{aligned}\tag{4.10}$$

Then

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|v\|^2+(c_1+\delta^2-\alpha\delta)\|u\|^2+\|\nabla u\|^2)+\delta(\|v\|^2+(c_1+\delta^2-\alpha\delta)\|u\|^2+\|\nabla u\|^2)\\
\leq\ & (\delta|\varepsilon|\cdot|z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2\cdot|z(\theta_t\omega)|^2)(\|u\|^2+\|v\|^2)+\frac{3\delta-\alpha}{2}\|v\|^2+\frac{\|g\|^2}{2(\alpha-\delta)}\\
&+|\varepsilon||z(\theta_t\omega)|(\|v\|^2+(c_1+\delta^2-\alpha\delta)\|u\|^2+\|\nabla u\|^2). \qquad (4.11)
\end{aligned}
$$

From (4.11), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|v\|^2+(c_1+\delta^2-\alpha\delta)\|u\|^2+\|\nabla u\|^2)\\
\leq\ & -[\delta-|\varepsilon|\cdot|z(\theta_t\omega)|-\beta_1(\frac{1}{2}\varepsilon^2\cdot|z(\theta_t\omega)|^2+\beta_2|\varepsilon||z(\theta_t\omega)|)](\|v\|^2+(c_1+\delta^2-\alpha\delta)\|u\|^2+\|\nabla u\|^2)\\
&+\frac{\|g\|^2}{2(\alpha-\delta)}, \qquad (4.12)
\end{aligned}
$$

where $\beta_1=1+\frac{1}{c_1+\delta^2-\alpha\delta}$, $\beta_2=\frac{3\delta+\alpha}{2}$.

Denote

$$
\Gamma(t,\omega)=\delta-|\varepsilon|\cdot|z(\theta_t\omega)|-\beta_1(\frac{1}{2}\varepsilon^2\cdot|z(\theta_t\omega)|^2+\beta_2|\varepsilon||z(\theta_t\omega)|). \qquad (4.13)
$$

Using the Gronwall inequality to integrate (4.12) over $(\tau-t,\tau)$ with $t\geq 0$, we get

$$
\begin{aligned}
&\|v(\tau,\tau-t,\omega,v_0)\|^2+(c_1+\delta^2-\alpha\delta)\|u(\tau,\tau-t,\omega,u_0)\|^2+\|\nabla u(\tau,\tau-t,\omega,u_0)\|^2\\
\leq\ & (\|v_0\|^2+(c_1+\delta^2-\alpha\delta)\|u_0\|^2+\|\nabla u_0)\|^2 e^{2\int_\tau^{\tau-t}\Gamma(s,\omega)ds}\\
&+c\int_\tau^{\tau-t} e^{2\int_\tau^s\Gamma(r,\omega)dr}\|g(\cdot,s)\|^2ds. \qquad (4.14)
\end{aligned}
$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.14), we obtain, for every $t\in\mathbb{R}^+,\tau\in\mathbb{R},\omega\in\Omega$,

$$
\begin{aligned}
&\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(c_1+\delta^2-\alpha\delta)\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
\leq\ & (\|v_0\|^2+(c_1+\delta^2-\alpha\delta)\|u_0\|^2+\|\nabla u_0)\|^2 e^{2\int_\tau^{\tau-t}\Gamma(s-\tau,\omega)ds}\\
&+c\int_\tau^{\tau-t} e^{2\int_\tau^s\Gamma(r-\tau,\omega)dr}\|g(\cdot,s)\|^2ds. \qquad (4.15)
\end{aligned}
$$

then

$$
\begin{aligned}
&\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(c_1+\delta^2-\alpha\delta)\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
\leq\ & (\|v_0\|^2+(c_1+\delta^2-\alpha\delta)\|u_0\|^2+\|\nabla u_0)\|^2 e^{2\int_0^{-t}\Gamma(s,\omega)ds}\\
&+c\int_{-t}^{0} e^{2\int_0^s\Gamma(r,\omega)dr}\|g(\cdot,s+\tau)\|^2ds. \qquad (4.16)
\end{aligned}
$$

Since $|z(\theta_t\omega)|$ is stationary and ergodic (see [8]), we get from (3.3) and the ergodic theorem that

$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}|z(\theta_r\omega)|dr=\mathbf{E}(|z(\theta_r\omega)|)=\frac{1}{\sqrt{\pi\delta}},$$
$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}|z(\theta_r\omega)|^2dr=\mathbf{E}(|z(\theta_r\omega)|^2)=\frac{1}{2\delta}. \tag{4.17}$$

By (4.16), there exists $T_1(\omega)>0$ such that for all $t\ge T_1(\omega)$,

$$\int_{-t}^{0}|z(\theta_r\omega)|dr=\frac{2}{\sqrt{\pi\delta}}t,$$
$$\int_{-t}^{0}|z(\theta_r\omega)|^2dr=\frac{1}{\delta}t. \tag{4.18}$$

Let ε satisfy

$$|\varepsilon|<\frac{2\sqrt{\delta}(\beta_1\beta_2+1)+\sqrt{4\delta(\beta_1\beta_2+1)^2+\pi\beta_1\delta^2}}{\beta_1\sqrt{\pi}}, \tag{4.19}$$

We have

$$e^{2\int_0^s\Gamma(r,\omega)dr}\le e^{2(\frac{\delta}{2})s}=e^{\delta s},\quad \forall s\le -T_1. \tag{4.20}$$

Since $|z(\theta_s\omega)|$ is tempered, by (3.9) and (4.17), we have the following integral is convergent,

$$R_1^2(\tau,\omega)\quad=\quad 2c\int_{-\infty}^{0}e^{2\int_0^s\Gamma(r,\omega)dr}(\|g(\cdot,s+\tau)\|^2)ds. \tag{4.21}$$

Since $D\in\mathcal{D}$ and $(u_0,v_0)\in D(\tau-t,\theta_{-t}\omega)$, for all $t\ge T_1$, we get from (4.18)-(4.20),

$$\begin{aligned}&(\|v_0\|^2+(c_1+\delta^2-\alpha\delta)\|u_0\|^2+\|\nabla u_0)\|^2e^{2\int_0^{-t}\Gamma(s,\omega)ds}\\ \le\;& ce^{-\delta t}(\|v_0\|^2+\|u_0\|^2+\|\nabla u_0\|)\\ \le\;& ce^{-\delta t}(\|D(\tau-t,\theta_{-t}\omega\|^2)\to 0,\quad as\quad t\to+\infty.\end{aligned} \tag{4.22}$$

From (4.1), (4.16), (4.21) and (4.22), there exists $T_2=T_2(\tau,\omega,D)\ge T_1$ such that for all $t\ge T_2$,

$$\|Y(\tau,\tau-t,\theta_{-\tau}\omega,Y_0(\theta_{-\tau}\omega))\|_E^2\le R_1^2(\tau,\omega). \tag{4.23}$$

So, the proof is completed. □

References

[1] L. Arnold, Random Dynamical Systems, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[2] X. Fan, Attractors for a damped stochastic wave equation of sine-Gordon type with sublinear multiplicative noise, Stochastic Analysis and Applications, 24(2006), 767-793.

[3] Y. Lv and W. Wang, Limiting dynamics for stochastic wave equations, Journal Differential Equations, 244(2008), 1-23.

[4] T. Caraballo, G. Lukaszewicz, J. Real, Pullback attractors for asymptotically compact nonautonomous dynamical systems, Nonlinear Analysis: Theory, Methods and Applications, 64(2006), 484-498.

[5] R. Jones, B. Wang, Asymptotic behavior of a class of stochastic nonlinear wave equations with dispersive and dissipative terms, Nonlinear Analysis: Real World Applications, 14(2013), 1308-1322.

[6] B. Wang, Random attractors for non-autonomous stochasitic wave equations with multiplicative noise, Discrete and continuous dynamical systems, 34(1)(2014), 269-300.

[7] Z. Wang, S. Zhou, A. Gu, Random attractor for a stochastic damped wave equation with multiplicative noise on unbounded domains, Nonlinear Analysis: Real World Applications, 12(2011), 3468-3482.

[8] H. Li, Y. You, J. Tu, Random attractors and averaging for non-autonomous stochastic wave equations with nonlinear damping, J. Differential Equations, 258 (2015): 148-190.