**Original Research Article**

# On the Analytical Approximation of the Nonlinear Cubic Oscillator by an Iteration Method


## Abstract

A modified approximate analytic solution of the cubic nonlinear oscillator "$\ddot{x} + x^3 = 0$" has been obtained based on an iteration procedure. Here we have used the truncated Fourier series in each iterative step. The approximate frequencies obtained by this technique show a good agreement with the exact frequency. The percentage of error between exact frequency and our fifth approximate frequency is as low as 0.009%. The calculation with this technique is very easy. The modified technique accelerates the rapid convergence, reduces the error and increases the validity range of the solution.



***Keywords:*** Iteration procedure; Cubic oscillator; Nonlinearity; Nonlinear oscillations.

**AMS subject classification**: 34A34, 34B99.




## 1. Introduction:

Most nonlinear phenomena are models of our real-life problems. Nonlinear evolution of equations is widely used as models to describe complex physical phenomena in various fields of science, especially in fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, vibrations, heat transfer and so on. Nonlinear systems are classified differently and 'nonlinear cubic oscillator' is one of them and has its own merit. In this situation perturbation method, homotopy method, homotopy perturbation method, Harmonic balance method, Rational harmonic balance method, Parameter expansion, Iteration method, etc are used to find approximate solutions to nonlinear problems.

The perturbation method is the most widely used method in which the nonlinear term is small. The method of Lindstedt-Poincare (LP) (Nayfeh, 1973; Krylov & Bogoliubov, 1947; Bogoliubov & Mitropolskii, 1961), Homotopy method (Wu *et al.,* 2006; Turkyilmazoglu, 2010, 2011, 2012), Homotopy perturbation method (Gottlieb, 2006) and Differential Transform method (Alquran & Al-Khamaiseh, 2010; Alquran & Doğan, 2010; Alquran & Al-Khaled, 2012) are most important among all perturbation methods. An important aspect of various perturbation methods is their relationship with each other. Among them, Krylov and Bogoliubov (1947) are certainly to be found most active. In most treatments of nonlinear oscillations by perturbation methods only periodic oscillations are treated, transients are not considered. Krylov & Bogoliubov (1947) have introduced a new perturbation method to discuss transients.

Harmonic balance method is another technique for finding the periodic solutions of a nonlinear system. If a periodic solution does not exist of an oscillator, it may be sought in the form of Fourier series and its coefficients are determined by requiring the series to satisfy the equation of motion. HB method which is originated by Mickens (1984) and farther work has been done by Mickens, 1991, 1996, 2005 ; Lim & Wu, 2002 ; Hu, 2004; Hu & Tang, 2007; Wu *et al.*, 2006; Gottlieb, 2006; Alam *et al.*, 2011; Haque *et al.* 2011;  Hosen , 2013 and so on for solving the strong nonlinear problems. However, in order to avoid solving an infinite system of algebraic equations, it is better to approximate the solution by a suitable finite sum of the trigonometric function. This is the main task of the harmonic balance method. Thus approximate solutions of an oscillator are obtained by harmonic balance method using a suitable truncation Fourier series. The method is capable of determining an analytic approximate solution to the nonlinear oscillator valid even for the case where the nonlinear terms are not small i.e., no particular parameter need exist.

The parameter expansion methodology was introduced in a paper by Senator & Bapat (1993). Subsequently, it was extended in a publication of Mickens (1999). However, the full generalization of this concept was done by He (2002). Recently Mickens (2010) use this method in his book Truly Nonlinear Oscillation and also use this method by Xu, 2007; Zengin *et al.,* 2008, etc.

Rational harmonic balance approximation (Mickens, 1986; Mickens & Semwogerere, 1996; Beléndez *et al.,* 2009) is a useful alternative procedure for calculating a second-order harmonic balance approximation is a rational approximation. This technique was introduced by Mickens (1986) and has been extended in its applications by Beléndez *et al.,* (2009). A major advantage of rational approximation is that it gives an implicit inclusion of all the harmonics contributing to the periodic solutions.

Recently, some authors use an iteration procedure (Mickens, 1987, 2010; Haque *et al.,* 2013, 2014; Haque, 2014; Haque & Hossain, 2016) which is valid for small together with large amplitude of oscillation, to attain the approximate frequency and the harmonious periodic solution of such nonlinear problems. Besides this method, there are some methods (Matko & Šafarič, 2009; Matko, 2011; Matko & Milanović, 2014) which are used to find an approximate solution in the case of the large amplitude of oscillations.

The iterative technique is also used as a technique for calculating approximate periodic solutions and corresponding frequencies of truly nonlinear oscillators for small and as well as large amplitude of oscillation. The method was originated by R.E. Mickens in 1987. In the article Xu & Cang (2008) provided a general basis for iteration method as they are currently used to calculate the approximate periodic solutions of various nonlinear oscillatory successfully. Further, Mickens used the iterative technique to calculate a higher-order approximation to the periodic solutions of a conservative oscillator.

Here, the iteration technique for determining the approximate solution of a cubic nonlinear oscillator is presented. The obtained result will be compared with the existing various results.

## 2. Methodology

Let us suppose that the nonlinear oscillator

$$\ddot{x} + f(\ddot{x}, x) = 0, \; x(0) = A, \; \dot{x}(0) = 0, \tag{1}$$

Where over dots denote differentiation with respect to time, t.

We choose the natural frequency $\Omega$ of this system. Then adding $\Omega^2 x$ to both sides of Eq. (1), we obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\ddot{x}, x) \equiv G(x, \ddot{x}) \,. \tag{2}$$

Now, we formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \ddot{x}_k); \;\; k = 0,1,2,3,............ \tag{3}$$

Together with initial condition

$$x_0(t) = A\cos(\Omega_0 t) \tag{4}$$

Hence $x_{k+1}$ satisfies the initial conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{5}$$

At each stage of the iteration, $\Omega_k$ is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$.

The above procedure gives the sequence of solutions: $x_0(t),\ x_1(t),\ x_2(t),\cdots$.

The method can proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second.

At this point, the following observations should be noted:

(a) The solution for $x_{k+1}(t)$ depends on having the solutions for k less than $(k+1)$

(b) The linear differential equation for $x_{k+1}(t)$ allows the determination of $\Omega_k$ by the requirement that secular terms be absent. Therefore, the angular frequency, "$\Omega$" appearing on the right-hand side of Eq. (5) in the function $x_k(t)$, is $\Omega_k$.

## 3. Solution procedure

Let us consider the cubic nonlinear oscillator

$$\ddot{x} + x^3 = 0 \tag{6}$$

Now adding $\Omega^2 x$ to both sides of Equation (6), we obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x^3 \tag{7}$$

Now the iteration scheme is according to Eq. (3)

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - x_k^3 \tag{8}$$

The initial condition is rewritten as

$$x_0(t) = A\cos\theta \tag{9}$$

where $\theta = \Omega_0 t.$ For $k = 0.$ the Eq. (8) becomes

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A cos\theta - A^3\, cos^3\theta \tag{10}$$

Now expanding $cos^3\theta$ in a Fourier Cosine series, the Eq. (10) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = ({\Omega_0}^2 - 0.75A^2)A\cos\theta - 0.25A^3\cos 3\theta \tag{11}$$

To check secular terms in the solution, we have to remove $Cos\theta$ from the right-hand side of Eq. (11).

Thus we have

$$\Omega_0 = 0.8660254037844386A \tag{12}$$

Then solving Eq. (11) and satisfying the initial condition $x_1(0) = A$, we obtain

$$x_1(t) = 0.958333295A\cos\theta + 0.041666705A\cos 3\theta \qquad (13)$$

This is the first approximate solution of Eq. (6) and the related $\Omega_1$ is to be determined. The value of $\Omega_1$ will be obtained from the solution of

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - x_1^3 \qquad (14)$$

Substituting $x_1(t)$ from Eq. (13) into the right hand side of Eq. (14), we obtain

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 = {} & \Omega_1^2(0.95833295A\cos\theta + 0.041666705A\cos 3\theta) \\ & -(0.6912976924435A^3\cos\theta + 0.2774884478877286A^3\cos 3\theta \\ & +0.02994794302083226A^3\cos 5\theta)\end{aligned} \qquad (15)$$

Again avoiding secular terms in the solution of Eq. (15), now we obtain

$$\Omega_1 = 0.8493257129433129A \qquad (16)$$

Then solving Eq. (15) and satisfying initial condition, we obtain the second approximate solution,

$$x_2(t) = 0.955393886A\cos\theta + 0.04287627A\cos 3\theta + 0.0017298439A\cos 5\theta \qquad (17)$$

This is the second approximate solution of Eq. (6)

In similar way we have the third and fourth approximate solutions are

$$x_3(t) = 0.955116283A\cos\theta + 0.043038747A\cos 3\theta + 0.00184497A\cos 5\theta \qquad (18)$$

$$x_4(t) = 0.9550932806A\cos\theta + 0.043050742A\cos 3\theta + 0.001855971403A\cos 5\theta \qquad (19)$$

Whereas the frequencies $\Omega_2$, $\Omega_3$ and $\Omega_4$ are

$$\Omega_2 = 0.8474560185405289A \qquad (20)$$

$$\Omega_3 = 0.8473021830725166A \qquad (21)$$

$$\Omega_4 = 0.8472887677067594A \qquad (22)$$

Thus $\Omega_0, \Omega_1, \Omega_2, \Omega_3, \Omega_4$ respectively obtained by Eqs. (12), (16), (20), (21), (22) represent the approximation of frequencies of oscillator (6).

## 4. Results and Discussion

An Iteration method is developed based on Mickens [30] to solve ‘cubic nonlinear oscillator’. In this section, we express the accuracy of the modified technique of iteration method by comparing with the existing results from different methods and with the exact frequency of the oscillator. To show the accuracy, we have calculated the percentage errors (denoted by Er (%)) by the definitions.

$$Er = \left| 100\{\Omega_e(A) - \Omega_i(A)\} / \Omega_e(A) \right| \quad ; i = 0, 1, 2, 3, \ldots ,$$

where $\Omega_i$ represents the approximate frequencies obtained by the adopted method and $\Omega_e$ represents the corresponding exact frequency of the oscillator.

Herein we have calculated the first, second, third, fourth and fifth approximate frequencies which are denoted by $\Omega_0, \Omega_1, \Omega_2, \Omega_3$ and $\Omega_4$ respectively. A comparison among the existing results showed by Mickens (2010) with the obtained results in the following table.

It is noted that Mickens (2010) found only first approximate frequency by Parameter Expansion, and the second approximate frequencies by harmonic balance method. Mickens (2010) also presented only the second approximate frequencies by iteration method.

**Table:** Comparison of the approximate frequencies obtained by the presented technique and other existing results with exact frequency $\Omega_e$ (Mickens, 2010) of cubic nonlinear oscillator.

| Exact Frequency $\Omega_e$ | $0.847213\,A$ | | | | |
|---|---|---|---|---|---|
| Amplitude $A$ | First Approximate Frequency $\Omega_0$ Er(%) | Second Approximate Frequency $\Omega_1$ Er(%) | Third Approximate Frequency $\Omega_2$ Er(%) | Fourth Approximate Frequency $\Omega_3$ Er(%) | Fifth Approximate Frequency $\Omega_4$ Er(%) |
| Mickens(2010) Parameter Expansion | $0.866025\,A$ 2.2 | – | – | – | – |
| Mickens(2010) HB Method | $0.866025\,A$ 2.2 | 0.8489 0.20 | – | – | – |
| Mickens(2010) Iteration method | $0.866025\,A$ 2.2 | 0.849326 0.2 | – | – | – |

| | | | | | |
|---|---|---|---|---|---|
| Adopted Method | $0.866025A$ 2.2 | $0.849326A$ 0.25 | $0.847456A$ 0.03 | $0.847302A$ 0.01 | $0.847289A$ 0.009 |

## 5. Convergence and Consistency Analysis

We know the basic idea of iteration methods is to construct a sequence of solutions $x_k$ (as well as frequencies $\Omega_k$) that have the property of convergence

$$x_e = \lim_{k \to \infty} x_k \quad \text{Or,} \quad \Omega_e = \lim_{k \to \infty} \Omega_k$$

Here $x_e$ is the exact solution of the given nonlinear oscillator.

In the present method, it has been shown that the solution yield the less error in each iterative step compared to the previous iterative step and finally $|\Omega_4 - \Omega_e| = |0.847289 - 0.847213| < \varepsilon$, where $\varepsilon$ is a small positive number and $A$ is chosen to be unity. From this, it is clear that the adopted method is convergent.

An iterative method of the form represented by Eq. (3) with initial guesses given in Eq. (4) and Eq. (5) is said to be consistent if

$$\lim_{k \to \infty} |x_k - x_e| = 0 \quad \text{Or,} \quad \lim_{k \to \infty} |\Omega_k - \Omega_e| = 0.$$

In the present analysis we see that

$$\lim_{k \to \infty} |\Omega_k - \Omega_e| = 0, \text{ as } |\Omega_4 - \Omega_e| = 0.$$

Thus the consistency of the method is achieved.

## 6. Conclusion

An iteration method has been used to solve nonlinear oscillations of conservative single-degree of freedom systems with odd nonlinearity. The iteration method is a powerful and effective mathematical tool in solving nonlinear differential equations of mathematical physics, applied mathematics, and engineering. The iteration procedure can be carried on if solutions of a higher degree of accuracy are required. In this paper, an iteration method has been employed for analytic treatment of the cubic nonlinear differential equation. The adopted method is convergent and obtained solution consistent. The performance of this method is reliable, simple and gives many new solutions.

## 7. References


Alam, A., Rahman, H., Haque, B. I., & Akbar, M. A. (2011). Perturbation technique for analytical solutions of fourth order near critically damped nonlinear systems. *International Journal of Basic & Applied Sciences*, *11*(1).

Alquran, M. T., & Doğan, N. (2010). Variational Iteration method for solving two-parameter singularly perturbed two point boundary value problem. *Applications and Applied Mathematics: An International Journal (AAM)*, *5*(1), 81-95.

Alquran, M., & Al-Khaled, K. (2012). Effective approximate methods for strongly nonlinear differential equations with oscillations. *Mathematical Sciences*, *6*(1), 32.

Alquran, M., & Al-Khamaiseh, B. M. (2010). Algorithms to solve nonlinear singularly perturbed two point boundary value problems. *Applied Mathematical Sciences*, *4*(57), 2809-27.

Beléndez, A., Gimeno, E., Alvarez, M. L., Gallego, S., Ortuño, M., & Méndez, D. I. (2009). A novel rational harmonic balance approach for periodic solutions of conservative nonlinear oscillators. *International Journal of Nonlinear Sciences and Numerical Simulation*, *10*(1), 13-26.

Bogoliubov, N. N. & Mitropolskii, Y. A. (1961). Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordan and Breach, New York.

Gottlieb, H. P. W. (2006). Harmonic balance approach to limit cycles for nonlinear jerk equations. *Journal of Sound and Vibration*, *297*(1-2), 243-250.

Haque, B. I. (2014). A New Approach of Mickens' Extended Iteration Method for Solving Some Nonlinear Jerk Equations. *Journal of Advances in Mathematics and Computer Science*, 3146-3162.

Haque, B. I., & Hossain, M. R. (2016). An analytic investigation of the quadratic nonlinear oscillator by an iteration method. *Journal of Advances in Mathematics and Computer Science*, 1-8.

Haque, B. I., Alam M. S., Rahman M. M. & Yeasmin I. A. (2014). Iterative technique of periodic solutions to a class of non-linear conservative systems. *International Journal of Conceptions on Computation and Information Technology*, 2(1), 92-97.

Haque, B. I., Alam, M. S., & Rahman, M. M. (2013). Modified solutions of some oscillators by iteration procedure. *Journal of the Egyptian Mathematical Society*, *21*(2), 142-147.

Haque, B. I., Alom, M. A., & Rahman, M. H. (2011). Perturbation Solution of Fourth Order Critically Damped Oscillatory Nonlinear Systems. *International Journal of Basic and Applied Science*, *11*, 68-77.

He, J.H. (2002). Homotopy perturbation method for bifurcation of nonlinear problems. *Journal of Non-Linear Mechanics*, 37, 309-314.

Hosen, M. A. (2013). Accurate approximate analytical solutions to an anti-symmetric quadratic nonlinear oscillator. *African Journal of Mathematics and Computer Science Research*, *6*(5), 77-81.

Hu, H. (2004). A modified method of equivalent linearization that works even when the non-linearity is not small. *Journal of Sound and Vibration*, *3*(276), 1145-1149.

Hu, H., & Tang, J. H. (2007). A classical iteration procedure valid for certain strongly nonlinear oscillators. *Journal of sound and vibration*, *299*(1-2), 397-402.

Krylov, N. N. & Bogoliubov, N. N. (1947). Introduction to Nonlinear Mechanics, Princeton University Press, New Jersey.

Lim, C. W., & Wu, B. S. (2002). A modified Mickens procedure for certain nonlinear oscillators. *Journal of Sound and Vibration*, *257*(1), 202.

Matko, V. (2011). Next generation AT-cut quartz crystal sensing devices. *Sensors*, *11*(5), 4474-4482.

Matko, V., & Milanović, M. (2014). Temperature-compensated capacitance–frequency converter with high resolution. *Sensors and Actuators A: Physical*, *220*, 262-269.

Matko, V., & Šafarič, R. (2009). Major improvements of quartz crystal pulling sensitivity and linearity using series reactance. *Sensors*, *9*(10), 8263-8270.

Mickens R. E. (1991). Nonlinear Oscillations. Cambridge University Press, New York.

Mickens, R. E. (1984). Comments on the method of harmonic balance. *Journal of Sound and Vibration*, *94*(3), 456-460.

Mickens, R. E. (1996). *Oscillations in planar dynamic systems* (Vol. 37). World Scientific.

Mickens, R. E. (2005). A generalized iteration procedure for calculating approximations to periodic solutions of "truly nonlinear oscillators". *Journal of Sound and Vibration*, *287*(4-5), 1045-1051.

Mickens, R. E., & Semwogerere, D. (1996). Fourier analysis of a rational harmonic balance approximation for periodic solutions. *Journal of Sound and Vibration*, *3*(195), 528-550.

Mickens, R.E. (1986). A generalization of the method of harmonic balance. *Journal of Sound and Vibration*, 111, 515-518.

Mickens, R.E. (1987). Iteration procedure for determining approximate solutions to non-linear oscillator equations. *Journal of Sound and Vibration*, 116, 185-187.

Mickens, R.E. (1999). Generalization of the senator-bapat method to systems having limit cycles. *Journal of Sound and Vibration*, 224, 167-171.

Mickens, R.E. (2010). Truly nonlinear oscillations: harmonic balance, parameter expansions, iteration, and averaging methods. World Scientific Pub Co Inc.

Nayfeh, A.H. (1973). Perturbation Methods, John Wiley, New York.

Senator, M., & Bapat, C. N. (1993). A perturbation technique that works even when the non-linearity is not small. *Journal of Sound and Vibration*, *164*(1), 1-27.

Turkyilmazoglu, M. (2010). Analytic approximate solutions of rotating disk boundary layer flow subject to a uniform suction or injection. *International Journal of Mechanical Sciences*, *52*(12), 1735-1744.

Turkyilmazoglu, M. (2011). An optimal analytic approximate solution for the limit cycle of Duffing–van der Pol equation. *Journal of Applied Mechanics*, *78*(2), 021005.

Turkyilmazoglu, M. (2012). An effective approach for approximate analytical solutions of the damped Duffing equation. *Physica Scripta*, *86*(1), 015301.

Wu, B. S., Sun, W. P., & Lim, C. W. (2006). An analytical approximate technique for a class of strongly non-linear oscillators. *International Journal of Non-Linear Mechanics*, *41*(6-7), 766-774.

Xu, H., & Cang, J. (2008). Analysis of a time fractional wave-like equation with the homotopy analysis method. *Physics Letters A*, *372*(8), 1250-1255.

Xu, L. (2007). Application of He's parameter-expansion method to an oscillation of a mass attached to a stretched elastic wire. *Physics Letters A*, *368*(3-4), 259-262.

Zengin, F. Ö., Kaya, M. O., & Demirbağ, S. A. (2008). Application of parameter expanding methods to nonlinear oscillators with discontinuities. *International Journal of Nonlinear Sciences and Numerical Simulation*, *9*(3), 267-270.