Original Research Article

Linear Summing Formulas of Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers

Abstract. In this paper, we present linear summation formulas for generalized Pentanacci numbers and generalized Gaussian Pentanacci numbers. Also, as special cases, we give linear summation formulas of Pentanacci and Pentanacci-Lucas numbers; Gaussian Pentanacci and Gaussian Pentanacci-Lucas numbers.

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1. Introduction and Preliminaries

In this work, we investigate linear summation formulas of generalized Pentanacci numbers and generalized Gaussian Pentanacci numbers. First, in this section, we present some background about generalized Pentanacci numbers.

There have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these type of sequences are the sequences of Pentanacci and Pentanacci-Lucas which are special case of generalized Pentanacci numbers. A generalized Pentanacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4)\}_{n\geq 0}$ is defined by the fifth-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5}, \quad (1.1)$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} - V_{-(n-4)} + V_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The first few generalized Pentanacci numbers with positive subscript and negative subscript are given in the following Table 1:

Table 1. A few generalized Pentanacci numbers

n	V_n	V_{-n}
0	c_0	c_0
1	c_1	$-c_0 - c_1 - c_2 - c_3 + c_4$
2	c_2	$2c_3 - c_4$
3	c_3	$2c_2 - c_3$
4	c_4	$2c_1 - c_2$
5	$c_0 + c_1 + c_2 + c_3 + c_4$	$2c_0 - c_1$
6	$c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4$	$-3c_0 - 2c_1 - 2c_2 - 2c_3 + 2c_4$
7	$2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4$	$c_0 + c_1 + c_2 + 5c_3 - 3c_4$
8	$4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4$	$4c_2 - 4c_3 + c_4$
9	$8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4$	$4c_1 - 4c_2 + c_3$
10	$16c_0 + 24c_1 + 28c_2 + 30c_3 + 31c_4$	$4c_0 - 4c_1 + c_2$

We consider two special cases of $\{V_n\}_{n\geq 0}$. Pentanacci sequence $\{P_n\}_{n\geq 0}$ and Pentanacci-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined by the fifth-order recurrence relations

$$(1.2) \qquad P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, \qquad P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4$$

and

$$(1.3) \qquad Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4} + Q_{n-5}, \qquad Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15$$

respectively. Note that P_n is the sequence $A001591$ in [2] and Q_n is the sequence $A074048$ in [2].

Next, we present the first few values of the Pentanacci and Pentanacci-Lucas numbers with positive and negative subscripts in the following Table 2:

Table 2. A few Pentanacci and Pentanacci-Lucas Numbers

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
P_n	2	0	0	0	-1	1	0	0	0	0	1	1	2	4	8	16	31	61	120
Q_n	-1	-1	-1	-7	9	-1	-1	-1	-1	5	1	3	7	15	31	57	113	223	439

2. Linear Sums of Generalized Pentanacci Numbers

For linear sums of Tribonacci and Tetrancci numbers, see [1] and [4], respectively. The following Theorem present some summation formulas of generalized Pentanacci numbers.

THEOREM 2.1. *For $n \geq 0$, we have the following lineer sum identities:*

(a): $\sum_{k=0}^{n} V_k = \frac{1}{4}(V_{n+4} - V_{n+2} - 2V_{n+1} + V_n - V_4 + V_2 + 2V_1 + 3V_0)$

(b): $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{8}(3V_{2n+2} + 4V_{2n+1} + V_{2n} + 2V_{2n-1} - V_{2n-2} - 3V_4 + 4V_3 - V_2 + 6V_1 + V_0)$

(c): $\sum_{k=0}^{n} V_{2k} = \frac{1}{8}(-V_{2n+2} + 4V_{2n+1} + 5V_{2n} + 2V_{2n-1} + 3V_{2n-2} + V_4 - 4V_3 + 3V_2 - 2V_1 + 5V_0)$

(d): $\sum_{k=0}^{n} V_{3k} = \frac{1}{4}(-V_{3n+3} + 2V_{3n+2} + V_{3n+1} + 2V_{3n} + V_{3n-1} - V_4 + 2V_3 - V_2 + 3V_0)$

(e): $\sum_{k=0}^{n} V_{3k+1} = \frac{1}{4}(V_{3n+3} + V_{3n+1} - V_{3n-1} + V_4 - 2V_3 - V_2 + 2V_1 - V_0)$

(f): $\sum_{k=0}^{n} V_{3k+2} = \frac{1}{4}(V_{3n+3} + 2V_{3n+2} + V_{3n+1} + V_{3n-1} - V_4 + 3V_2 + V_0)$

(g): $\sum_{k=0}^{n} V_{4k} = \frac{1}{16}\left(-5V_{4n+4} + 4V_{4n+3} + 9V_{4n+2} + 10V_{4n+1} + 7V_{4n} + 5V_4 - 4V_3 - 9V_2 - 10V_1 + 9V_0\right)$

(h): $\sum_{k=0}^{n} V_{4k+1} = \frac{1}{16}\left(-V_{4n+4} + 4V_{4n+3} + 5V_{4n+2} + 2V_{4n+1} - 5V_{4n} + V_4 - 4V_3 - 5V_2 + 14V_1 + 5V_0\right)$

(i): $\sum_{k=0}^{n} V_{4k+2} = \frac{1}{16}\left(3V_{4n+4} + 4V_{4n+3} + V_{4n+2} - 6V_{4n+1} - V_{4n} - 3V_4 - 4V_3 + 15V_2 + 6V_1 + V_0\right)$

(j): $\sum_{k=0}^{n} V_{4k+3} = \frac{1}{16}\left(7V_{4n+4} + 4V_{4n+3} - 3V_{4n+2} + 2V_{4n+1} + 3V_{4n} - 7V_4 + 12V_3 + 3V_2 - 2V_1 - 3V_0\right)$

(k): $\sum_{k=0}^{n} V_{5k} = \frac{1}{4}\left(-3V_{5n+5} + 4V_{5n+4} + 3V_{5n+3} + 2V_{5n+2} + V_{5n+1} - V_4 + V_2 + 2V_1 + 7V_0\right)$

(l): $\sum_{k=0}^{n} V_{5k+1} = \frac{1}{4}\left(V_{5n+5} - V_{5n+3} - 2V_{5n+2} - 3V_{5n+1} - V_4 + V_2 + 6V_1 - V_0\right)$

(m): $\sum_{k=0}^{n} V_{5k+2} = \frac{1}{4}\left(V_{5n+5} - V_{5n+3} - 2V_{5n+2} + V_{5n+1} - V_4 + 5V_2 - 2V_1 - V_0\right)$

(n): $\sum_{k=0}^{n} V_{5k+3} = \frac{1}{4}\left(V_{5n+5} - V_{5n+3} + 2V_{5n+2} + V_{5n+1} - V_4 + 4V_3 - 3V_2 - 2V_1 - V_0\right)$

(o): $\sum_{k=0}^{n} V_{5k+4} = \frac{1}{4}\left(V_{5n+5} + 3V_{5n+3} + 2V_{5n+2} + V_{5n+1} + 3V_4 - 4V_3 - 3V_2 - 2V_1 - V_0\right).$

Proof.

(a): Using the recurrence relation

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5}$$

i.e.

$$V_n - V_{n-1} = V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} = V_{n-5} + V_{n-4} + V_{n-3} + V_{n-2}$$

we obtain

$$\begin{aligned}
V_5 - V_4 &= V_0 + V_1 + V_2 + V_3 \\
V_6 - V_5 &= V_1 + V_2 + V_3 + V_4 \\
V_7 - V_6 &= V_2 + V_3 + V_4 + V_5 \\
V_8 - V_7 &= V_3 + V_4 + V_5 + V_6 \\
V_9 - V_8 &= V_4 + V_5 + V_6 + V_7 \\
&\vdots \\
V_n - V_{n-1} &= V_{n-5} + V_{n-4} + V_{n-3} + V_{n-2} \\
V_{n+1} - V_n &= V_{n-4} + V_{n-3} + V_{n-2} + V_{n-1} \\
V_{n+2} - V_{n+1} &= V_{n-3} + V_{n-2} + V_{n-1} + V_n \\
V_{n+3} - V_{n+2} &= V_{n-2} + V_{n-1} + V_n + V_{n+1} \\
V_{n+4} - V_{n+3} &= V_{n-1} + V_n + V_{n+1} + V_{n+2} \\
V_{n+5} - V_{n+4} &= V_n + V_{n+1} + V_{n+2} + V_{n+3}
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
V_{n+5} - V_4 &= \sum_{k=0}^{n} V_k + \left(V_{n+1} - V_0 + \sum_{k=0}^{n} V_k\right) + \left(V_{n+2} + V_{n+1} - V_1 - V_0 + \sum_{k=0}^{n} V_k\right) \\
&+ \left(V_{n+3} + V_{n+2} + V_{n+1} - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k\right)
\end{aligned}$$

or

$$4\sum_{k=0}^{n} V_k = V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} - V_4 + V_2 + 2V_1 + 3V_0$$

which maybe reduced easily to (a) by using (1.1) and dividing both sides by 4. Note that

$$\begin{aligned}
V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} &= (V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} + V_n) - V_{n+3} - 2V_{n+2} - 3V_{n+1} \\
&= V_{n+4} - V_{n+2} - 2V_{n+1} + V_n
\end{aligned}$$

(b),(c): We write the following obvious equations;

$$\begin{aligned}
V_3 &= V_4 - V_2 - V_1 - V_0 - V_{-1} \\
V_5 &= V_6 - V_4 - V_3 - V_2 - V_1 \\
V_7 &= V_8 - V_6 - V_5 - V_4 - V_3 \\
V_9 &= V_{10} - V_8 - V_7 - V_6 - V_5 \\
V_{11} &= V_{12} - V_{10} - V_9 - V_8 - V_7 \\
V_{13} &= V_{14} - V_{12} - V_{11} - V_{10} - V_9 \\
V_{15} &= V_{16} - V_{14} - V_{13} - V_{12} - V_{11} \\
&\vdots \\
V_{2n-1} &= V_{2n} - V_{2n-2} - V_{2n-3} - V_{2n-4} - V_{2n-5} \\
V_{2n+1} &= V_{2n+2} - V_{2n} - V_{2n-1} - V_{2n-2} - V_{2n-3}.
\end{aligned}$$

Now, adding these equations we have

$$\begin{aligned}
-V_1 + \sum_{k=0}^{n} V_{2k+1} &= \left(-V_0 - V_2 + V_{2n+2} + \sum_{k=0}^{n} V_{2k}\right) + \left(V_0 - \sum_{k=0}^{n} V_{2k}\right) \\
&+ \left(V_{2n+1} - \sum_{k=0}^{n} V_{2k+1}\right) + \left(V_{2n} - \sum_{k=0}^{n} V_{2k}\right) + \left(V_{2n+1} + V_{2n-1} - V_{-1} - \sum_{k=0}^{n} V_{2k+1}\right)
\end{aligned}$$

or

$$3\sum_{k=0}^{n} V_{2k+1} = V_1 - V_{-1} - V_2 + V_{2n+2} + 2V_{2n+1} + V_{2n} + V_{2n-1} - \sum_{k=0}^{n} V_{2k}.$$

or using $V_{-1} = V_4 - V_3 - V_2 - V_1 - V_0$,

$$3\sum_{k=0}^{n} V_{2k+1} = -V_4 + V_3 + 2V_1 + V_0 + V_{2n+2} + 2V_{2n+1} + V_{2n} + V_{2n-1} - \sum_{k=0}^{n} V_{2k}.$$

Note that

$$V_1 - V_{-1} - V_2 = V_1 - (V_4 - V_3 - V_2 - V_1 - V_0) - V_2 = -V_4 + V_3 + 2V_1 + V_0.$$

Similarly, we write the following obvious equations;

$$\begin{aligned}
V_2 &= V_3 - V_1 - V_0 - V_{-1} - V_{-2} \\
V_4 &= V_5 - V_3 - V_2 - V_1 - V_0 \\
V_6 &= V_7 - V_5 - V_4 - V_3 - V_2 \\
V_8 &= V_9 - V_7 - V_6 - V_5 - V_4 \\
V_{10} &= V_{11} - V_9 - V_8 - V_7 - V_6 \\
V_{12} &= V_{13} - V_{11} - V_{10} - V_9 - V_8 \\
V_{14} &= V_{15} - V_{13} - V_{12} - V_{11} - V_{10} \\
&\vdots \\
V_{2n-2} &= V_{2n-1} - V_{2n-3} - V_{2n-4} - V_{2n-5} - V_{2n-6} \\
V_{2n} &= V_{2n+1} - V_{2n-1} - V_{2n-2} - V_{2n-3} - V_{2n-4}.
\end{aligned}$$

Now, adding these equations, we have

$$\begin{aligned}
-V_0 + \sum_{k=0}^{n} V_{2k} &= \left(-V_1 + \sum_{k=0}^{n} V_{2k+1}\right) + \left(V_{2n+1} - \sum_{k=0}^{n} V_{2k+1}\right) + \left(V_{2n} - \sum_{k=0}^{n} V_{2k}\right) \\
&\quad + \left(V_{2n+1} + V_{2n-1} - V_{-1} - \sum_{k=0}^{n} V_{2k+1}\right) + \left(V_{2n-2} + V_{2n} - V_{-2} - \sum_{k=0}^{n} V_{2k}\right)
\end{aligned}$$

or

$$3\sum_{k=0}^{n} V_{2k} = (-V_{-2} - V_{-1} + V_0 - V_1) + 2V_{2n+1} + 2V_{2n} + V_{2n-1} + V_{2n-2} - \sum_{k=0}^{n} V_{2k+1}$$

or using $V_{-2} = V_3 - V_2 - V_1 - V_0 - V_{-1}$,

$$3\sum_{k=0}^{n} V_{2k} = -V_3 + V_2 + 2V_0 + 2V_{2n+1} + 2V_{2n} + V_{2n-1} + V_{2n-2} - \sum_{k=0}^{n} V_{2k+1}.$$

Note that

$$-V_{-2} - V_{-1} + V_0 - V_1 = -(V_3 - V_2 - V_1 - V_0 - V_{-1}) - V_{-1} + V_0 - V_1 = -V_3 + V_2 + 2V_0.$$

Solving the following system

$$\begin{aligned}
3\sum_{k=0}^{n} V_{2k+1} &= -V_4 + V_3 + 2V_1 + V_0 + V_{2n+2} + 2V_{2n+1} + V_{2n} + V_{2n-1} - \sum_{k=0}^{n} V_{2k}, \\
3\sum_{k=0}^{n} V_{2k} &= -V_3 + V_2 + 2V_0 + 2V_{2n+1} + 2V_{2n} + V_{2n-1} + V_{2n-2} - \sum_{k=0}^{n} V_{2k+1},
\end{aligned}$$

we find that

$$\sum_{k=0}^{n} V_{2k+1} = \frac{1}{8}(3V_{2n+2} + 4V_{2n+1} + V_{2n} + 2V_{2n-1} - V_{2n-2} + V_0 + 6V_1 - V_2 + 4V_3 - 3V_4)$$

$$\sum_{k=0}^{n} V_{2k} = \frac{1}{8}(-V_{2n+2} + 4V_{2n+1} + 5V_{2n} + 2V_{2n-1} + 3V_{2n-2} + V_4 - 4V_3 + 3V_2 - 2V_1 + 5V_0).$$

(d),(e),(f): Using the recurrence relation

$$V_k = V_{k-1} + V_{k-2} + V_{k-3} + V_{k-4} + V_{k-5}$$

i.e.

$$V_{k-1} = V_k - V_{k-2} - V_{k-3} - V_{k-4} - V_{k-5}$$

we write the obvious equations

$$\begin{aligned}
V_0 &= V_1 - V_{-1} - V_{-2} - V_{-3} - V_{-4} \\
V_3 &= V_4 - V_2 - V_1 - V_0 - V_{-1} \\
V_6 &= V_7 - V_5 - V_4 - V_3 - V_2 \\
V_9 &= V_{10} - V_8 - V_7 - V_6 - V_5 \\
V_{12} &= V_{13} - V_{11} - V_{10} - V_9 - V_8 \\
V_{15} &= V_{16} - V_{14} - V_{13} - V_{12} - V_{11} \\
V_{18} &= V_{19} - V_{17} - V_{16} - V_{15} - V_{14} \\
V_{21} &= V_{22} - V_{20} - V_{19} - V_{18} - V_{17} \\
V_{24} &= V_{25} - V_{23} - V_{22} - V_{21} - V_{20} \\
V_{27} &= V_{28} - V_{26} - V_{25} - V_{24} - V_{23} \\
&\vdots \\
V_{3n-6} &= V_{3n-5} - V_{3n-7} - V_{3n-8} - V_{3n-9} - V_{3n-10} \\
V_{3n-3} &= V_{3n-2} - V_{3n-4} - V_{3n-5} - V_{3n-6} - V_{3n-7} \\
V_{3n} &= V_{3n+1} - V_{3n-1} - V_{3n-2} - V_{3n-3} - V_{3n-4}
\end{aligned}$$

Now, adding these equations, we have

$$\begin{aligned}
\sum_{k=0}^{n} V_{3k} &= \left(\sum_{k=0}^{n} V_{3k+1}\right) + \left(-\sum_{k=0}^{n} V_{3k+2} - V_{-1} + V_{3n+2}\right) + \left(-\sum_{k=0}^{n} V_{3k+1} - V_{-2} + V_{3n+1}\right) \\
&\quad + \left(-\sum_{k=0}^{n} V_{3k} - V_{-3} + V_{3n}\right) + \left(-\sum_{k=0}^{n} V_{3k+2} - V_{-4} - V_{-1} + V_{3n-1} + V_{3n+2}\right) \\
&\Rightarrow \\
2\sum_{k=0}^{n} V_{3k} &= 2V_{3n+2} + V_{3n+1} + V_{3n} + V_{3n-1} - V_{-4} - V_{-3} - V_{-2} - 2V_{-1} - 2\sum_{k=0}^{n} V_{3k+2}
\end{aligned}$$

Similarly, we write the obvious equations

$$\begin{aligned}
V_{-1} &= V_0 - V_{-2} - V_{-3} - V_{-4} - V_{-5} \\
V_2 &= V_3 - V_1 - V_0 - V_{-1} - V_{-2} \\
V_5 &= V_6 - V_4 - V_3 - V_2 - V_1 \\
V_8 &= V_9 - V_7 - V_6 - V_5 - V_4 \\
V_{11} &= V_{12} - V_{10} - V_9 - V_8 - V_7 \\
V_{14} &= V_{15} - V_{13} - V_{12} - V_{11} - V_{10} \\
V_{17} &= V_{18} - V_{16} - V_{15} - V_{14} - V_{13} \\
V_{20} &= V_{21} - V_{19} - V_{18} - V_{17} - V_{16} \\
V_{23} &= V_{24} - V_{22} - V_{21} - V_{20} - V_{19} \\
V_{26} &= V_{27} - V_{25} - V_{24} - V_{23} - V_{22} \\
&\vdots \\
V_{3n-4} &= V_{3n-3} - V_{3n-5} - V_{3n-6} - V_{3n-7} - V_{3n-8} \\
V_{3n-1} &= V_{3n} - V_{3n-2} - V_{3n-3} - V_{3n-4} - V_{3n-5} \\
V_{3n+2} &= V_{3n+3} - V_{3n+1} - V_{3n} - V_{3n-1} - V_{3n-2}
\end{aligned}$$

Now, adding these equations, we obtain

$$\begin{aligned}
V_{-1} + \sum_{k=0}^{n} V_{3k+2} &= \left(V_{3n+3} + \sum_{k=0}^{n} V_{3k}\right) + \left(-V_{-2} - \sum_{k=0}^{n} V_{3k+1}\right) + \left(-V_{-3} - \sum_{k=0}^{n} V_{3k}\right) \\
&\quad + \left(V_{3n+2} - V_{-1} - V_{-4} - \sum_{k=0}^{n} V_{3k+2}\right) + \left(V_{3n+1} - V_{-2} - V_{-5} - \sum_{k=0}^{n} V_{3k+1}\right) \\
&\Rightarrow \\
2\sum_{k=0}^{n} V_{3k+2} &= -V_{-5} - V_{-4} - V_{-3} - 2V_{-2} - 2V_{-1} + V_{3n+3} + V_{3n+2} + V_{3n+1} - 2\sum_{k=0}^{n} V_{3k+1}.
\end{aligned}$$

Similarly, we write the obvious equations

$$\begin{aligned}
V_{-2} &= V_{-1}-V_{-3}-V_{-4}-V_{-5}-V_{-6} \\
V_{1} &= V_{2}-V_{0}-V_{-1}-V_{-2}-V_{-3} \\
V_{4} &= V_{5}-V_{3}-V_{2}-V_{1}-V_{0} \\
V_{7} &= V_{8}-V_{6}-V_{5}-V_{4}-V_{3} \\
V_{10} &= V_{11}-V_{9}-V_{8}-V_{7}-V_{6} \\
V_{13} &= V_{14}-V_{12}-V_{11}-V_{10}-V_{9} \\
V_{16} &= V_{17}-V_{15}-V_{14}-V_{13}-V_{12} \\
V_{19} &= V_{20}-V_{18}-V_{17}-V_{16}-V_{15} \\
V_{22} &= V_{23}-V_{21}-V_{20}-V_{19}-V_{18} \\
V_{25} &= V_{26}-V_{24}-V_{23}-V_{22}-V_{21} \\
&\vdots \\
V_{3n-5} &= V_{3n-4}-V_{3n-6}-V_{3n-7}-V_{3n-8}-V_{3n-9} \\
V_{3n-2} &= V_{3n-1}-V_{3n-3}-V_{3n-4}-V_{3n-5}-V_{3n-6} \\
V_{3n+1} &= V_{3n+2}-V_{3n}-V_{3n-1}-V_{3n-2}-V_{3n-3}
\end{aligned}$$

Now, adding these equations, we obtain

$$\begin{aligned}
V_{-2}+\sum_{k=0}^{n}V_{3k+1} &= \left(V_{-1}+\sum_{k=0}^{n}V_{3k+2}\right)+\left(-V_{-3}-\sum_{k=0}^{n}V_{3k}\right)+\left(V_{3n+2}-V_{-4}-V_{-1}-\sum_{k=0}^{n}V_{3k+2}\right) \\
&\quad +\left(V_{3n+1}-V_{-5}-V_{-2}-\sum_{k=0}^{n}V_{3k+1}\right)+\left(V_{3n.}-V_{-6}-V_{-3}-\sum_{k=0}^{n}V_{3k}\right) \\
&\Rightarrow \\
2\sum_{k=0}^{n}V_{3k+1} &= -2V_{-2}-V_{-6}-V_{-5}-V_{-4}-2V_{-3}+V_{3n+2}+V_{3n+1}+V_{3n.}-2\sum_{k=0}^{n}V_{3k}.
\end{aligned}$$

Solving the following system

$$\begin{aligned}
2\sum_{k=0}^{n}V_{3k} &= 2V_{3n+2}+V_{3n+1}+V_{3n}+V_{3n-1}-V_{-4}-V_{-3}-V_{-2}-2V_{-1}-2\sum_{k=0}^{n}V_{3k+2} \\
2\sum_{k=0}^{n}V_{3k+2} &= -V_{-5}-V_{-4}-V_{-3}-2V_{-2}-2V_{-1}+V_{3n+3}+V_{3n+2}+V_{3n+1}-2\sum_{k=0}^{n}V_{3k+1} \\
2\sum_{k=0}^{n}V_{3k+1} &= -2V_{-2}-V_{-6}-V_{-5}-V_{-4}-2V_{-3}+V_{3n+2}+V_{3n+1}+V_{3n.}-2\sum_{k=0}^{n}V_{3k}
\end{aligned}$$

we find

$$\begin{aligned}\sum_{k=0}^{n} V_{3k} &= \frac{1}{4}(-V_{3n+3} + 2V_{3n+2} + V_{3n+1} + 2V_{3n} + V_{3n-1} - V_4 + 2V_3 - V_2 + 3V_0)\\ \sum_{k=0}^{n} V_{3k+1} &= \frac{1}{4}(V_{3n+3} + V_{3n+1} - V_{3n-1} + V_4 - 2V_3 - V_2 + 2V_1 - V_0)\\ \sum_{k=0}^{n} V_{3k+2} &= \frac{1}{4}(V_{3n+3} + 2V_{3n+2} + V_{3n+1} + V_{3n-1} - V_4 + 3V_2 + V_0).\end{aligned}$$

(g),(h),(i),(j): As in the cases (d),(e),(f), solving the following system

$$\begin{aligned}2\sum_{k=0}^{n} V_{4k} &= V_{4n+3} + V_{4n+2} + V_{4n+1} + V_{4n} - V_1 + V_0 - \sum_{k=0}^{n} V_{4k+2} - \sum_{k=0}^{n} V_{4k+3}\\ 2\sum_{k=0}^{n} V_{4k+1} &= V_{4n+3} + V_{4n+2} + V_{4n+1} - V_2 + V_1 + V_0 - \sum_{k=0}^{n} V_{4k+3} - \sum_{k=0}^{n} V_{4k}\\ 2\sum_{k=0}^{n} V_{4k+2} &= V_{4n+3} + V_{4n+2} - V_3 + V_2 + V_1 + V_0 - \sum_{k=0}^{n} V_{4k+1} - \sum_{k=0}^{n} V_{4k}\\ 2\sum_{k=0}^{n} V_{4k+3} &= V_{4n+4} + V_{4n+3} - V_4 + V_3 + V_2 + V_1 - \sum_{k=0}^{n} V_{4k+2} - \sum_{k=0}^{n} V_{4k+1}\end{aligned}$$

we find

$$\begin{aligned}\sum_{k=0}^{n} V_{4k} &= \frac{1}{16}\left(-5V_{4n+4} + 4V_{4n+3} + 9V_{4n+2} + 10V_{4n+1} + 7V_{4n} + 5V_4 - 4V_3 - 9V_2 - 10V_1 + 9V_0\right)\\ \sum_{k=0}^{n} V_{4k+1} &= \frac{1}{16}\left(-V_{4n+4} + 4V_{4n+3} + 5V_{4n+2} + 2V_{4n+1} - 5V_{4n} + V_4 - 4V_3 - 5V_2 + 14V_1 + 5V_0\right)\\ \sum_{k=0}^{n} V_{4k+2} &= \frac{1}{16}\left(3V_{4n+4} + 4V_{4n+3} + V_{4n+2} - 6V_{4n+1} - V_{4n} - 3V_4 - 4V_3 + 15V_2 + 6V_1 + V_0\right)\\ \sum_{k=0}^{n} V_{4k+3} &= \frac{1}{16}\left(7V_{4n+4} + 4V_{4n+3} - 3V_{4n+2} + 2V_{4n+1} + 3V_{4n} - 7V_4 + 12V_3 + 3V_2 - 2V_1 - 3V_0\right)\end{aligned}$$

(k),(l),(m),(n),(o): As in the cases (d),(e),(f), solving the following system

$$\begin{aligned}
\sum_{k=0}^{n} V_{5k} &= V_{5n+4}+V_{5n+3}+V_{5n+2}+V_{5n+1}-V_1+V_0-\sum_{k=0}^{n} V_{5k+4}-\sum_{k=0}^{n} V_{5k+3}-\sum_{k=0}^{n} V_{5k+2} \\
\sum_{k=0}^{n} V_{5k+1} &= V_{5n+4}+V_{5n+3}+V_{5n+2}-V_2+V_1+V_0-\sum_{k=0}^{n} V_{5k+4}-\sum_{k=0}^{n} V_{5k+3}-\sum_{k=0}^{n} V_{5k} \\
\sum_{k=0}^{n} V_{5k+2} &= V_{5n+4}+V_{5n+3}-V_3+V_2+V_1+V_0-\sum_{k=0}^{n} V_{5k+4}-\sum_{k=0}^{n} V_{5k+1}-\sum_{k=0}^{n} V_{5k} \\
\sum_{k=0}^{n} V_{5k+3} &= V_{5n+4}-V_4+V_3+V_2+V_1+V_0-\sum_{k=0}^{n} V_{5k+2}-\sum_{k=0}^{n} V_{5k+1}-\sum_{k=0}^{n} V_{5k} \\
\sum_{k=0}^{n} V_{5k+4} &= \sum_{k=0}^{n} V_{5k+5}-\sum_{k=0}^{n} V_{5k+3}-\sum_{k=0}^{n} V_{5k+2}-\sum_{k=0}^{n} V_{5k+1}-\sum_{k=0}^{n} V_{5k} \\
\sum_{k=0}^{n} V_{5k+4} &= V_{5n+5}-V_0+\sum_{k=0}^{n} V_{5k}-\sum_{k=0}^{n} V_{5k+3}-\sum_{k=0}^{n} V_{5k+2}-\sum_{k=0}^{n} V_{5k+1}-\sum_{k=0}^{n} V_{5k}
\end{aligned}$$

we find

$$\begin{aligned}
\sum_{k=0}^{n} V_{5k} &= \frac{1}{4}(4V_{5n+4}+3V_{5n+3}+2V_{5n+2}+V_{5n+1}-3V_{5n+5}+7V_0-V_4+V_2+2V_1) \\
\sum_{k=0}^{n} V_{5k+1} &= \frac{1}{4}(V_{5n+5}-V_{5n+3}-2V_{5n+2}-3V_{5n+1}-V_4+V_2+6V_1-V_0) \\
\sum_{k=0}^{n} V_{5k+2} &= \frac{1}{4}(V_{5n+5}-V_{5n+3}-2V_{5n+2}+V_{5n+1}-V_4+5V_2-2V_1-V_0) \\
\sum_{k=0}^{n} V_{5k+3} &= \frac{1}{4}(V_{5n+5}-V_{5n+3}+2V_{5n+2}+V_{5n+1}-V_4+4V_3-3V_2-2V_1-V_0) \\
\sum_{k=0}^{n} V_{5k+4} &= \frac{1}{4}(V_{5n+5}+3V_{5n+3}+2V_{5n+2}+V_{5n+1}-4V_3-3V_2+3V_4-2V_1-V_0).
\end{aligned}$$

For a different proof of (a),(b),(c) see [3]. As special cases of above Theorem, we have the following two Corollaries. First one present some summation formulas of Pentanacci numbers.

COROLLARY 2.2. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} P_k = \frac{1}{4}(P_{n+4}-P_{n+2}-2P_{n+1}+P_n-1)$

(b): $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{8}(3P_{2n+2}+4P_{2n+1}+P_{2n}+2P_{2n-1}-P_{2n-2}+1)$

(c): $\sum_{k=0}^{n} P_{2k} = \frac{1}{8}(-P_{2n+2}+4P_{2n+1}+5P_{2n}+2P_{2n-1}+3P_{2n-2}-3)$

(d): $\sum_{k=0}^{n} P_{3k} = \frac{1}{4}(-P_{3n+3}+2P_{3n+2}+P_{3n+1}+2P_{3n}+P_{3n-1}-1)$

(e): $\sum_{k=0}^{n} P_{3k+1} = \frac{1}{4}(P_{3n+3}+P_{3n+1}-P_{3n-1}+P_4-3)$

(f): $\sum_{k=0}^{n} P_{3k+2} = \frac{1}{4}(P_{3n+3}+2P_{3n+2}+P_{3n+1}+P_{3n-1}-1)$

(g): $\sum_{k=0}^{n} P_{4k} = \frac{1}{16}(-5P_{4n+4}+4P_{4n+3}+9P_{4n+2}+10P_{4n+1}+7P_{4n}-7)$

(h): $\sum_{k=0}^{n} P_{4k+1} = \frac{1}{16}(-P_{4n+4}+4P_{4n+3}+5P_{4n+2}+2P_{4n+1}-5P_{4n}+5)$

(i): $\sum_{k=0}^{n} P_{4k+2} = \frac{1}{16}(3P_{4n+4}+4P_{4n+3}+P_{4n+2}-6P_{4n+1}-P_{4n}+1)$

(j): $\sum_{k=0}^{n} P_{4k+3} = \frac{1}{16}\left(7P_{4n+4} + 4P_{4n+3} - 3P_{4n+2} + 2P_{4n+1} + 3P_{4n} - 3\right)$

(k): $\sum_{k=0}^{n} P_{5k} = \frac{1}{4}\left(-3P_{5n+5} + 4P_{5n+4} + 3P_{5n+3} + 2P_{5n+2} + P_{5n+1} - 1\right)$

(l): $\sum_{k=0}^{n} P_{5k+1} = \frac{1}{4}\left(P_{5n+5} - P_{5n+3} - 2P_{5n+2} - 3P_{5n+1} + 3\right)$

(m): $\sum_{k=0}^{n} P_{5k+2} = \frac{1}{4}\left(P_{5n+5} - P_{5n+3} - 2P_{5n+2} + P_{5n+1} - 1\right)$

(n): $\sum_{k=0}^{n} P_{5k+3} = \frac{1}{4}\left(P_{5n+5} - P_{5n+3} + 2P_{5n+2} + P_{5n+1} - 1\right)$

(o): $\sum_{k=0}^{n} P_{5k+4} = \frac{1}{4}\left(P_{5n+5} + 3P_{5n+3} + 2P_{5n+2} + P_{5n+1} - 1\right)$

Next Corollary gives some summation formulas of Pentanacci-Lucas numbers.

Corollary 2.3. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} Q_{k} = \frac{1}{4}(Q_{n+4} - Q_{n+2} - 2Q_{n+1} + Q_{n} + 5)$

(b): $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{8}(3Q_{2n+2} + 4Q_{2n+1} + Q_{2n} + 2Q_{2n-1} - Q_{2n-2} - 9)$

(c): $\sum_{k=0}^{n} Q_{2k} = \frac{1}{8}(-Q_{2n+2} + 4Q_{2n+1} + 5Q_{2n} + 2Q_{2n-1} + 3Q_{2n-2} + 19)$

(d): $\sum_{k=0}^{n} Q_{3k} = \frac{1}{4}(-Q_{3n+3} + 2Q_{3n+2} + Q_{3n+1} + 2Q_{3n} + Q_{3n-1} + 11)$

(e): $\sum_{k=0}^{n} Q_{3k+1} = \frac{1}{4}(Q_{3n+3} + Q_{3n+1} - Q_{3n-1} - 5)$

(f): $\sum_{k=0}^{n} Q_{3k+2} = \frac{1}{4}(Q_{3n+3} + 2Q_{3n+2} + Q_{3n+1} + Q_{3n-1} - 1)$

(g): $\sum_{k=0}^{n} Q_{4k} = \frac{1}{16}\left(-5Q_{4n+4} + 4Q_{4n+3} + 9Q_{4n+2} + 10Q_{4n+1} + 7Q_{4n} + 55\right)$

(h): $\sum_{k=0}^{n} Q_{4k+1} = \frac{1}{16}\left(-Q_{4n+4} + 4Q_{4n+3} + 5Q_{4n+2} + 2Q_{4n+1} - 5Q_{4n} + 11\right)$

(i): $\sum_{k=0}^{n} Q_{4k+2} = \frac{1}{16}\left(3Q_{4n+4} + 4Q_{4n+3} + Q_{4n+2} - 6Q_{4n+1} - Q_{4n} - 17\right)$

(j): $\sum_{k=0}^{n} Q_{4k+3} = \frac{1}{16}\left(7Q_{4n+4} + 4Q_{4n+3} - 3Q_{4n+2} + 2Q_{4n+1} + 3Q_{4n} - 29\right)$

(k): $\sum_{k=0}^{n} Q_{5k} = \frac{1}{4}\left(-3Q_{5n+5} + 4Q_{5n+4} + 3Q_{5n+3} + 2Q_{5n+2} + Q_{5n+1} + 25\right)$

(l): $\sum_{k=0}^{n} Q_{5k+1} = \frac{1}{4}\left(Q_{5n+5} - Q_{5n+3} - 2Q_{5n+2} - 3Q_{5n+1} - 11\right)$

(m): $\sum_{k=0}^{n} Q_{5k+2} = \frac{1}{4}\left(Q_{5n+5} - Q_{5n+3} - 2Q_{5n+2} + Q_{5n+1} - 7\right)$

(n): $\sum_{k=0}^{n} Q_{5k+3} = \frac{1}{4}\left(Q_{5n+5} - Q_{5n+3} + 2Q_{5n+2} + Q_{5n+1} - 3\right)$

(o): $\sum_{k=0}^{n} Q_{5k+4} = \frac{1}{4}\left(Q_{5n+5} + 3Q_{5n+3} + 2Q_{5n+2} + Q_{5n+1} + 1\right)$

3. Linear Sums of Generalized Gaussian Pentanacci Numbers

A Gaussian integer z is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib$, $a, b \in \mathbb{Z}$. If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers. Gaussian generalized Pentanacci numbers $\{GV_n\}_{n\geq 0} = \{GV_n(GV_0, GV_1, GV_2, GV_3, GV_4)\}_{n\geq 0}$ are defined by

$$GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3} + GV_{n-4} + GV_{n-5}, \tag{3.1}$$

with the initial conditions

$$\begin{aligned} GV_0 &= c_0 + (-c_0 - c_1 - c_2 - c_3 + c_4)i, GV_1 = c_1 + c_0 i, GV_2 = c_2 + c_1 i, \\ GV_3 &= c_3 + c_2 i, GV_4 = c_4 + c_3 i \end{aligned}$$

not all being zero. The sequences $\{GV_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GV_{-n} = -GV_{-(n-1)} - GV_{-(n-2)} - GV_{-(n-3)} - GV_{-(n-4)} + GV_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (3.1) hold for all integer n. Note that for $n \geq 0$

$$GV_n = V_n + iV_{n-1} \tag{3.2}$$

and

$$GV_{-n} = V_{-n} + iV_{-n-1}$$

We consider two special cases of $GV_n : GV_n(0, 1, 1+i, 2+i, 4+2i) = GP_n$ is the sequence of Gaussian Pentanacci numbers and $GV_n(5-i, 1+5i, 3+i, 7+3i, 15+7i) = GQ_n$ is the sequence of Gaussian Pentanacci-Lucas numbers. We formally define them as follows:

Gaussian Pentanacci numbers are defined by

$$GP_n = GP_{n-1} + GP_{n-2} + GP_{n-3} + GP_{n-4} + GP_{n-5}, \tag{3.3}$$

with the initial conditions

$$GP_0 = 0, GP_1 = 1, GP_2 = 1+i, GP_3 = 2+i, GP_4 = 4+2i$$

and Gaussian Pentanacci-Lucas numbers are defined by

$$GQ_n = GQ_{n-1} + GQ_{n-2} + GQ_{n-3} + GQ_{n-4} + GQ_{n-5} \tag{3.4}$$

with the initial conditions

$$GQ_0 = 5-i, GQ_1 = 1+5i, GQ_2 = 3+i, GQ_3 = 7+3i, GQ_4 = 15+7i.$$

Note that for $n \geq 0$

$$GP_n = M_n + iM_{n-1},\ \ GQ_n = R_n + iR_{n-1}$$

and

$$GP_{-n} = M_{-n} + iM_{-n-1}, GQ_{-n} = R_{-n} + iR_{-n-1}.$$

The following Theorem present some summation formulas of Gaussian generalized Pentanacci numbers.

THEOREM 3.1. *For $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} GV_k = \frac{1}{4}(GV_{n+4} - GV_{n+2} - 2GV_{n+1} + GV_n - GV_4 + GV_2 + 2GV_1 + 3GV_0)$

(b): $\sum_{k=0}^{n} GV_{2k+1} = \frac{1}{8}(3GV_{2n+2} + 4GV_{2n+1} + GV_{2n} + 2GV_{2n-1} - GV_{2n-2} - 3GV_4 + 4GV_3 - GV_2 + 6GV_1 + GV_0)$

(c): $\sum_{k=0}^{n} GV_{2k} = \frac{1}{8}(-GV_{2n+2} + 4GV_{2n+1} + 5GV_{2n} + 2GV_{2n-1} + 3GV_{2n-2} + GV_4 - 4GV_3 + 3GV_2 - 2GV_1 + 5GV_0)$

(d): $\sum_{k=0}^{n} GV_{3k} = \frac{1}{4}(-GV_{3n+3} + 2GV_{3n+2} + GV_{3n+1} + 2GV_{3n} + GV_{3n-1} - GV_4 + 2GV_3 - GV_2 + 3GV_0)$

(e): $\sum_{k=0}^{n} GV_{3k+1} = \frac{1}{4}(GV_{3n+3} + GV_{3n+1} - GV_{3n-1} + GV_4 - 2GV_3 - GV_2 + 2GV_1 - GV_0)$

(f): $\sum_{k=0}^{n} GV_{3k+2} = \frac{1}{4}(GV_{3n+3} + 2GV_{3n+2} + GV_{3n+1} + GV_{3n-1} - GV_4 + 3GV_2 + GV_0)$

(g): $\sum_{k=0}^{n} GV_{4k} = \frac{1}{16}(-5GV_{4n+4} + 4GV_{4n+3} + 9GV_{4n+2} + 10GV_{4n+1} + 7GV_{4n} + 5GV_4 -4GV_3 - 9GV_2 - 10GV_1 + 9GV_0)$

(h): $\sum_{k=0}^{n} GV_{4k+1} = \frac{1}{16}(-GV_{4n+4} + 4GV_{4n+3} + 5GV_{4n+2} + 2GV_{4n+1} - 5GV_{4n} + GV_4 -4GV_3 - 5GV_2 + 14GV_1 + 5GV_0)$

(i): $\sum_{k=0}^{n} GV_{4k+2} = \frac{1}{16}(3GV_{4n+4} + 4GV_{4n+3} + GV_{4n+2} - 6GV_{4n+1} - GV_{4n} - 3GV_4 -4GV_3 + 15GV_2 + 6GV_1 + GV_0)$

(j): $\sum_{k=0}^{n} GV_{4k+3} = \frac{1}{16}(7GV_{4n+4} + 4GV_{4n+3} - 3GV_{4n+2} + 2GV_{4n+1} + 3GV_{4n} - 7GV_4 +12GV_3 + 3GV_2 - 2GV_1 - 3GV_0)$

(k): $\sum_{k=0}^{n} GV_{5k} = \frac{1}{4}(-3GV_{5n+5} + 4GV_{5n+4} + 3GV_{5n+3} + 2GV_{5n+2} + GV_{5n+1} -GV_4 + GV_2 + 2GV_1 + 7GV_0)$

(l): $\sum_{k=0}^{n} GV_{5k+1} = \frac{1}{4}(GV_{5n+5} - GV_{5n+3} - 2GV_{5n+2} - 3GV_{5n+1} - GV_4 + GV_2 + 6GV_1 - GV_0)$

(m): $\sum_{k=0}^{n} GV_{5k+2} = \frac{1}{4}(GV_{5n+5} - GV_{5n+3} - 2GV_{5n+2} + GV_{5n+1} - GV_4 + 5GV_2 - 2GV_1 - GV_0)$

(n): $\sum_{k=0}^{n} GV_{5k+3} = \frac{1}{4}(GV_{5n+5} - GV_{5n+3} + 2GV_{5n+2} + GV_{5n+1} - GV_4 +4GV_3 - 3GV_2 - 2GV_1 - GV_0)$

(o): $\sum_{k=0}^{n} GV_{5k+4} = \frac{1}{4}(GV_{5n+5} + 3GV_{5n+3} + 2GV_{5n+2} + GV_{5n+1} +3GV_4 - 4GV_3 - 3GV_2 - 2GV_1 - GV_0).$

Proof. (a)-(o) can be proved exactly as in the proof of Theorem 2.1.

As special cases of the above Theorem, we have the following two Corollaries. First one present summation formulas of Gaussian Pentanacci numbers.

COROLLARY 3.2. *For $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} GP_k = \frac{1}{4}(GP_{n+4} - GP_{n+2} - 2GP_{n+1} + GP_n - 1 - i)$

(b): $\sum_{k=0}^{n} GP_{2k+1} = \frac{1}{8}(3GP_{2n+2} + 4GP_{2n+1} + GP_{2n} + 2GP_{2n-1} - GP_{2n-2} + 1 - 3i)$

(c): $\sum_{k=0}^{n} GP_{2k} = \frac{1}{8}(-GP_{2n+2} + 4GP_{2n+1} + 5GP_{2n} + 2GP_{2n-1} + 3GP_{2n-2} - 3 + i)$

(d): $\sum_{k=0}^{n} GP_{3k} = \frac{1}{4}(-GP_{3n+3} + 2GP_{3n+2} + GP_{3n+1} + 2GP_{3n} + GP_{3n-1} - 1 - i)$

(e): $\sum_{k=0}^{n} GP_{3k+1} = \frac{1}{4}(GP_{3n+3} + GP_{3n+1} - GP_{3n-1} + 1 - i)$

(f): $\sum_{k=0}^{n} GP_{3k+2} = \frac{1}{4}(GP_{3n+3} + 2GP_{3n+2} + GP_{3n+1} + GP_{3n-1} - 1 + i)$

(g): $\sum_{k=0}^{n} GP_{4k} = \frac{1}{16}(-5GP_{4n+4} + 4GP_{4n+3} + 9GP_{4n+2} + 10GP_{4n+1} + 7GP_{4n} - 7 - 3i)$

(h): $\sum_{k=0}^{n} GP_{4k+1} = \frac{1}{16}(-GP_{4n+4} + 4GP_{4n+3} + 5GP_{4n+2} + 2GP_{4n+1} - 5GP_{4n} + 5 - 7i)$

(i): $\sum_{k=0}^{n} GP_{4k+2} = \frac{1}{16}(3GP_{4n+4} + 4GP_{4n+3} + GP_{4n+2} - 6GP_{4n+1} - GP_{4n} + 1 + 5i)$

(j): $\sum_{k=0}^{n} GP_{4k+3} = \frac{1}{16}(7GP_{4n+4} + 4GP_{4n+3} - 3GP_{4n+2} + 2GP_{4n+1} + 3GP_{4n} - 3 + i)$

(k): $\sum_{k=0}^{n} GP_{5k} = \frac{1}{4}\left(-3GP_{5n+5} + 4GP_{5n+4} + 3GP_{5n+3} + 2GP_{5n+2} + GP_{5n+1} - 1 - i\right)$

(l): $\sum_{k=0}^{n} GP_{5k+1} = \frac{1}{4}\left(GP_{5n+5} - GP_{5n+3} - 2GP_{5n+2} - 3GP_{5n+1} + 3 - i\right)$

(m): $\sum_{k=0}^{n} GP_{5k+2} = \frac{1}{4}\left(GP_{5n+5} - GP_{5n+3} - 2GP_{5n+2} + GP_{5n+1} - 1 + 3i\right)$

(n): $\sum_{k=0}^{n} GP_{5k+3} = \frac{1}{4}\left(GP_{5n+5} - GP_{5n+3} + 2GP_{5n+2} + GP_{5n+1} - 1 - i\right)$

(o): $\sum_{k=0}^{n} GP_{5k+4} = \frac{1}{4}\left(GP_{5n+5} + 3GP_{5n+3} + 2GP_{5n+2} + GP_{5n+1} - 1 - i\right)$

Second Corollary gives summation formulas of Gaussian Pentanacci-Lucas numbers.

Corollary 3.3. *For $n \geq 0$ we have the following formulas:*

(a): $\sum_{k=0}^{n} GQ_{k} = \frac{1}{4}(GQ_{n+4} - GQ_{n+2} - 2GQ_{n+1} + GQ_{n} + 5 + i)$

(b): $\sum_{k=0}^{n} GQ_{2k+1} = \frac{1}{8}(3GQ_{2n+2} + 4GQ_{2n+1} + GQ_{2n} + 2GQ_{2n-1} - GQ_{2n-2} - 9 + 19i)$

(c): $\sum_{k=0}^{n} GQ_{2k} = \frac{1}{8}(-GQ_{2n+2} + 4GQ_{2n+1} + 5GQ_{2n} + 2GQ_{2n-1} + 3GQ_{2n-2} + 19 - 17i)$

(d): $\sum_{k=0}^{n} GQ_{3k} = \frac{1}{4}(-GQ_{3n+3} + 2GQ_{3n+2} + GQ_{3n+1} + 2GQ_{3n} + GQ_{3n-1} + 11 - 5i)$

(e): $\sum_{k=0}^{n} GQ_{3k+1} = \frac{1}{4}(GQ_{3n+3} + GQ_{3n+1} - GQ_{3n-1} - 5 + 11i)$

(f): $\sum_{k=0}^{n} GQ_{3k+2} = \frac{1}{4}(GQ_{3n+3} + 2GQ_{3n+2} + GQ_{3n+1} + GQ_{3n-1} - 1 - 5i)$

(g): $\sum_{k=0}^{n} GQ_{4k} = \frac{1}{16}\left(-5GQ_{4n+4} + 4GQ_{4n+3} + 9GQ_{4n+2} + 10GQ_{4n+1} + 7GQ_{4n} + 55 - 45i\right)$

(h): $\sum_{k=0}^{n} GQ_{4k+1} = \frac{1}{16}\left(-GQ_{4n+4} + 4GQ_{4n+3} + 5GQ_{4n+2} + 2GQ_{4n+1} - 5GQ_{4n} + 11 + 55i\right)$

(i): $\sum_{k=0}^{n} GQ_{4k+2} = \frac{1}{16}\left(3GQ_{4n+4} + 4GQ_{4n+3} + GQ_{4n+2} - 6GQ_{4n+1} - GQ_{4n} - 17 + 11i\right)$

(j): $\sum_{k=0}^{n} GQ_{4k+3} = \frac{1}{16}\left(7GQ_{4n+4} + 4GQ_{4n+3} - 3GQ_{4n+2} + 2GQ_{4n+1} + 3GQ_{4n} - 29 - 17i\right)$

(k): $\sum_{k=0}^{n} GQ_{5k} = \frac{1}{4}\left(-3GQ_{5n+5} + 4GQ_{5n+4} + 3GQ_{5n+3} + 2GQ_{5n+2} + GQ_{5n+1} + 25 - 3i\right)$

(l): $\sum_{k=0}^{n} GQ_{5k+1} = \frac{1}{4}\left(GQ_{5n+5} - GQ_{5n+3} - 2GQ_{5n+2} - 3GQ_{5n+1} - 11 + 25i\right)$

(m): $\sum_{k=0}^{n} GQ_{5k+2} = \frac{1}{4}\left(GQ_{5n+5} - GQ_{5n+3} - 2GQ_{5n+2} + GQ_{5n+1} - 7 - 11i\right)$

(n): $\sum_{k=0}^{n} GQ_{5k+3} = \frac{1}{4}\left(GQ_{5n+5} - GQ_{5n+3} + 2GQ_{5n+2} + GQ_{5n+1} - 3 - 7i\right)$

(o): $\sum_{k=0}^{n} GQ_{5k+4} = \frac{1}{4}\left(GQ_{5n+5} + 3GQ_{5n+3} + 2GQ_{5n+2} + GQ_{5n+1} + 1 - 3i\right)$

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