The first integrals of a second order ordinary differential equation and application

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Abstract   The first integrals of second order ordinary differential equations are considered. The necessary conditions of the existence of analytical first integrals for the equation are presented. Then, the first integrals of the equation are obtained using Lie symmetry method. The results of the first integrals are applied to certain classes of partial differential equations, the conditions of nonexistence of the traveling wave solutions of the partial differential equations are obtained, and traveling wave solutions of the equations under the certain parametric conditions are also obtained.

Key words   First integral, Lie symmetry, Traveling wave solutions, Partial differential equations

1   Introduction

It is well known that the study of the integrability of differential equations has been one of the main topics in mathematics and physics, and other subjects. The integrability of system of ordinary differential equations has been studied by many authors[1]-[5]. First integrals are the powerful tool in the study of the integrability of ordinary differential equations and partial differential equations (see for instance Refs.[6]-[9] and the references therein). As we know, searching for first integrals of a differential equations system plays a very important role for studying the system. Many different methods have been used for studying the existence and searching for first integrals of ordinary differential systems. For example, the Lie groups ([10]-[12]), the Darboux theory of integrability ([13]), the Painlevéanalysis ([14]), the use of Lax pairs ([15]), the Kudryashov method([16]), etc. In [2], some simple criteria for the nonexistence of analytic integrals of general nonlinear systems are given. There provided a link between the number of first integrals and the resonant relations for a quasi-periodic vector field in [17]. In the paper, we consider first integrals of the differential equation,

$$y'' = ay' + by^2 + cy, \tag{1}$$

a, b, c are constants, and $abc \neq 0$. Let $y' = z$, equation (1) can be written as the system,

$$\begin{cases} \dot{y} &= z \\ \dot{z} &= az + by^2 + cy. \end{cases} \tag{2}$$

(2) has two equilibrium points $O_1(0,0)$ and $O_2(-\frac{c}{b}, 0)$. Let us denote the Jacobi matrix of the vector field of (2) at O_i as $A_i, (i = 1, 2)$, and

$$A_1 = \begin{pmatrix} 0 & 1 \\ c & a \end{pmatrix},$$

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$$A_2 = \begin{pmatrix} 0 & 1 \\ -c & a \end{pmatrix}.$$

So, it is easy to get the eigenvalues of A_1 are $\lambda_{1,2} = \dfrac{a \pm \sqrt{a^2 + 4c}}{2}$, those of A_2 are $\lambda_{1,2} = \dfrac{a \pm \sqrt{a^2 - 4c}}{2}$. In [18]-[19], the existence of first integrals of plane systems are studied. If the system has nontrivial analytic first integrals in a neighbourhood of a trivial solution, then the eigenvalues of the Jacobi matrix of the system at the trivial solution have to satisfy certain resonant condition. In the paper, we first consider the existence of analytical first integrals of (1) and obtain the first integrals of (1) using Lie sysmmetry method. Then, we consider the certain partial differential equations, and present the traveling wave solutions of the equations. The paper is organized as follows. In Section 2, The sufficient conditions of the nonexistence for first integrals of system (2) are given. In Section 3, Lie symmetries admitted by (2) are found by differentiating the symmetry condition, and first integrals of (2) are deduced by constructing an algebraic equations system using Lie symmetries admitted by (2). In Section 4, the parametric conditions of the nonexistence of traveling wave solutions of the certain partial differential equations are given. Some classes of traveling wave solutions of the partial differential equations are presented in Section 5. Section 6 is conclusions.

2 The necessary conditions of the existence of first integrals

As we know, if the system $\dot{x} = f(x), x \in D \subset C^n$, has nontrivial analytic first integrals in a neighbourhood of a trivial solution, then the eigenvalues of the matrix $\dfrac{\partial f}{\partial x}(0)$ have to satisfy certain resonant conditions[18], where $f(0) = 0$. So, for system (1), we have the following result.

Theorem 1. 1) $c > 0$ and $\dfrac{a^2 + 2c - a\sqrt{a^2 + 4c}}{2c}$ is not a nonnegative rational number.

2) $c < 0$, $a^2 + 4c > 0$.

3) $a^2 + 4c < 0$.

If one of above conditions is satisfied, then system (2) has no nontrivial analytic first integrals in a neighbourhood of $O_1(0,0)$.

Proof. We will use the proof by contradiction. Let us suppose there is an analytic first integral $\Omega(y, z)$ of system (2) in a neighbourhood of $O_1(0,0)$. Without loss of generality, we assume that $\Omega(0,0) = 0$. Let us expand the first integral into the Taylor series

$$\Omega(y, z) = \Omega_{(1)}(y, z) + \Omega_{(2)}(y, z) + ... + \Omega_{(k)}(y, z) + ...,$$

where $\Omega_{(k)}(y, z), k = 1, 2, ...$ are homogeneous polynomials in (y, z),

$$\Omega_{(k)}(y, z) = \frac{1}{k!}(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})^k \Omega|_{(0,0)}.$$

We can rewrite the field vector of (2) in a neighborhood of $O_1(0,0)$ as

$$\begin{pmatrix} z \\ az + by^2 + cy \end{pmatrix} = A_1 \begin{pmatrix} y \\ z \end{pmatrix} + o\left(\| \begin{pmatrix} y \\ z \end{pmatrix} \| \right).$$

After a nonsingular linear transformation, A_1 can be changed to a Jordan canonical form J_1. For simplicity, we rewrite the factor field of system (2) as the following form,

$$\begin{pmatrix} z \\ az + by^2 + cy \end{pmatrix} = J_1 \begin{pmatrix} y \\ z \end{pmatrix} + o\left(\| \begin{pmatrix} y \\ z \end{pmatrix} \| \right).$$

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1) $c > 0$.

In the case, $a^2 + 4c > 0$, and $J_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. So, in a neighbourhood of O_1, we have

$$\lambda_1 y \frac{\partial(\Omega_{(1)} + \Omega_{(2)} + ...)}{\partial y} + \lambda_2 z \frac{\partial(\Omega_{(1)} + \Omega_{(2)} + ...)}{\partial z} = 0. \tag{3}$$

Let us equate all the terms in (3) of the same order with respect to the variables y, z to 0, we have

$$\lambda_1 y \frac{\partial \Omega_{(1)}}{\partial y} + \lambda_2 z \frac{\partial \Omega_{(1)}}{\partial z} = 0,$$

that is,

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \frac{\partial \Omega_{(1)}}{\partial y}|_{(0,0)} y \\ \frac{\partial \Omega_{(1)}}{\partial z}|_{(0,0)} z \end{pmatrix} = 0.$$

That maens J_1 has a zero eigenvalue, which contradicts the condition that $\lambda_1 \lambda_2 \neq 0$. So, one can have $\Omega_{(1)} = 0$. We suppose that we have proved that $\Omega_{(1)} = \Omega_{(2)} = ... = \Omega_{(k-1)} = 0$. Then it follows from (3) that $\Omega_{(k)}$ has to satisfy the following the equation,

$$\lambda_1 y \frac{\partial \Omega_{(k)}}{\partial y} + \lambda_2 z \frac{\partial \Omega_{(k)}}{\partial z} = 0.$$

Because $\Omega_{(k)}$ is a sum of elementary monomials, $\Omega_{(k)} = \Sigma_{k_1+k_2=k} \Omega_{k_1 k_2} y^{k_1} z^{k_2}$, as follows from (3), the formula can be obtained,

$$(\lambda_1 k_1 + \lambda_2 k_2) \Sigma_{k_1+k_2=k} \Omega_{k_1 k_2} y^{k_1} z^{k_2} = 0. \tag{4}$$

So,

$$\lambda_1 k_1 + \lambda_2 k_2 = 0,$$

that is,

$$\frac{k_1}{k_2} = -\frac{\lambda_2}{\lambda_1} = \frac{a^2 + 2c - a\sqrt{a^2 + 4c}}{2c}. \tag{5}$$

when $c > 0$, (5) contradicts the condition that $\dfrac{a^2 + 2c - a\sqrt{a^2 + 4c}}{2c}$ is not a nonnegative rational number.

2) $c < 0$.

When $a^2 + 4c > 0$, the left part of (5) is nonnegative number, and the right part of (5) is negative number. That is a contradictory.

When $a^2 + 4c < 0$, A_1 has a pair of conjugate imaginary eigenvalues $\lambda_{1,2} = \dfrac{a \pm \sqrt{-(a^2 + 4c)}i}{2}$. There is the nonsingular linear transformation in C^2, which changes A_1 to the Jordan canonical $J_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Similarly, from (4), one get

$$(\lambda_1 k_1 + \lambda_2 k_2) = (k_1 + k_2)a + (k_1 - k_2)\sqrt{-(a^2 + 4c)}i = 0.$$

This is not valid for $k_i, i = 1, 2 \in N, k_1 k_2 \neq 0$.

Therefore, system (2) has no analytic first integrals in a neighborhood of O_1.

Theorem 2. 1) $c < 0$ and $\dfrac{-a^2 + 2c + a\sqrt{a^2 - 4c}}{2c}$ is not a nonnegative rational number.

2) $c > 0$, $a^2 - 4c > 0$.

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3) $a^2 - 4c < 0$.

If one of above conditions is satisfied, then system (2) has no nontrivial analytic first integrals in a neighbourhood of $O_2(-\frac{c}{b}, 0)$.

Proof. Similarly, we can proof the result.

3 Finding first integrals in other parametric conditions using Lie symmetry

In the section, we attempt to look for the first integrals of system (2) in other regions using Lie symmetry.

3.1 Infinitesimal generators

Let $X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ be the infinitesimal generator of the symmetry group G admitted by (1).

$$X^{(2)} = \xi(x, y)\partial_x + \eta(x, y)\partial_y + \eta^{(1)}\partial_{y'} + \eta^{(2)}\partial_{y''}$$

is the prolonged infinitesimal generator, where

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2,$$

$$\eta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y''.$$

By the linearized symmetry condition, we have

$$
\begin{aligned}
&\eta(-2by - c) + \{\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2\}(-a) + \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' \\
&+ (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')(ay' + by^2 + cy) = 0.
\end{aligned}
\tag{6}
$$

After setting the coefficients of the powers $(y')^i$ $(i = 1, 2, 3)$ in (6) to zero, one can get the determining equations system

$$
\begin{aligned}
&\xi_{yy} = 0, \\
&a\xi_y + (\eta_{yy} - 2\xi_{xy}) - 3a\xi_y = 0, \\
&-a\xi_x + 2\eta_{xy} - \xi_{xx} - 3\xi_y(by^2 + cy) = 0, \\
&\eta(-2by - c) - a\eta_x + \eta_{xx} + (\eta_y - 2\xi_x)(by^2 + cy) = 0.
\end{aligned}
\tag{7}
$$

The first equation of (7) gives

$$\xi(x, y) = a_1(x)y + a_2(x).
\tag{8}$$

After substituting (8) into the second equation of (7), one can have

$$\eta(x, y) = [a_1'(x) + aa_1(x)]y^2 + a_3(x)y + a_4(x),
\tag{9}$$

where $a_1(x), a_2(x), a_3(x)$ and $a_4(x)$ are functions of x to be determined. Inserting (8) and (9) into the third equation of (7), we have a polynomial of y with degree 2 which is zero if and only if each variable coefficient is set to zero

$$
\begin{aligned}
&ba_1(x) = 0, \\
&-aa_2'(x) + 2a_3'(x) - a_2''(x) = 0.
\end{aligned}
\tag{10}
$$

Owing to $b \neq 0$, we have $a_1(x) = 0$. We deduce that

$$\xi(x) = a_2(x), \quad \eta(x, y) = a_3(x)y + a_4(x).$$

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Similarly, substituting $\xi(x)$ and $\eta(x, y)$ into the last equation of (7), we obtain a polynomial of y with degree 2 which is zero if and only if the following equations are satisfied

$$b(a_3(x) + 2a_2'(x)) = 0,$$
$$-2ba_4(x) - aa_3'(x) + a_3''(x) - 2ca_2'(x) = 0. \tag{11}$$
$$-ca_4(x) - aa_4'(x) + a_4''(x) = 0.$$

Analyzing the first equation of (11) and the second equation of (10), we have

$$a_2(x) = c_1 e^{-\frac{a}{5}x} + c_2$$

and

$$a_3(x) = \frac{2a}{5} c_1 e^{-\frac{a}{5}x},$$

where c_1, c_2 are integration constants. Substituting $a_2(x)$ and $a_3(x)$ into the second equation of (11), we have

$$a_4(x) = \frac{c_1 a}{b} (\frac{6a^2}{125} + \frac{c}{5}) e^{-\frac{ax}{5}}.$$

Substituting $a_4(x)$ into the last equation of (11), we obtain one parametric condition:

$$6a^2 \pm 25c = 0. \tag{12}$$

Because c_1 and c_2 are arbitrary constants, for simplicity, we may assume $c_1 = 0$, $c_2 = 1$, Then we find

$$\xi = 1, \eta = 0.$$

Hence, one infinitesimal generator is generated as $X_1 = \partial_x$. We also assume $c_1 = 1$, $c_2 = 0$, then we obtain

$$a_2(x) = e^{-\frac{ax}{5}}, \ a_3(x) = \frac{2a}{5} e^{-\frac{ax}{5}}$$

and

$$a_4(x) = \frac{a}{b} (\frac{6a^2}{125} + \frac{c}{5}) e^{-\frac{ax}{5}}.$$

So, we have two expressions

$$\xi = e^{-\frac{ax}{5}}, \ \eta = \frac{2a}{5} e^{-\frac{ax}{5}} y + \frac{a}{b} (\frac{6a^2}{125} + \frac{c}{5}) e^{-\frac{ax}{5}}.$$

Two infinitesimal generators are obtained as follows

$$X_1 = \partial_x \ X_2 = e^{-\frac{ax}{5}} \partial_x + e^{-\frac{ax}{5}} [\frac{2a}{5} y + \frac{a}{b} (\frac{6a^2}{125} + \frac{c}{5})] \partial_y. \tag{13}$$

3.2 Obtaining first integrals

Let us return to consider equation (1). (1) can also be rewritten as the following third order autonomous system,

$$\begin{cases} \dot{x} &= 1 \\ \dot{y} &= z \\ \dot{z} &= az + by^2 + cy. \end{cases} \tag{14}$$

The partial differential operator associated to (14) is

$$X = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + (az + by^2 + cy) \frac{\partial}{\partial z}. \tag{15}$$

Based on Lie symmetry method in [20]-[22], we can have the following result.

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Theorem 3. If system (1) admits the Lie symmetry with generators (13) under the condition (12), then the generator of the corresponding Lie symmetry admitted by system (14) is

$$
\begin{aligned}
V_1 &= \frac{\partial}{\partial x}, \\
V_2 &= e^{-\frac{ax}{5}}\frac{\partial}{\partial x} + e^{-\frac{ax}{5}}\Big[\frac{2a}{5}y + \frac{6a^3}{125b} + \frac{ac}{5b}\Big]\frac{\partial}{\partial y} \\
&\quad + e^{-\frac{ax}{5}}\Big[-\frac{a}{5}\Big(\frac{2a}{5}y + \frac{6a^3}{125b} + \frac{ac}{5b}\Big) + \frac{3a}{5}z\Big]\frac{\partial}{\partial z}.
\end{aligned}
\tag{16}
$$

Proof. It is easy to prove the result based on Theorem 1 in [21].

Next, we will apply the above results to (14) for obtaining first integrals under the condition (12).

3.2.1 The first integral under the condition $6a^2 = 25c$

(14) admits two one-parameter Lie symmetries with generators

$$
\begin{aligned}
V_1 &= \frac{\partial}{\partial x}, \\
V_2 &= e^{-\frac{ax}{5}}\frac{\partial}{\partial x} + e^{-\frac{ax}{5}}\Big[\frac{2a}{5}y + \frac{2ac}{5b}\Big]\frac{\partial}{\partial y} \\
&\quad + e^{-\frac{ax}{5}}\Big[-\frac{c}{3}y - \frac{c^2}{3b} + \frac{3a}{5}z\Big]\frac{\partial}{\partial z}.
\end{aligned}
\tag{17}
$$

It is easy to find that

$$
[V_1, V_2] = -\frac{a}{5}V_2.
$$

So we can obtain the corresponding structural coefficients

$$
C_{1,2}^1 = -C_{2,1}^1 = 0, \quad C_{1,2}^2 = -C_{2,1}^2 = -\frac{a}{5}.
$$

We can let $b_1 = 1, b_0 = 0$, and get the solution f_1, f_2, f_3 from the corresponding algebraic system[20],

$$
\begin{bmatrix}
1 & z & az + by^2 + cy \\
1 & 0 & 0 \\
e^{-\frac{ax}{5}} & e^{-\frac{ax}{5}}\big[\frac{2a}{5}y + \frac{2ac}{5b}\big] & e^{-\frac{ax}{5}}\big[-\frac{c}{3}y - \frac{c^2}{3b} + \frac{3a}{5}z\big]
\end{bmatrix}
\begin{bmatrix}
f_1 \\ f_2 \\ f_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\ b_1 \\ b_0
\end{bmatrix},
\tag{18}
$$

and a first integral $\Omega(x)$ of (14) is given by the following line integral

$$
\Omega(x) = \int_{x_0}^x f_1 dx + f_2 dy + f_3 dz,
$$

$$
\begin{aligned}
f_1 &= 1, \\
f_2 &= \frac{by^2 + \frac{4cy}{3} + \frac{2az}{5} + \frac{c^2}{3b}}{-2cyz - \frac{2c^2z}{b} + \frac{3az^2}{5} - \frac{2aby^3}{5} - \frac{4acy^2}{5} - \frac{2ac^2y}{5b}}, \\
f_3 &= \frac{\frac{2a}{5}y + \frac{2ac}{5b} - z}{-2cyz - \frac{2c^2z}{b} + \frac{3az^2}{5} - \frac{2aby^3}{5} - \frac{4acy^2}{5} - \frac{2ac^2y}{5b}}.
\end{aligned}
\tag{19}
$$

Then, we can obtain a first integral of (14),

$$
\Omega(x, y, z) = -\frac{5}{6a}\ln\Big(-2cyz - \frac{2c^2z}{b} + \frac{3az^2}{5} - \frac{2aby^3}{5} - \frac{4acy^2}{5} - \frac{2ac^2y}{5b}\Big) + x.
$$

It can be rewritten as

$$
\Omega_1 = e^{-\frac{6ax}{5}}\Big(-2cyz - \frac{2c^2z}{b} + \frac{3az^2}{5} - \frac{2aby^3}{5} - \frac{4acy^2}{5} - \frac{2ac^2y}{5b}\Big) = I_1,
\tag{20}
$$

where I_1 is an arbitrary constant.

3.2.2 The first integral under the condition $6a^2 = -25c$

In this case, (14) admits two one-parameter Lie symmetries with generators is

$$
\begin{aligned}
V_1 &= \frac{\partial}{\partial x}, \\
V_3 &= e^{-\frac{ax}{5}}\frac{\partial}{\partial x} + e^{-\frac{ax}{5}}\frac{2a}{5}y\frac{\partial}{\partial y} \\
&\quad + e^{-\frac{ax}{5}}(\frac{c}{3}y + \frac{3a}{5}z)\frac{\partial}{\partial z}.
\end{aligned}
\tag{21}
$$

It is easy to find that

$$
[V_1, V_3] = -\frac{a}{5}V_3.
$$

So we can obtain the corresponding structural coefficients

$$
C_{1,3}^1 = -C_{3,1}^1 = 0, \quad C_{1,3}^2 = -C_{3,1}^2 = -\frac{a}{5}.
$$

We can let $b_1 = 1, b_0 = 0$, and get the solution f_1, f_2, f_3 from the corresponding algebraic system[20],

$$
\begin{bmatrix}
1 & z & az + by^2 + cy \\
1 & 0 & 0 \\
e^{-\frac{ax}{5}} & e^{-\frac{ax}{5}}\frac{2a}{5}y & e^{-\frac{ax}{5}}(\frac{c}{3}y + \frac{3a}{5}z)
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\
b_1 \\
b_0
\end{bmatrix},
\tag{22}
$$

and a first integral $\Omega(x)$ of (14) is given by the following line integral

$$
\Omega(x) = \int_{x_0}^{x} f_1 dx + f_2 dy + f_3 dz,
$$

$$
\begin{aligned}
f_1 &= 1, \\
f_2 &= \frac{az + by^2 + cy - \frac{c}{3}y - \frac{3az}{5}}{\frac{c}{3}yz + \frac{3a}{5}z^2 - \frac{2a^2}{5}yz - \frac{2ab}{5}y^3 - \frac{2ac}{5}y^2}, \\
f_3 &= \frac{\frac{2ay}{5} - z}{\frac{c}{3}yz + \frac{3a}{5}z^2 - \frac{2a^2}{5}yz - \frac{2ab}{5}y^3 - \frac{2ac}{5}y^2}.
\end{aligned}
\tag{23}
$$

Then, we can obtain a first integral of (14),

$$
\Omega = x - \frac{5}{6a}\ln(\frac{3az^2}{5} + 2cyz - \frac{2aby^3}{5} - \frac{2acy^2}{5}).
$$

It can be rewritten as

$$
\Omega_2 = e^{-\frac{6ax}{5}}(\frac{3az^2}{5} + 2cyz - \frac{2aby^3}{5} - \frac{2acy^2}{5}) = I_2,
\tag{24}
$$

where I_2 is an arbitrary constant.

Here, the obtained first integrals (20) and (24) are identical to first integrals in corresponding parametric condition in [23].

4 Application to traveling wave solutions of the certain PDEs

The standard form of the Burgers-KdV equation is

$$
u_t + uu_x + \beta u_{xx} + su_{xxx} = 0,
\tag{25}
$$

where β and s are real constants with $\beta s \neq 0$. In [24], author surveys some recent advances in the study of traveling wave solutions to (25), a class of traveling solitray wave solutions in terms of elliptic

functions with arbitrary velocity is obtained by using the first integral method(is first presented by Feng in 2003) as well as the method of compatible vector fields. The relevant research results of (25) can be referred to [24, 25] and the references therein.

The nonlinear reaction-diffusion equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \mu + u - \beta u^2, \ \mu, \beta \in R. \tag{26}$$

When $\mu = 0$, it is the so-called Fisher equation suggested by Fisher[26]. In [23], Feng applied the Divisor Theorem for two variables in the complex domain, to find a first integral of an equivalent autonomous system. Then, a class of traveling wave solutions is obtained accordingly. In [27], the traveling wave solutions of a similar equation to the nonlinear reaction-diffusion equation(26) were systematically studied.

In [28], the two-dimensional Burgers-Korteweg-de Vries equation

$$(u_t + uu_x + \beta u_{xx} + su_{xxx})_x + \gamma u_{yy} = 0, \tag{27}$$

where β, s and γ are real constants, is considered using the first integral method. The introduction of the two-dimensional Burgers-Korteweg-de Vries equation can be referred to [29] and the references therein.

Assume that equation (25) and (26) have traveling wave solutions of the form $u = u(\xi)$, $\xi = x - vt$. After substitution and performing one integration accordingly, one can have

$$u'' = au' + bu^2 + ru + d. \tag{28}$$

For (25), $a = -\frac{\beta}{s}$, $b = -\frac{1}{2s}$, $r = \frac{v}{s}$, $d = \frac{k}{s}$ and k is an arbitrary integration constant in (28). For (26), $a = -v$, $b = \beta$, $r = -1$, $d = -\mu$ in (28).

Assume equation (27) has the solution in the form $u = u(\xi)$, $\xi = hx + ly - vt$. After substituting the formula to (27), one can have

$$sh^4u^{(4)} + \beta h^3 u''' + \alpha h^2 (uu')' + \gamma l^2 u'' - vhu'' = 0.$$

Integrating the above equation twice with respect to ξ, then we can have (28), where $d = \frac{k}{sh^4}$, k is second integration constant and the first one is set to zero, and $a = -\frac{\beta}{sh}$, $b = -\frac{1}{2sh^2}$, $r = \frac{vh - \gamma l^2}{sh^4}$.

Under the transformation $y = u + \frac{-r \pm \sqrt{r^2 - 4bd}}{2b}$, (28) can be changed to (1), where $c = \pm\sqrt{r^2 - 4bd}$.

In this section, we will consider the traveling wave solutions of equation (25), (26) and (27) based on the results of the first integral of (1). We can use first integrals Ω_1, Ω_2 to derive traveling wave solutions of equation (25), (26) and (27) under corresponding parametric conditions. Comparison with the existing results will also be provided at the end of this section.

4.1 The nonexistence of traveling wave solutions of PDEs

Based on Theorem 1 and Theorem 2, it is not difficult to obtain the following results for the certain partial differential equations.

Theorem 4. 1) $c = \sqrt{r^2 - 4bd}$ and $\frac{a^2 - a\sqrt{a^2 + 4c}}{2c}$ is not a rational number.

2) $c = -\sqrt{r^2 - 4bd}$ and $a^2 + 4c > 0$.

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3) $a^2 + 4c < 0$.

If one of the above conditions is satisfied, then (25), (26) and (27) have no traveling wave solutions in a neighbourhood of $(u, u') = (\dfrac{r - \sqrt{r^2 - 4bd}}{2b}, 0)$.

Theorem 5. 1) $c = -\sqrt{r^2 - 4bd}$ and $\dfrac{-a^2 + a\sqrt{a^2 - 4c}}{2c}$ is not a rational number.

2) $c = \sqrt{r^2 - 4bd}$ and $a^2 - 4c > 0$.

3) $a^2 - 4c < 0$.

If one of the above conditions is satisfied, then (25), (26) and (27) have no traveling wave solutions in a neighbourhood of $(u, u') = (-\dfrac{c}{b} + \dfrac{r - \sqrt{r^2 - 4bd}}{2b}, 0)$.

4.2 Traveling wave solutions of PDEs under the condition $6a^2 = 25c$

From the first integral (20), we can deduce traveling wave solutions of the above PDEs.

Case 1. $I_1 = 0$.

Inserting $z = y'$ into (20), one has

$$2cyy' + \frac{2c^2y'}{b} - \frac{3ay'^2}{5} + \frac{2aby^3}{5} + \frac{4acy^2}{5} + \frac{2ac^2y}{5b} = 0.$$

Solving the above quadratic equation of variable y' and getting the following formula,

$$y' = \frac{-(10bcy + 10c^2) \pm \sqrt{(10bcy + 10c^2)^2 + 12ab(2ab^2y^3 + 4abcy^2 + 2ac^2y)}}{-6ab}.$$

In consideration of the condition $6a^2 = 25c$, we can obtain

$$y' = \frac{5[2c(c + by) \pm 2(c + by)\sqrt{c(c + by)}]}{6ab}. \tag{29}$$

Making a substitution $Y = \sqrt{c + by}$, an exact solution to (29) can be deduced,

$$y = \frac{c}{b}[c_1^2(\frac{e^{\frac{5c}{6a}\xi}}{1 \mp c_1 e^{\frac{5c}{6a}\xi}})^2 - 1],$$

where c_1 is an arbitrary constant. Using the identity

$$\frac{1}{1 + e^{-2x}} = \frac{1}{2}(1 + \tanh x)$$

and choosing $c_1 = \mp 1$, one can get the exact solution to (29)

$$y = \frac{c}{b}[\frac{1}{4}(1 + \tanh \frac{5c}{12a}\xi)^2 - 1].$$

Utilizing the identity $sech^2 t + \tanh^2 t = 1$, the above solution can be expressed as follows,

$$y = -\frac{c}{4b}sech^2 \frac{5c}{12a}\xi + \frac{c}{2b}\tanh \frac{5c}{12a}\xi - \frac{c}{2b}. \tag{30}$$

Owning to (30), one can get the traveling wave solution to (25) under the condition $6a^2 = 25c$ as follows,

$$u = \frac{3\beta^2}{25\alpha}sech^2[\frac{\beta(x - vt)}{10s}] - \frac{6\beta^2}{25\alpha}\tanh[\frac{\beta(x - vt)}{10s}] + v,$$

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the traveling wave solution to (26) under the condition $6a^2 = 25c$ as follows,

$$u = -\frac{3v^2}{50\beta}sech^2[\frac{v}{10}(x-vt)] - \frac{3v^2}{25\beta}\tanh[\frac{3v}{10}(x-vt)] - \frac{\alpha}{2\beta},$$

and the solution to (27) is

$$u = \frac{3\beta^2}{25s}sech^2[\frac{\beta}{10sh}(hx+ly-vt)] + \frac{6\beta^2}{25s}\tanh[\frac{\beta}{10sh}(hx+ly-vt)] + \frac{l^2\gamma - vh}{h^2}.$$

Case 2. $I_1 < 0$

Using the similar method in the literature [23], inserting $z = y'$ into the first integral (20), and the first integral can be expressed as

$$[e^{\frac{-3a}{5}x}(z - \frac{5c}{3a}y - \frac{5c^2}{3ab})]^2 - \frac{2b}{3}[e^{-\frac{2a}{5}x}(y+\frac{c}{b})]^3$$
$$= \{\frac{d}{dx}[(y+\frac{c}{b})e^{-\frac{5c}{3a}x}]\}^2 e^{-\frac{5c}{3a}x} - \frac{2b}{3}[e^{-\frac{2a}{5}x}(y+\frac{c}{b})]^3$$
$$= I_1.$$

Owing to $6a^2 = 25c$, one has $-\frac{5c}{3a} = -\frac{2a}{5}$ and inserts it to the above formula, one has

$$\{\frac{d}{dx}[(y+\frac{c}{b})e^{-\frac{2a}{5}x}]\}^2 e^{-\frac{2a}{5}x} - \frac{2b}{3}[e^{-\frac{2a}{5}x}(y+\frac{c}{b})]^3 = I_1. \tag{31}$$

Let $\Phi = \frac{b}{6}(y+\frac{c}{b})e^{-\frac{2a}{5}x}$, $q = \frac{5}{a}e^{\frac{a}{5}x}$, (31) can be rewritten as the following equation

$$(\frac{d\Phi}{dq})^2 - 4\Phi^3 - I_1 = 0. \tag{32}$$

Its solution can be expressed in terms of the Weierstrass function $\wp(q;g_2,g_3)$ with $g_2 = 0$ and $g_3 = -I_1$. We know that the Weierstrass function $\wp(q;0,-I_1)$ for the standard equation $(\Phi')^2 - 4\Phi^3 - I_1 = 0$ can be expressed by the Jacobian elliptic cosine function[30],

$$\Phi(q) = R + H\frac{1 + cn(2\sqrt{H}q + c_2; \frac{2-\sqrt{3}}{4})}{1 - cn(2\sqrt{H}q + c_2; \frac{2-\sqrt{3}}{4})},$$

where c_2 is an arbitrary constant, $R = -\sqrt[3]{\frac{I_1}{4}}$ and $H = \sqrt{3}R$. Consequently, changing to the original variables and using the inverse transformations of Φ and q, one can get the following formula,

$$y = \frac{3c_3^2a^2}{50b}e^{\frac{2ax}{5}}[\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3e^{\frac{ax}{5}} + c_2; \frac{2-\sqrt{3}}{4})}{1 - cn(c_3e^{\frac{ax}{5}} + c_2; \frac{2-\sqrt{3}}{4})}] - \frac{c}{b},$$

where c_3 is an arbitrary constant.

Accordingly, one can obtain a traveling wave solution to (25) as follows,

$$u(x,t) = -\frac{3c_3^2\beta^2}{25s}e^{-\frac{2\beta}{5s}(x-vt)}[\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3e^{-\frac{\beta}{5s}(x-vt)} + c_2; \frac{2-\sqrt{3}}{4})}{1 - cn(c_3e^{-\frac{\beta}{5s}(x-vt)} + c_2; \frac{2-\sqrt{3}}{4})}] + v \pm \sqrt{v^2 + 2k},$$

one can obtain a traveling wave solution to (26) as follows,

$$u(x,t) = \frac{3c_3^2v^2}{50\beta}e^{-\frac{2v}{5}(x-vt)}[\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3e^{-\frac{v}{5}(x-vt)} + c_2; \frac{2-\sqrt{3}}{4})}{1 - cn(c_3e^{-\frac{v}{5}(x-vt)} + c_2; \frac{2-\sqrt{3}}{4})}] + \frac{\alpha - \sqrt{\alpha^2 + 4\beta\mu}}{2\beta}$$

and the solution to (27) is

$$
\begin{aligned}
u(x,t) &= \frac{-3c_3^2\beta^2}{25s}e^{-\frac{2\beta}{5sh}(hx+ly-vt)}\Big[\frac{\sqrt{3}}{3}+\frac{1+cn(c_3e^{-\frac{\beta}{5sh}(hx+ly-vt)}+c_2;\frac{2-\sqrt{3}}{4})}{1-cn(c_3e^{-\frac{\beta}{5sh}(hx+ly-vt)}+c_2;\frac{2-\sqrt{3}}{4})}\Big]\\
&\pm\frac{\sqrt{(vh-\gamma l^2)^2+2h^2k}}{h^2}.
\end{aligned}
$$

Case 3. $I_1>0$.

It is known([30]) that $\wp(q;0,-I_1)=-\wp(iq;0,I_1)$ and $cn(iq;\frac{2-\sqrt{3}}{4})cn(q;\frac{2+\sqrt{3}}{4})=1$. These relations let us apply the result of (32) for $I_1>0$, and the corresponding solution can be obtained,

$$
\Phi(q)=-R+H\frac{1+cn(2\sqrt{H}q+c_2;\frac{2+\sqrt{3}}{4})}{1-cn(2\sqrt{H}q+c_2;\frac{2+\sqrt{3}}{4})},
$$

where c_2 is an arbitrary constant, $R=\sqrt[3]{\frac{I_1}{4}}$ and $H=\sqrt{3}R$. Consequently, changing to the original variables and using the inverse transformations of Φ and q, one can get the following formula,

$$
y=\frac{3c_3^2a^2}{50b}e^{\frac{2ax}{5}}\Big[-\frac{\sqrt{3}}{3}+\frac{1+cn(c_3e^{\frac{ax}{5}}+c_2;\frac{2+\sqrt{3}}{4})}{1-cn(c_3e^{\frac{ax}{5}}+c_2;\frac{2+\sqrt{3}}{4})}\Big]-\frac{c}{b},
$$

where c_3 is an arbitrary constant.

Similarly, one can obtain a traveling wave solution to (25) as follows,

$$
u(x,t)=-\frac{3c_3^2\beta^2}{25\alpha s}e^{-\frac{2\beta}{5s}(x-vt)}\Big[-\frac{\sqrt{3}}{3}+\frac{1+cn(c_3e^{-\frac{\beta}{5s}(x-vt)}+c_2;\frac{2+\sqrt{3}}{4})}{1-cn(c_3e^{-\frac{\beta}{5s}(x-vt)}+c_2;\frac{2+\sqrt{3}}{4})}\Big]+\frac{12\beta^2}{25s\alpha},
$$

one can obtain a traveling wave solution to (26) as follows,

$$
u(x,t)=\frac{3c_3^2v^2}{50\beta}e^{-\frac{2v}{5}(x-vt)}\Big[-\frac{\sqrt{3}}{3}+\frac{1+cn(c_3e^{-\frac{v}{5}(x-vt)}+c_2;\frac{2+\sqrt{3}}{4})}{1-cn(c_3e^{-\frac{v}{5}(x-vt)}+c_2;\frac{2+\sqrt{3}}{4})}\Big]-\frac{6v^2}{25\beta},
$$

and the solution to (27) is

$$
\begin{aligned}
u(x,t) &= \frac{-3c_3^2\beta^2}{25\alpha s}e^{-\frac{2\beta}{5sh}(hx+ly-vt)}\Big[-\frac{\sqrt{3}}{3}+\frac{1+cn(c_3e^{-\frac{\beta}{5sh}(hx+ly-vt)}+c_2;\frac{2+\sqrt{3}}{4})}{1-cn(c_3e^{-\frac{\beta}{5sh}(hx+ly-vt)}+c_2;\frac{2+\sqrt{3}}{4})}\Big]\\
&+\frac{2\sqrt{(vh-\gamma l^2)sh^2+2\alpha k}}{\alpha h}.
\end{aligned}
$$

4.3 Traveling wave solutions of PDEs under the condition $6a^2=-25c$

Case 1. $I_2=0$.

Inserting $z=y'$ to (24), one has

$$
\frac{3ay'^2}{5}+2cyy'-\frac{2aby^3}{5}-\frac{2acy^2}{5}=0.
$$

Solving the above quadratic equation of variable y' and getting the following formula,

$$
y'=\frac{-10cy\pm\sqrt{(10cy)^2-12a(-2aby^3-2acy^2)}}{6a}.
$$

In consideration of the condition $6a^2=-25c$, we can obtain

$$
y'=\frac{-5cy\pm5y\sqrt{-bcy}}{3a}. \tag{33}
$$

Making a substitution $Y = \sqrt{-bcy}$, an exact solution to (33) can be deduced,

$$y = -\frac{1}{bc}\left(\frac{c}{1 - c_1 e^{\frac{5c}{6a}\xi}}\right)^2,$$

where c_1 is arbitrary constant. Using the identity

$$\frac{1}{1 + e^{-2x}} = \frac{1}{2}(1 + \tanh x)$$

and choosing $c_1 = -1$, one can get the exact solution to (33)

$$y = \frac{c}{4b}[1 + \tanh(-\frac{5c}{12a}\xi)]^2. \tag{34}$$

Utilizing the identity $sech^2 t + \tanh^2 t = 1$, the above solution can be expressed as follows,

$$y = -\frac{c}{4b} sech^2 \frac{5c}{12a}\xi - \frac{c}{2b}\tanh\frac{5c}{12a}\xi + \frac{c}{2b}. \tag{35}$$

Owning to (35), one can get the traveling wave solution to (25) under the condition $6a^2 = -25c$ as follows,

$$u = -\frac{3\beta^2}{25s\alpha} sech^2[\frac{\beta}{10s}(x - vt)] - \frac{6\beta^2}{25s\alpha}\tanh[\frac{\beta}{10s}(x - vt)] + \frac{6\beta^2}{25s\alpha},$$

the traveling wave solution to (26) under the condition $6a^2 = -25c$ as follows,

$$u = \frac{3v^2}{50\beta} sech^2[\frac{v}{10}(x - vt)] + \frac{3v^2}{25\beta}\tanh[\frac{v}{10}(x - vt)] - \frac{3v^2}{25\beta}$$

and the solution to (27) is

$$\begin{aligned}
u &= \frac{\sqrt{(vh - \gamma l^2)sh^2 + 2\alpha k}}{2\alpha h} sech^2[\frac{-5\sqrt{(vh - \gamma h^2)sh^2 + 2\alpha k}}{12\beta h^2}(hx + ly - vt)] \\
&+ \frac{\sqrt{(vh - \gamma l^2)sh^2 + 2\alpha k}}{\alpha h}\tanh[\frac{-5\sqrt{(vh - \gamma l^2)sh^2 + 2\alpha k}}{12\beta h^2}(hx + ly - vt)] \\
&- \frac{\sqrt{(vh - \gamma l^2)sh^2 + 2\alpha k}}{\alpha h}.
\end{aligned}$$

Case 2. $I_2 < 0$

Similarly, using $z = y'$, the first integral (24) can be expressed as

$$[e^{\frac{-3a}{5}x}(z + \frac{5c}{3a}y)]^2 - \frac{2b}{3a}(e^{-\frac{2a}{5}x}y)^3 = I_2.$$

Owning to $6a^2 = -25c$, one has $\frac{5c}{3a} = -\frac{2a}{5}$ and inserts it to the above formula, one has

$$[\frac{d}{dx}(ye^{-\frac{2a}{5}x})]^2 e^{-\frac{2a}{5}x} - \frac{2b}{3a}(ye^{-\frac{2a}{5}x})^3 = I_2. \tag{36}$$

Let $\Phi = \frac{b}{6a}ye^{-\frac{2a}{5}x}$, $q = \frac{5}{a}e^{\frac{a}{5}x}$, (36) can be rewritten as the following equation

$$(\frac{d\Phi}{dq})^2 - 4\Phi^3 - I_2 = 0. \tag{37}$$

Similarly, its solution can be expressed in terms of the Weierstrass function $\wp(q; g_2, g_3)$ with $g_2 = 0$ and $g_3 = -I_2$. We know that the Weierstrass function $\wp(q; 0, -I_2)$ for the standard equation $(\Phi')^2 - 4\Phi^3 - I_2 = 0$ can be expressed by the Jacobian elliptic cosine function[30],

$$\Phi(q) = R + H\frac{1 + cn(2\sqrt{H}q + c_2; \frac{2-\sqrt{3}}{4})}{1 - cn(2\sqrt{H}q + c_2; \frac{2-\sqrt{3}}{4})},$$

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where c_2 is an arbitrary constant, $R = -\sqrt[3]{\dfrac{I_2}{4}}$ and $H = \sqrt{3}R$. Consequently, changing to the original variables and using the inverse transformations of Φ and q, one can get the following formula,

$$y = \frac{3c_3^2 a^3}{50b} e^{\frac{2ax}{5}} \left[\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3 e^{\frac{ax}{5}} + c_2; \frac{2-\sqrt{3}}{4})}{1 - cn(c_3 e^{\frac{ax}{5}} + c_2; \frac{2-\sqrt{3}}{4})} \right],$$

where c_3 is an arbitrary constant.

Accordingly, one can obtain a traveling wave solution to (25) as follows,

$$u(x,t) = -\frac{3c_3^2 \beta^3}{25\alpha s^2} e^{-\frac{2\beta}{5s}(x-vt)} \left[\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3 e^{-\frac{\beta}{5s}(x-vt)} + c_2; \frac{2-\sqrt{3}}{4})}{1 - cn(c_3 e^{-\frac{\beta}{5s}(x-vt)} + c_2; \frac{2-\sqrt{3}}{4})} \right],$$

one can obtain a traveling wave solution to (26) as follows,

$$u(x,t) = \frac{3c_3^2 v^2}{50\beta^2} e^{-\frac{2v}{5}(x-vt)} \left[\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3 e^{-\frac{v}{5}(x-vt)} + c_2; \frac{2-\sqrt{3}}{4})}{1 - cn(c_3 e^{-\frac{v}{5}(x-vt)} + c_2; \frac{2-\sqrt{3}}{4})} \right],$$

and the solution to (27) is

$$u(x,t) = \frac{-3c_3^2 \beta^2}{25\alpha s} e^{-\frac{2\beta}{5sh}(hx+ly-vt)} \left[\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3 e^{-\frac{\beta}{5sh}(hx+ly-vt)} + c_2; \frac{2-\sqrt{3}}{4})}{1 - cn(c_3 e^{-\frac{\beta}{5sh}(hx+ly-vt)} + c_2; \frac{2-\sqrt{3}}{4})} \right].$$

Case 3. $I_2 > 0$.

Similarly, let us apply the result of (37) for $I_2 > 0$, and the corresponding solution can be obtained,

$$\Phi(q) = -R + H \frac{1 + cn(2\sqrt{H}q + c_2; \frac{2+\sqrt{3}}{4})}{1 - cn(2\sqrt{H}q + c_2; \frac{2+\sqrt{3}}{4})},$$

where c_2 is an arbitrary constant, $R = \sqrt[3]{\dfrac{I_2}{4}}$ and $H = \sqrt{3}R$. Consequently, changing to the original variables and using the inverse transformations of Φ and q, one can get the following formula,

$$y = \frac{3c_3^2 a^3}{50b} e^{\frac{2ax}{5}} \left[-\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3 e^{\frac{ax}{5}} + c_2; \frac{2+\sqrt{3}}{4})}{1 - cn(c_3 e^{\frac{ax}{5}} + c_2; \frac{2+\sqrt{3}}{4})} \right],$$

where c_3 is an arbitrary constant.

One can obtain a traveling wave solution to (25) as follows,

$$u(x,t) = -\frac{3c_3^2 \beta^3}{25\alpha s^2} e^{-\frac{2\beta}{5s}(x-vt)} \left[-\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3 e^{-\frac{\beta}{5s}(x-vt)} + c_2; \frac{2+\sqrt{3}}{4})}{1 - cn(c_3 e^{-\frac{\beta}{5s}(x-vt)} + c_2; \frac{2+\sqrt{3}}{4})} \right],$$

one can obtain a traveling wave solution to (26) as follows,

$$u(x,t) = \frac{3c_3^2 v^2}{50\beta^2} e^{-\frac{2v}{5}(x-vt)} \left[-\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3 e^{-\frac{v}{5}(x-vt)} + c_2; \frac{2+\sqrt{3}}{4})}{1 - cn(c_3 e^{-\frac{v}{5}(x-vt)} + c_2; \frac{2+\sqrt{3}}{4})} \right]$$

and the solution to (27) is

$$u(x,t) = \frac{-3c_3^2 \beta^2}{25\alpha s} e^{-\frac{2\beta}{5sh}(hx+ly-vt)} \left[-\frac{\sqrt{3}}{3} + \frac{1 + cn(c_3 e^{-\frac{\beta}{5sh}(hx+ly-vt)} + c_2; \frac{2+\sqrt{3}}{4})}{1 - cn(c_3 e^{-\frac{\beta}{5sh}(hx+ly-vt)} + c_2; \frac{2+\sqrt{3}}{4})} \right].$$

In the section, we deduce the traveling wave solutions of (25), (26) and (27) under parameters conditions $6a^2 = 25c$ and $6a^2 = -25c$ accordingly. The first integrals of (14) and first integrals in

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[23](formula (19) and (20)) are identical as the parameters conditions are changed accordingly. In [24], the author studied the Burgers-Korteweg-de vries equation using the first integral method, and got the traveling wave solution of the equation only in the case $I_1 = 0$ and the case $I_2 = 0$. Almendral and Sanjuan [31] investigated the invariance and integrability properties of the Helmholtz oscillator, and they used the Lie group theory of differential equations to find a first integral only under the parametric restraint $6a^2 = 25c$, which is identical to our formula (20). In [23], authors proposed an effective method(first integral method) to obtain traveling wave solutions of a reaction-diffusion equation (26), which are identical to the above obtained traveling wave solutions of (26) under the condition $6a^2 = \pm 25c$. To our knowledge, the traveling wave solutions of (25) and (27) obtained under the condition $I_1 \neq 0$ or $I_2 \neq 0$ are not found in the existing literature.

5 Conclusions

In this paper, we first presented the condition of nonexistence of the first integrals of a class of ordinary differential equations and the method for getting first integrals of the equations using Lie symmetry. Then, we obtain the conditions of the nonexistence of traveling wave solutions of certain partial differential equations, and we also apply the above first integrals to search for traveling wave solutions of the partial differential equations, which can be converted to the following form through the traveling wave transformation

$$u''(\xi) = au'(\xi) + R(u), \tag{38}$$

where $R(u) = bu^2 + ru + d$, $a \neq 0$, b, r, d are real. We proposed an effective method to deal with (38) using Lie symmetries admitted by (38). Through constructing an algebra equations system based on the generators of Lie symmetries admitted by (38), we obtained first integrals of an explicit form to an equivalent autonomous system under concert parametric conditions. Then using first integrals $I_i > 0, I_i = 0$ and $I_i < 0, i = 1, 2$, a class of traveling wave solutions were accordingly derived by solving this first order differential equation. The method in the paper to obtaining first integrals need not let $R(u)$ in (38) be a polynomial with real coefficients, it can be an arbitrary element function. The technique described herein can be applied to other partial differential equations in other subjects, in which (38) may be any order nonlinear ordinary differential equations and $R(u)$ can be any element functions.

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