On A Generalized Pentanacci Sequence

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Abstract. The well known Pentanacci sequence is a fifth order recurrence sequence. In this paper, it will be defined an other generalized Pentanacci sequence and established some properties of this sequence using matrix methods.

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1. Introduction and Preliminaries

In this work, we define a generalized Pentanacci sequence and establish some properties of this sequence using matrix methods. We use the matrix techniques as in [15].

Pentanacci sequence $\{P_n\}_{n\geq 0}$ and Pentanacci sequence $\{Q_n\}_{n\geq 0}$ are defined by the fifth-order recurrence relations

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, \qquad P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4 \tag{1.1}$$

and

$$Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4} + Q_{n-5}, \qquad Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15 \tag{1.2}$$

respectively. P_n is the sequence $A001591$ in [10] and Q_n is the sequence $A074048$ in [10]. Pentanacci sequence has been studied by many authors, see for example [7], [8], [9].

The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -P_{-(n-1)} - P_{-(n-2)} - P_{-(n-3)} - P_{-(n-4)} + P_{-(n-5)}$$

and

$$Q_{-n} = -Q_{-(n-1)} - Q_{-(n-2)} - Q_{-(n-3)} - Q_{-(n-4)} + Q_{-(n-5)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1) and (1.2) hold for all integer n.

Next, we present the first few values of the Pentanacci and Pentanacci-Lucas numbers with positive and negative subscripts in the following Table 1.

Table 1. A few Pentanacci and Pentanacci-Lucas Numbers

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
P_n	2	0	0	0	-1	1	0	0	0	0	1	1	2	4	8	16	31	61	120
Q_n	-1	-1	-1	-7	9	-1	-1	-1	-1	5	1	3	7	15	31	57	113	223	439

For all integers n, usual Pentanacci and Pentanacci-Lucas numbers can be expressed using Binet's formulas

$$\begin{aligned} P_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)} \end{aligned}$$

(see Theorem 2.2 in [14]) or

$$P_n = \frac{\alpha-1}{6\alpha-10}\alpha^{n-1} + \frac{\beta-1}{6\beta-10}\beta^{n-1} + \frac{\gamma-1}{6\gamma-10}\gamma^{n-1} + \frac{\delta-1}{6\delta-10}\delta^{n-1} + \frac{\lambda-1}{6\lambda-10}\lambda^{n-1} \tag{1.3}$$

(see for example [3])

and

$$Q_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n$$

respectively, where $\alpha, \beta, \gamma, \delta$ and λ are the roots of the equation

$$x^5 - x^4 - x^3 - x^2 - x - 1 = 0. \tag{1.4}$$

Moreover, the approximate value of $\alpha, \beta, \gamma, \delta$ and λ are given by

$$\begin{aligned} \alpha &= 1.9659 \\ \beta &= -0.67835 + 0.45854i \\ \gamma &= -0.67835 - 0.45854i \\ \delta &= 0.19538 + 0.84885i \\ \lambda &= 0.19538 - 0.84885i. \end{aligned}$$

In fact, there are no solutions of the characteristic equation (1.4) in terms of radicals, see [18].

The generating functions for the Pentanacci sequence $\{P_n\}_{n\geq 0}$ and Pentanacci-Lucas sequence $\{Q_n\}_{n\geq 0}$ are

$$f_{P_n}(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-x-x^2-x^3-x^4-x^5}$$

and

$$f_{Q_n}(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{5-4x-3x^2-2x^3-x^4}{1-x-x^2-x^3-x^4-x^5}$$

respectively, (see [14]).

2. Main Results

We consider the generalized Tribonacci sequence defined by

$$E_n = E_{n-1} + E_{n-2} + E_{n-3} + E_{n-4} + E_{n-5}, \qquad E_0 = 5, E_1 = 1, E_2 = 2, E_3 = 0, E_4 = 4. \tag{2.1}$$

The following Table 2 presents the first few values of the generalized Pentanacci numbers E_n with positive and negative subscripts:

Table 2. Generalized Pentanacci numbers E_n with non-negative and negative indices

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
E_n	-4	12	-4	-13	9	0	4	-4	-4	5	1	2	0	4	12	19	37	72	144

Obviously,

$$x^5 - x^4 - x^3 - x^2 - x - 1 = 0$$

is also the characteristic equation of the sequence (2.1) and it produces four roots as $\alpha, \beta, \gamma, \delta$ and λ which are given above. The following Theorem presents the generating function of generalized Pentanacci numbers E_n.

THEOREM 2.1. *The generating function of generalized Pentanacci numbers E_n is given as*

$$f_{E_n}(x) = \frac{5 - 4x - 4x^2 - 8x^3 - 4x^4}{1 - x - x^2 - x^3 - x^4 - x^5}. \tag{2.2}$$

Proof. Let

$$f_{E_n}(x) = \sum_{n=0}^{\infty} E_n x^n$$

be generating function of generalized Pentanacci numbers. Using (2.1) and some calculation, we obtain

$$\begin{aligned} & f_{E_n}(x) - x f_{E_n}(x) - x^2 f_{E_n}(x) - x^3 f_{E_n}(x) - x^4 f_{E_n}(x) - x^5 f_{E_n}(x) \\ = \; & E_0 + (E_1 - E_0)x + (E_2 - E_1 - E_0)x^2 + (E_3 - E_2 - E_1 - E_0)x^3 + (E_4 - E_3 - E_2 - E_1 - E_0)x^4 \end{aligned}$$

which gives (2.2).

We define the square matrix E of order 5 as:

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that

$$\det E = 1.$$

E is called the generating matrix for the sequence (2.1).

THEOREM 2.2.

(a): *For* $n \geq 1$, *we have*

$$\begin{pmatrix} E_{n+4} \\ E_{n+3} \\ E_{n+2} \\ E_{n+1} \\ E_{n} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} E_{n+4} \\ E_{n+2} \\ E_{n+1} \\ E_{n} \\ E_{n-1} \end{pmatrix}. \tag{2.3}$$

(b): *For* $n \geq 0$, *we have*

$$\begin{pmatrix} E_{n+4} \\ E_{n+3} \\ E_{n+2} \\ E_{n+1} \\ E_{n} \end{pmatrix} = E^{n} \begin{pmatrix} E_{4} \\ E_{3} \\ E_{2} \\ E_{1} \\ E_{0} \end{pmatrix}. \tag{2.4}$$

Proof. (a) and (b) can be proved by using induction on n.

Next we present Binet formula for the generalized Pentanacci sequence $\{E_n\}$.

THEOREM 2.3 (Binet Formula for the Generalized Tetranacci Sequence).

$$\begin{aligned} E_n &= 4\Big(\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\Big) \\ &\quad +8\Big(\frac{\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\Big) \\ &\quad +3\Big(\frac{\alpha^{n-2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n-2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^{n-2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n-2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n-2}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\Big) \\ &\quad +2\Big(\frac{\alpha^{n-3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n-3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^{n-3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n-3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n-3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\Big) \\ &= 4P_{n-3} + 8P_{n-4} + 3P_{n-5} + 2P_{n-6}. \end{aligned}$$

Proof. The general form of the generalized Tribonacci sequence can be expressed in the following form

$$E_n = A\alpha^n + B\beta^n + C\gamma^n + D\delta^n + E\lambda^n \tag{2.5}$$

where A, B, C and D are constants that can be determined by the initial conditions. Thus putting the values $n = 0$, $n = 1$, $n = 2$, $n = 3$ and $n = 4$ in (2.5), we obtain

$$\begin{aligned}
A + B + C + D + E &= 5 \\
A\alpha + B\beta + C\gamma + D\delta + E\lambda &= 1 \\
A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 + E\lambda^{n2} &= 2 \\
A\alpha^3 + B\beta^3 + C\gamma^3 + D\delta^3 + E\lambda^3 &= 0 \\
A\alpha^4 + B\beta^4 + C\gamma^4 + D\delta^4 + E\lambda^4 &= 4.
\end{aligned}$$

We can write above system in a matrix form as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta & \lambda \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \lambda^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 & \lambda^3 \\ \alpha^4 & \beta^4 & \gamma^4 & \delta^4 & \lambda^4 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}$$

or

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta & \lambda \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \lambda^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 & \lambda^3 \\ \alpha^4 & \beta^4 & \gamma^4 & \delta^4 & \lambda^4 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}.$$

Solving the above matrix system of equations for A, B, C and D, we get

$$\begin{aligned}
A &= \frac{2(\beta\lambda + \beta\gamma + \lambda\gamma + \beta\delta + \lambda\delta + \gamma\delta) + 4 - (\beta\lambda\gamma + \beta\lambda\delta + \beta\gamma\delta + \lambda\gamma\delta) + 5\beta\lambda\gamma\delta}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} \\
B &= \frac{2(\alpha\lambda + \alpha\gamma + \alpha\delta + \lambda\gamma + \lambda\delta + \gamma\delta) + 4 - (\alpha\lambda\gamma + \alpha\lambda\delta + \alpha\gamma\delta + \lambda\gamma\delta) + 5\alpha\lambda\gamma\delta}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\
C &= \frac{2(\alpha\beta + \alpha\lambda + \beta\lambda + \alpha\delta + \beta\delta + \lambda\delta) + 4 - (\alpha\beta\lambda + \alpha\beta\delta + \alpha\lambda\delta + \beta\lambda\delta) + 5\alpha\beta\lambda\delta}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\
D &= \frac{2(\alpha\beta + \alpha\lambda + \alpha\gamma + \beta\lambda + \beta\gamma + \lambda\gamma) + 4 - (\alpha\beta\lambda + \alpha\beta\gamma + \alpha\lambda\gamma + \beta\lambda\gamma) + 5\alpha\beta\lambda\gamma}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\
E &= \frac{2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) + 4 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) + 5\alpha\beta\gamma\delta}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}.
\end{aligned}$$

Note that we have the following identities:

$$
\begin{aligned}
\alpha+\beta+\gamma+\delta+\lambda &= 1,\\
\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta &= -1,\\
\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta &= 1,\\
\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta &= -1\\
\alpha\beta\gamma\delta\lambda &= 1.
\end{aligned}
$$

It now follows that

$$
\begin{aligned}
& 2(\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)+4-(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)+5\beta\lambda\gamma\delta\\
= & \frac{1}{\alpha^3}(4\alpha^3+8\alpha^2+3\alpha+2)=(4+8\alpha^{-1}+3\alpha^{-2}+2\alpha^{-3})
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
2(\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)+4-(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)+5\alpha\lambda\gamma\delta &= (4+8\beta^{-1}+3\beta^{-2}+2\beta^{-3})\\
2(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)+4-(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)+5\alpha\beta\lambda\delta &= (4+8\gamma^{-1}+3\gamma^{-2}+2\gamma^{-3})\\
2(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)+4-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)+5\alpha\beta\lambda\gamma &= (4+8\delta^{-1}+3\delta^{-2}+2\delta^{-3})\\
2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)+4-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)+5\alpha\beta\gamma\delta &= (4+8\lambda^{-1}+3\lambda^{-2}+2\lambda^{-3}).
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
E_n &= A\alpha^n+B\beta^n+C\gamma^n+D\delta^n+E\lambda^n\\
&= 4(\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad+\frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)})\\
&\quad+8(\frac{\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad+\frac{\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)})\\
&\quad+3(\frac{\alpha^{n-2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^{n-2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad+\frac{\gamma^{n-2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^{n-2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^{n-2}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)})\\
&\quad+2(\frac{\alpha^{n-3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^{n-3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad+\frac{\gamma^{n-3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^{n-3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^{n-3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)})\\
&= 4P_{n-3}+8P_{n-4}+3P_{n-5}+2P_{n-6}.
\end{aligned}
$$

Identities which is given in the following Lemma can be established using by matrix methods.

LEMMA 2.4.

(a): $P_n = \frac{1}{16\,857}(1761E_{n+4} - 801E_{n+3} - 1317E_{n+2} - 105E_{n+1} - 861E_n)$,

(b): $E_n = 4P_{n+4} - 8P_{n+3} - 4P_{n+2} + 9P_{n+1}$,

(c): $E_n = \frac{1}{9584}(-296Q_{n+4} + 6192Q_{n+3} - 8552Q_{n+2} - 7568Q_{n+1} + 8448Q_n)$,

(d): $Q_n = \frac{1}{16\,857}(39E_{n+4} - 1176E_{n+3} + 10\,041E_{n+2} + 4602E_{n+1} + 11\,889E_n)$.

We now present a matrix formula for E_n which is called Simson formula.

THEOREM 2.5 (Simson formula). *For $n \geq 0$ we have*

$$\begin{vmatrix} E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \end{vmatrix} = 16\,857$$

Proof. The proof follows from

$$\begin{vmatrix} E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \end{vmatrix} = \begin{vmatrix} E_4 & E_3 & E_2 & E_1 & E_0 \\ E_3 & E_2 & E_1 & E_0 & E_{-1} \\ E_2 & E_1 & E_0 & E_{-1} & E_{-2} \\ E_1 & E_0 & E_{-1} & E_{-2} & E_{-3} \\ E_0 & E_{-1} & E_{-2} & E_{-3} & E_{-4} \end{vmatrix}. \tag{2.6}$$

The formula (2.6) is given in Soykan [13].

We now obtain the result of Theorem 2.3 (Binet formula for the generalized Tribonacci sequence $\{E_n\}$) using matrix method.

Second Proof of Theorem 2.3 using matrix method (diagonalization).

The characteristic equation of the generating matrix E is

$$0 = |E - xI_4| = \begin{vmatrix} 1-x & 1 & 1 & 1 & 1 \\ 1 & -x & 0 & 0 & 0 \\ 0 & 1 & -x & 0 & 0 \\ 0 & 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 1 & -x \end{vmatrix} = -(x^5 - x^4 - x^3 - x^2 - x - 1)$$

where x is the eigenvalue of E and I_5 is the 5×5 unite matrix. Note that $\alpha, \beta, \gamma, \delta$ and λ are the roots of the characteristic equation $x^5 - x^4 - x^3 - x^2 - x - 1 = 0$ and also $\alpha, \beta, \gamma, \delta$ and λ are the five eigenvalues of the square matrix E. Next we find the eigenvectors corresponding to the eigenvalues $\alpha, \beta, \gamma, \delta$ and λ. We can find the eigenvector by solving the following system of linear equations:

$$(E - xI_5)u_x = 0$$

where u_x is the column vector of order 4×1. First we find the eigenvector corresponding to the eigenvalue α. Then from

$$(E - \alpha I_5)u_\alpha = \begin{pmatrix} 1-\alpha & 1 & 1 & 1 & 1 \\ 1 & -\alpha & 0 & 0 & 0 \\ 0 & 1 & -\alpha & 0 & 0 \\ 0 & 0 & 1 & -\alpha & 0 \\ 0 & 0 & 0 & 1 & -\alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = 0$$

we have the system

$$\begin{aligned} u_2 + u_3 + u_4 + u_5 - u_1(\alpha - 1) &= 0 \\ u_1 - \alpha u_2 &= 0 \\ u_2 - \alpha u_3 &= 0 \\ u_3 - \alpha u_4 &= 0 \\ u_4 - \alpha u_5 &= 0. \end{aligned}$$

If we take $u_5 = c$ in above system we obtain $u_1 = c\alpha^4, u_2 = c\alpha^3, u_3 = c\alpha^2, u_4 = c\alpha$. Thus the eigenvectors corresponding to α are of the form $\begin{pmatrix} c\alpha^4 \\ c\alpha^3 \\ c\alpha^2 \\ c\alpha \\ c \end{pmatrix}$ and in particular if we take $c = 1$ then the eigenvectors corresponding to α is $\begin{pmatrix} \alpha^4 \\ \alpha^3 \\ \alpha^2 \\ \alpha \\ 1 \end{pmatrix}$. Similarly, using the same technique, we see that the eigenvectors corresponding to β, γ, δ and λ are $\begin{pmatrix} \beta^4 \\ \beta^3 \\ \beta^2 \\ \beta \\ 1 \end{pmatrix}$, $\begin{pmatrix} \gamma^4 \\ \gamma^3 \\ \gamma^2 \\ \gamma \\ 1 \end{pmatrix}$, $\begin{pmatrix} \delta^4 \\ \delta^3 \\ \delta^2 \\ \delta \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \lambda^4 \\ \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{pmatrix}$, respectively. Let

$$P = \begin{pmatrix} \alpha^4 & \beta^4 & \gamma^4 & \delta^4 & \lambda^4 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 & \lambda^3 \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \lambda^2 \\ \alpha & \beta & \gamma & \delta & \lambda \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Now let

$$D = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

i.e., D is the diagonal matrix in which the eigenvalues of E are on the main diagonal. Then using the diagonalization of the generating matrix E we obtain $E = PDP^{-1}$. So we get

$$E^n = (PDP^{-1})^n = PD^nP^{-1}.$$

Using the above last equality and (2.4) and comparing the fourth row entries of the matrices we obtain desired result.

3. Conclusion

In this paper, we defined a generalized Pentanacci sequence and proved some properties of this sequence using matrix methods. The method used in this paper can be used for the other linear recurrence sequences, too.

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. Many authors use matrix methods in their work. For example, in the articles [1], [2], [4], [5] and [6], the authors defined some linear recurrence sequences and gave their various properties by matrix methods. On the other hand, the matrix sequences have taken so much interest for different type of numbers. See, for example, [11], [12], [16] and [17]. It is our intention to continue the study and explore some properties of some type of matrix sequences, such as matrix sequences of Tetranacci and Tetranacci-Lucas numbers.

Competing Interests

Author have declared that no competing interests exist.

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