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FIXED POINTS OF CONTRACTIVE TYPE MAPS IN CONE METRIC SPACE OVER BANACH ALGEBRA

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Authors' contributions

This work was carried out in collaboration between all authors. Authors AUR and MU designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author AUR, KQ and GM managed the analyses of the study. Author MU managed the literature searches. All authors read and approved the final manuscript.

ABSTRACT

Our goal in this paper is to prove some fixed point and common fixed theorems for contractive type maps in aCMS over Banach algebra, which unify, extend and generalize most of the existing relevant fixed point theorems from Shaoyuan Xu and StojanRadenovic [20]. We provide illustrative example to verify our results.

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1. INTRODUCTION

As a generalization of metric spaces, cone metric spaces were scrutinized by Huang and Zhang in 2007(see [1]). In CMS (cone metric space) X , $d(u, v)$ for $u, v \in X$ is a vector in an ordered Banach space E , quite apart from that which is a non-negative real number in general metric spaces. They presented the version of the Banach contraction principle and other fundamental theorems in the setting of cone metric spaces. Afterwards, by omitting the assumption of normality in the theorems of [1], Rezapour and Hambarani [2] established some fixed point theorems, as the generalizations and extensions of the analogous results in [1]. Besides, they gave a number of examples to vouch the existence of non-normal cones, which proves that such generalizations are significant. For more details, we refer the reader to [2-14].

Newly, Liu and Xu [15] familiarized the idea of CMS over Banach algebras (which were called CMS over Banach algebras in [15]), replacing Banach spaces by Banach algebras as the underlying spaces of CMS. They replaced the Banach space E by a Banach algebra A and familiarized the idea of CMS over Banach algebras. In this manner, they vouched some fixed point theorems of generalized Lipschitz mappings with natural and weaker conditions on generalized Lipschitz constant h by means of spectral radius. For more details, we refer the reader to [15, 20, and 24].

2. PRELIMINARIES

For the sake of reciter, we shall recollect some fundamental concepts and lemmas. We begin with the following definition as a recall from [15].

Let A always is a real Banach algebra. Then $\forall u, v, w \in A, \alpha \in \mathbb{R}$, we have

1. $(uv)w = u(vw)$;
2. $u(v + w) = uv + uw$ and $(u + v)w = uw + vw$;
3. $\alpha(uv) = (\alpha u)v = u(\alpha v)$;
4. $\|uv\| \leq \|u\|\|v\|$.

We shall assume that a Banach algebra has a multiplicative identity e such that $eu = ue = u, \forall u \in A$. An element $u \in A$ is said to be invertible if there is an inverse element $v \in$

Asuch that $uv = vu = e$. The inverse of u is denoted by u^{-1} . For more details, we refer the reader to [16].

The following proposition is given in [16].

Proposition 2.1 Let A be a Banach algebra with a unit e , and $u \in A$. If the spectral radius $\rho(u)$ of u is less than 1, i.e.,

$$\rho(u) = \lim_{n \rightarrow \infty} \|u^n\|^{\frac{1}{n}} = \inf \|u^n\|^{\frac{1}{n}} < 1,$$

then $e - u$ is invertible. Actually,

$$(e - u)^{-1} = \sum_{i=0}^{\infty} u^i$$

Remark 2.2 From [16] we see that the spectral radius $\rho(u)$ of u satisfies $\rho(u) \leq \|u\|, \forall u \in A$, where A is a Banach algebra with a unit e .

Remark 2.3 (see [16]) In Proposition 2.1, if the condition ' $\rho(u) < 1$ ' is replaced by ' $\|u\| < 1$ ', then the conclusion remains true.

A subset P of A is called a cone of A if

1. P is nonempty closed and $\{\theta, e\} \subset P$;
2. $\delta P + \mu P \subset P$ for all nonnegative real numbers δ, μ ;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra A . For a given cone $P \subset A$, we can define a partial ordering \preceq with respect to P by $u \preceq v$ if and only if $v - u \in P$. $u \prec v$ will stand for $u \preceq v$ and $u \neq v$, while $u \ll v$ will stand for $v - u \in \text{int}P$, where $\text{int}P$ denotes the interior of P . If $\text{int}P \neq \emptyset$, then P is called a solid cone.

The cone P is called normal if there is a number $M > 0$ such that, $\forall u, v \in A, \theta \preceq u \preceq v \Rightarrow \|u\| \leq M\|v\|$. The least positive number satisfying the above is called the normal constant of P (see [1]).

In the following we always assume that A is a Banach algebra with a unit e , P is a solid cone in A and \preceq is the partial ordering with respect to P .

Definition 2.4 (see [1, 15, 17]) Let X be a nonempty set. Suppose that the mapping $d_c: X \times X \rightarrow A$ satisfies

1. $\theta \preceq d_c(u, v), \forall u, v \in X$ and $d_c(u, v) = \theta \Leftrightarrow u = v$;
2. $d_c(u, v) = d_c(v, u), \forall u, v \in X$;
3. $d_c(u, v) \preceq d_c(u, w) + d_c(w, v), \forall u, v, w \in X$.

Then d_c is called a cone metric on X , and (X, d_c) is called a cone metric space (CMS) over Banach algebra A .

Definition 2.5 (see [1, 15, 17]) Let (X, d_c) be a CMS over a Banach algebra A , $x \in X$ and let $\{u_n\}_{n=0}^{\infty} \subset X$ be a sequence. Then:

1. $\{u_n\}_{n=0}^{\infty}$ converges to u whenever for each $c \in A$ with $c \gg \theta \exists$ a natural number N such that $d_c(u_n, u) \ll c$ for all $n \geq N$. We write $\lim_{n \rightarrow \infty} u_n = u$ or $u_n \rightarrow u (n \rightarrow \infty)$.
2. $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence whenever for each $c \in A$ with $c \gg \theta \exists$ a natural number N such that $d_c(u_n, u_m) \ll c$ for all $n, m \geq N$.
3. (X, d_c) is a complete CMS if every Cauchy sequence is convergent in X .

Now, we shall appeal to the following lemmas in the sequel.

Lemma 2.6 (see [18]) If E is a real Banach space with a cone P and if $b \preceq \mu b$ with $b \in P$ and $\theta \preceq b \ll 1$, then $b = \theta$.

Lemma 2.7 (see [9]) If E is a real Banach space with a solid cone P and if $\theta \preceq x \ll c$ for each $\theta \ll c$, then $x = \theta$.

Lemma 2.8(see [9]) If E is a real Banach space with a solid cone P and $\{x_n\} \subset P$ is a sequence with $\|x_n\| \rightarrow 0$ ($n \rightarrow \infty$) then for any $\theta \ll c$, $\exists N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$, i.e. x_n is a c-sequence

Finally, let us recall the concept of generalized Lipschitz mapping defining on the cone metric spaces over Banach algebras, which is introduced in [15].

Definition 2.9(see [15]) Let (X, d_c) be a CMS over a Banach algebra A . A mapping $T: X \rightarrow X$ is called a generalized Lipschitz mapping if there exists a vector $h \in P$ with $\rho(h) < 1$ and for all $u, v \in X$, one has $d_c(Tu, Tv) \leq h d_c(u, v)$

Remark 2.10 In Definition 2.9, we only suppose the spectral radius of h is less than 1, while $\|h\| < 1$ is not assumed. Generally speaking, it is meaningful since by Remark 2.2, the condition $\rho(h) < 1$ is weaker than that $\|h\| < 1$.

Remark 2.11(see [16]) If $\rho(u) < 1$, then $\|u\|^n \rightarrow 0$ ($n \rightarrow \infty$).

Lemma 2.12(see [16]) Let A be a Banach algebra with a unit e , $h, k \in A$. If h commutes with k , then

$$\rho(h + k) \leq \rho(h) + \rho(k),$$

$$\rho(hk) \leq \rho(h)\rho(k).$$

Lemma 2.13(see [16]) If E is a real Banach space with a solid cone P

- (1). If $a_1, a_2, a_3 \in E$ and $a_1 \leq a_2 \ll a_3$, then $a_1 \ll a_3$.
- (2). If $a_1 \in P$ and $a_1 \ll a_3$ for each $a_3 \gg \theta$, then $a_1 = \theta$.

Lemma 2.14(see [20]) Let P be a solid cone in a Banach algebra A . Suppose that $h \in P$ and $\{x_n\} \subset P$ is a c-sequence. Then $\{hx_n\}$ is a c-sequence.

Proposition 2.15(see [20]) Let P be a solid cone in a Banach space A and let $\{u_n\}, \{v_n\} \subset X$ be sequences. If $\{u_n\}$ and $\{v_n\}$ are c-sequences and $\gamma, \delta > 0$ then $\{\gamma u_n + \delta v_n\}$ is a c-sequence.

Proposition 2.16 (see [20]) Let P be a solid cone in a Banach algebra A . and let $\{u_n\} \subset P$ is a sequence.. Then the following conditions are equivalent:

1. $\{u_n\}$ is a c-sequence.
2. For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n < c$ for $n \geq n_0$.
3. For each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $u_n \leq c$ for $n \geq n_1$.

Lemma 2.17 (see [16]) Let A be a Banach algebra with a unit $e, h \in A$, then $\lim_{n \rightarrow \infty} \|h^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(h)$ satisfies

$$\rho(h) = \lim_{n \rightarrow \infty} \|h^n\|^{\frac{1}{n}} = \inf \|h^n\|^{\frac{1}{n}}$$

If $\rho(h) < |\lambda|$, then $(\lambda e - h)$ is invertible in A ; moreover,

$$(\lambda e - h)^{-1} = \sum_{i=0}^{\infty} \frac{h^i}{\lambda^{i+1}}$$

where λ is a constant.

Lemma 2.18 (see [16]) Let A be a Banach algebra with a unit e and $h \in A$. If λ is a complex constant and $\rho(h) < |\lambda|$, then

$$\rho((\lambda e - h)^{-1}) \leq \frac{1}{|\lambda| - \rho(h)}$$

Lemma 2.19 (see [16]) Let A be a Banach algebra with a unit e and P be a solid cone in A . Let $h \in A$ and $u_n = h^n$. If $\rho(h) < 1$, then $\{u_n\}$ is a c-sequence.

Definition 2.20 (see [6]) A pair of maps S and T is called weakly compatible pair if they commute at coincidence points.

Example 2.21 Let $X = [0, 3]$ be equipped with the usual metric space $d(x, y) = |x - y|$. Define $S, T: [0, 3] \rightarrow [0, 3]$ by

$$Su = \begin{cases} u & \text{if } u \in [0,1) \\ 3 & \text{if } u \in [1,3] \end{cases} \text{ and } Tu = \begin{cases} 3 - u & \text{if } u \in [0,1) \\ 3 & \text{if } u \in [1,3] \end{cases}$$

Then for any $u \in [0, 3]$, $STu = TSu$, showing that S, T are weakly compatible maps on $[0, 3]$.

Lemma 2.22 (see[22]). Let T and S be weakly compatible self-maps of a set X . If T and S have a unique point of coincidence $w = Tu = Su$, then w is the unique common fixed point of T and S .

3. MAIN RESULTS

Theorem 3.1 Let (X, d_c) be a CMS over Banach algebra A and P be a solid cone in A . Let $h_i \in P$ ($i = 1, \dots, 5$) be generalized Lipschitz constants with $\rho(h_1) + \rho(h_2 + h_3 + h_4 + h_5) <$

1. Suppose that h_1 commutes with $h_2 + h_3 + h_4 + h_5$ and the mappings $T, S : X \rightarrow X$ satisfy that

$$d_c(Tu, Tv) \leq h_1 d_c(Su, Sv) + h_2 d_c(Tu, Su) + h_3 d_c(Tv, Sv) + h_4 d_c(Su, Tv) + h_5 d_c(Tu, Sv) \quad (3.1)$$

for all $u, v \in X$. If the range of S contains the range of T and $S(X)$ is a complete subspace, then T and S have a unique point of coincidence in X . Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point.

Proof. Suppose $u_0 \in X$ be an arbitrary point. Since $T(X) \subset S(X)$, $\exists u_1 \in X$ such that $Tu_0 = Su_1$. By induction, a sequence $\{Tu_n\}$ can be chosen such that $Tu_n = Su_{n+1}$ ($n = 0, 1, 2, \dots$). Thus, by (3.1), for any natural number n , on the one hand, we obtain

$$\begin{aligned} d_c(Su_{n+1}, Su_n) &= d_c(Tu_n, Tu_{n-1}) \\ &\leq h_1 d_c(Su_n, Su_{n-1}) + h_2 d_c(Tu_n, Su_n) + h_3 d_c(Tu_{n-1}, Su_{n-1}) \\ &\quad + h_4 d_c(Su_n, Tu_{n-1}) + h_5 d_c(Tu_n, Su_{n-1}) \\ &= h_1 d_c(Su_n, Su_{n-1}) + h_2 d_c(Su_{n+1}, Su_n) + h_3 d_c(Su_n, Su_{n-1}) \\ &\quad + h_4 d_c(Su_n, Su_n) + h_5 d_c(Su_{n+1}, Su_{n-1}) \\ &\leq (h_1 + h_3 + h_5) d_c(Su_n, Su_{n-1}) + (h_2 + h_5) d_c(Su_{n+1}, Su_n) \end{aligned}$$

This \Rightarrow

$$(e - h_2 - h_5)d_c(Su_{n+1}, Su_n) \leq (h_1 + h_3 + h_5)d_c(Su_n, Su_{n-1})(3.2)$$

For another thing,

$$\begin{aligned} d_c(Su_n, Su_{n+1}) &= d_c(Tu_{n-1}, Tu_n) \\ &\leq h_1 d_c(Su_{n-1}, Su_n) + h_2 d_c(Tu_{n-1}, Su_{n-1}) + h_3 d_c(Tu_n, Su_n) \\ &\quad + h_4 d_c(Su_{n-1}, Tu_n) + h_5 d_c(Tu_{n-1}, Su_n) \\ &\leq h_1 d_c(Su_{n-1}, Su_n) + h_2 d_c(Su_n, Su_{n-1}) + h_3 d_c(Su_{n+1}, Su_n) \\ &\quad + h_4 d_c(Su_{n-1}, Su_{n+1}) + h_5 d_c(Su_n, Su_n) \\ &\leq (h_1 + h_2 + h_4)d_c(Su_{n-1}, Su_n) + (h_3 + h_4)d_c(Su_n, Su_{n+1}) \end{aligned}$$

This \Rightarrow

$$(e - h_3 - h_4)d_c(Su_n, Su_{n+1}) \leq (h_1 + h_2 + h_4)d_c(Su_{n-1}, Su_n)(3.3)$$

Add up (3.2) and (3.3) produces that

$$(2e - h_2 - h_3 - h_4 - h_5)d_c(Su_n, Su_{n+1}) \leq (2h_1 + h_2 + h_3 + h_4 + h_5)d_c(Su_{n-1}, Su_n)(3.4)$$

Taking $k = h_2 + h_3 + h_4 + h_5$, (3.4) yields that

$$(2e - h)d_c(Su_n, Su_{n+1}) \leq (2h_1 + h)d_c(Su_{n-1}, Su_n) \quad (3.5)$$

Because

$$\rho(h) \leq \rho(h_1) + \rho(h) < 1$$

leads to $\rho(h) < 1 < 2$, then from Lemma 2.17, it concludes that $2e - h$ is invertible.

Furthermore,

$$(2e - h)^{-1} = \sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}}$$

In both sides of (3.5), multiplying by $(2e - h)^{-1}$, we attain at

$$d_c(Su_n, Su_{n+1}) \leq (2e - h)^{-1}(2h_1 + h)d_c(Su_{n-1}, Su_n) \quad (3.6)$$

Taking $(2e - h)^{-1}(2h_1 + h) = k$, by (3.6), we reach

$$d_c(Su_n, Su_{n+1}) \leq kd_c(Su_{n-1}, Su_n) \leq \dots \leq k^n d_c(Su_0, Su_1) = k^n d_c(Su_0, Tu_0) \quad (3.7)$$

Because h_1 commutes with h , it follows that

$$\begin{aligned} (2e - h)^{-1}(2h_1 + h) &= \sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}} (2h_1 + h) \\ &= 2 \left(\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}} \right) h_1 + \left(\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}} \right) h \\ &= 2h_1 \left(\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}} \right) + \sum_{i=0}^{\infty} \frac{h^{i+1}}{2^{i+1}} \\ &= 2h_1 \left(\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}} \right) + h \left(\sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}} \right) \\ &= (2h_1 + h) \sum_{i=0}^{\infty} \frac{h^i}{2^{i+1}} \\ &= (2h_1 + h)(2e - h)^{-1} \end{aligned}$$

To say that $(2e - h)^{-1}$ commutes with $(2h_1 + h)$. By Lemma 2.12 and Lemma 2.18, we avail

$$\begin{aligned} \rho(k) &= \rho((2e - h)^{-1}(2h_1 + h)) \\ &\leq \rho((2e - h)^{-1})\rho(2h_1 + h) \\ &\leq \frac{1}{2 - \rho(h)} [2\rho(h_1) + \rho(h)] < 1 \end{aligned}$$

which shows that $e - k$ is invertible and $\|k^m\| \rightarrow 0$ ($m \rightarrow \infty$). Hence, for any $m \geq 1$; $p \geq 1$ and $k \in P$ with $\rho(k) < 1$, we obtain that

$$\begin{aligned}
d_c(Su_m, Su_{m+p}) &\leq d_c(Su_m, Su_{m+1}) + d(Su_{m+1}, Su_{m+p}) \\
&\leq d_c(Su_m, Su_{m+1}) + d_c(Su_{m+1}, Su_{m+2}) + d_c(Su_{m+2}, Su_{m+p}) \\
&\leq d_c(Su_m, Su_{m+1}) + d_c(Su_{m+1}, Su_{m+2}) + \dots + d_c(Su_{m+p-1}, Su_{m+p}) \\
&\leq k^m d_c(Su_0, Tu_0) + k^{m+1} d_c(Su_0, Tu_0) + \dots + k^{m+p-1} d_c(Su_0, Tu_0) \\
&= k^m [e + k + \dots + k^{p-1}] d_c(Su_0, Tu_0) \\
&\leq k^m (e - k)^{-1} d_c(Su_0, Tu_0) \tag{3.8}
\end{aligned}$$

With advantage of Lemma 2.19 and Lemma 2.14, we obtain $\{Su_n\}$ is a Cauchy sequence. Since $S(X)$ is complete, $\exists z \in S(X)$ such that $Su_n \rightarrow z$ ($n \rightarrow \infty$). Thus $\exists w \in X$ such that $Sw = z$. We shall certify $Tw = z$. In order to finish this, for one thing,

$$\begin{aligned}
d_c(Su_n, Tw) &= d_c(Tu_{n-1}, Tw) \\
&\leq h_1 d_c(Su_{n-1}, Sw) + h_2 d_c(Tu_{n-1}, Su_{n-1}) + h_3 d_c(Tw, Sw) \\
&\quad + h_4 d_c(Su_{n-1}, Tw) + h_5 d_c(Tu_{n-1}, Sw) \\
&= h_1 d_c(Su_{n-1}, z) + h_2 d_c(Su_n, Su_{n-1}) + h_3 d_c(Tw, z) \\
&\quad + h_4 d_c(Su_{n-1}, Tw) + h_5 d_c(Su_n, z) \\
&\leq h_1 d_c(Su_{n-1}, z) + h_2 [d_c(Su_n, z) + d_c(z, Su_{n-1})] \\
&\quad + h_3 [d_c(Tw, Su_n) + d_c(Su_n, z)] \\
&\quad + h_4 [d_c(Su_{n-1}, z) + d_c(z, Su_n) + d_c(Su_n, Tw)] \\
&\quad + h_5 d_c(Su_n, z)
\end{aligned}$$

This \Rightarrow

$$(e - h_3 - h_4)d_c(Su_n, Tw) \leq (h_1 + h_2 + h_4)d_c(Su_{n-1}, z) \\ + (h_2 + h_3 + h_4 + h_5)d_c(Su_n, z) \quad (3.9)$$

On the other hand, we obtain

$$d_c(Su_n, Tw) = d_c(Tu_{n-1}, Tw) = d_c(Tw, Tu_{n-1}) \\ \leq h_1 d_c(Sw, Su_{n-1}) + h_2 d_c(Tw, Sw) + h_3 d_c(Tu_{n-1}, Su_{n-1}) \\ + h_4 d_c(Sw, Tu_{n-1}) + h_5 d_c(Tw, Su_{n-1}) \\ = h_1 d_c(z, Su_{n-1}) + h_2 d_c(Tw, z) + h_3 d_c(Su_n, Su_{n-1}) \\ + h_4 d_c(z, Su_n) + h_5 d_c(Tw, Su_{n-1}) \\ \leq h_1 d_c(z, Su_{n-1}) + h_2 [d_c(Tw, Su_n) + d_c(Su_n, z)] \\ + h_3 [d_c(Su_n, z) + d_c(z, Su_{n-1})] + h_4 d_c(z, Su_n) \\ + h_5 [d_c(Tw, Su_n) + d_c(Su_n, z) + d_c(z, Su_{n-1})]$$

This \Rightarrow

$$(e - h_2 - h_5)d_c(Su_n, Tw) \leq (h_1 + h_3 + h_5)d_c(Su_{n-1}, z) \\ + (h_2 + h_3 + h_4 + h_5)d_c(Su_n, z) \quad (3.10)$$

Combine (3.9) and (3.10), it follows that

$$(2e - h_2 - h_3 - h_4 - h_5)d_c(Su_n, Tw) \leq (2h_1 + h_2 + h_3 + h_4 + h_5)d_c(Su_{n-1}, z) \\ + 2(h_2 + h_3 + h_4 + h_5)d_c(Su_n, z) \\ \Rightarrow (2e - h)d_c(Su_n, Tw) \leq (2h_1 + h)d_c(Su_{n-1}, z) + 2hd_c(Su_n, z)$$

Because

$$\rho(h) \leq \rho(h_1) + \rho(h) < 1 \quad (3.11)$$

thus by Lemma 2.17, it concludes that $2e - h$ is invertible. As a result, it follows immediately from (3.9) that

$$d_c(Su_n, Tw) \leq ((2e - h))^{-1} [(2h_1 + h)d_c(Su_{n-1}, z) + 2hd_c(Su_n, z)]$$

Since $\{d_c(Su_{n-1}, z)\}$ and $\{d_c(Su_n, z)\}$ are c-sequences, then by Lemma 2.14, we acquire that $\{d_c(Su_n, Tw)\}$ is ac-sequence, thus $Su_n \rightarrow Tw$ ($n \rightarrow \infty$). Hence $Tw = Sw = z$. In the following we shall show T and S have a unique point of coincidence.

If $\exists w' \neq w$ such that $Tw' = Sw'$. Then we obtain

$$\begin{aligned} d_c(Sw', Sw) &= d_c(Tw', Tw) \\ &\leq h_1 d_c(Sw', Sw) + h_2 d_c(Tw', Sw') + h_3 d_c(Tw, Sw) \\ &\quad + h_4 d_c(Sw', Tw) + h_5 d_c(Tw', Sw) \\ &= (h_1 + h_4 + h_5) d_c(Sw', Sw) \end{aligned}$$

Set $\gamma = h_1 + h_4 + h_5$, then it follows that

$$d_c(Sw', Sw) \leq \gamma d_c(Sw', Sw) \leq \dots \leq \gamma^n d_c(Sw', Sw) \quad (3.12)$$

Because of

$$\rho(h) \leq \rho(h_1) + \rho(h) < 1$$

Since h_1 commutes with h , then by Lemma 2.12,

$$\rho(h_1 + h) \leq \rho(h_1) + \rho(h) < 1$$

Accordingly, by Lemma 2.19, we speculate that $\{(h_1 + h)^n\}$ is a c-sequence. Noticing that $\gamma \leq h_1 + h$ leadstoy $\gamma^n \leq (h_1 + h)^n$, we claim that $\{\gamma^n\}$ is a c-sequence. Consequently, in view of (3.12), it is easy to see $d_c(Sw', Sw) = \theta$ that is, $Sw' = Sw$.

Finally, if (T, S) is weakly compatible, then by using Lemma 2.22, we claim that T and S have a unique common fixed point.

Corollary 3.2 Let (X, d) be a CMS over Banach algebra A and let P be the solid cone with $h \in P$ where $\rho(h) < 1$. Suppose the mappings $T, S : X \rightarrow X$ satisfy generalized Lipschitz condition:

$$d_c(Tu, Tv) \leq h d_c(Su, Sv) \quad (3.13)$$

for all $u, v \in X$. If the range of S contains the range of T and $S(X)$ is a complete subspace, then T and S have a unique point of coincidence in X . Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point.

Proof. Choose $h_1 = h$ and $h_2 = h_3 = h_4 = h_5 = 0$ in Theorem 3.1, we complete the proof.

Corollary 3.3 Let (X, d) be a CMS over Banach algebra A and let P be the solid cone with $h \in P$ where $\rho(h) < 1$. Suppose the mappings $T, S : X \rightarrow X$ satisfy generalized Lipschitz condition:

$$d_c(Tu, Tv) \leq h [d_c(Tu, Sv) + d_c(Tv, Su)] \quad (3.14)$$

for all $u, v \in X$. If the range of S contains the range of T and $S(X)$ is a complete subspace, then T and S have a unique point of coincidence in X . Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point.

Proof. Choose $h_4 = h_5 = h$ and $h_1 = h_2 = h_3 = 0$ in Theorem 3.1, the proof is valid.

Corollary 3.4 Let (X, d_c) be a CMS over Banach algebra A and let P be the solid cone with $h \in P$ where $\rho(h) < 1$. Suppose the mappings $T, S : X \rightarrow X$ satisfy generalized Lipschitz condition:

$$d_c(Tu, Tv) \leq h [d_c(Tu, Su) + d_c(Tv, Sv)] \quad (3.15)$$

for all $u, v \in X$. If the range of S contains the range of T and $S(X)$ is a complete subspace, then T and S have a unique point of coincidence in X . Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point.

Proof. Choose $h_2 = h_3 = h$ and $h_1 = h_4 = h_5 = 0$ in Theorem 3.1, the claim holds.

Corollary 3.5 Let (X, d_c) be a complete CMS over Banach algebra A and P be a solid cone in A . Let $h_i \in P$ ($i = 1, \dots, 5$) be generalized Lipschitz constants with $\rho(h_1) + \rho(h_2 + h_3 + h_4 + h_5) < 1$. Suppose that h_1 commutes with $h_2 + h_3 + h_4 + h_5$ and the mapping $T : X \rightarrow X$ satisfies that

$$d_c(Tu, Tv) \leq h_1 d_c(u, v) + h_2 d_c(Tu, u) + h_3 d_c(Tv, v)$$

$$+h_4d_c(u, Tv) + h_5d_c(Tu, v)(3.16)$$

for all $u, v \in X$. then T has a unique fixed point in X .

Proof. Taking $S = I_X$ (Identity mapping) in Theorem 3.1, the proof is valid.

Remark 3.6

1. If we take $S = I_X$ and choose $h_1 = h$ and $h_2 = h_3 = h_4 = h_5 = 0$ in Theorem 3.1, we get Theorem 3.1 of Shaoyuan Xu and StojanRadenovic [20].
2. If we take $S = I_X$ and choose $h_4 = h_5 = h$ and $h_1 = h_2 = h_3 = 0$ in Theorem 3.1, we have Theorem 3.2 of Shaoyuan Xu and StojanRadenovic [20].
3. If we take $S = I_X$ and choose $h_1 = h_4 = h_5 = 0$ and $h_2 = h_3 = h$ in Theorem 3.1, we obtain Theorem 3.3 of Shaoyuan Xu and StojanRadenovic [20].

Example 3.7 Let $X = [0,1]$ and A be the set of all real valued functions on X which also have continuous derivatives on X with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$ and the usual multiplication. Let $P = \{u \in A, u(t) \geq 0, t \in X\}$. It is clear that P is a nonnormal cone and A is a Banach algebra with a unit $e = 1$. Define a mapping $d_c: X \times X \rightarrow A$ by $d_c(u, v) = |u - v|e^t$. We make a conclusion that (X, d) is a complete CMS over Banach algebra A . Now define the mappings $T, S: X \rightarrow X$ by $Tu = \frac{u}{8}$ and $Su = \frac{u}{2}$. Choose $h_1 = \frac{1+t}{8}, h_2 = \frac{1+t}{12}, h_3 = \frac{1+t}{16}, h_4 = h_5 = 0$. Simple calculations show that all conditions of Theorem 3.1 are satisfied. Therefore, 0 is the unique common fixed point of T and S .

Conclusion

In this paper, we proved the some fixed point and common fixed point theorems under contractive type conditions in CMS over Banach algebras. Our results are more general than that of the results of Shaoyuan Xu and StojanRadenovic [20]. This result can be extended to other spaces. Example is constructed to support our result.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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