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Approximate Solution Technique for Singular Fredholm Integral Equations of the First Kind with Oscillatory Kernels

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ABSTRACT

An efficient quadrature formula was developed for evaluating numerically certain singular Fredholm integral equations of the first kind with oscillatory trigonometric kernels. The method is based on the Lagrange interpolation formula and the orthogonal polynomial considered are the Legendre polynomials whose zeros served as interpolation nodes. A test example was provided for the verification and validation of the rule developed. The results showed the convergence of the solution and can be improved by increasing n.

Keywords: Singular kernel, Oscillatory kernel, Lagrange interpolation, orthogonal polynomial, Legendre polynomial

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1. INTRODUCTION

The Fredholm integral equations of the first kind with oscillatory kernels

$$\int_{-1}^{1} \frac{e^{i\upsilon t}}{t-x} u(t)dt = f(x), \qquad \upsilon \geq 0, \qquad i^2 = -1, \qquad -1 < x < 1, \tag{1}$$

where f is a given continuous function, and u an unknown function, have wide applications in mathematics, physics, engineering and other applied and computational sciences [12]. If υ is large, the integrand is highly oscillatory and in most cases the integral equation cannot be solved analytically and so, there is need for numerical methods.

Many efficient methods have been developed for the evaluation of oscillatory integrals. The earliest numerical method for evaluating rapidly oscillatory functions were based on the piecewise approximation by second-degree polynomials over an even number of sub intervals and then integrating exactly out the crippling oscillatory factor [6]. An improvement on the Filon's method was done by Flinn [7] whose approximation used fifth-degree polynomials. Stetter [16] used the idea of approximating the transformed function by polynomials in $\frac{1}{t}$. Miklosko [10] proposed to use an interpolating quadrature formula with the Chebyshev's nodes. Piessens and Poleunis [13] approximated the function by a sum of Chebyshev polynomials. Ting and Luke [17] approximated integrals whose integrands are oscillatory and contain singularities at the end points of the interval of integration by expanding the function in series of orthogonal polynomials over the interval of integration with respect to the weight function. Okecha [12] developed algorithms based on the modified Lagrange interpolation formula, Legendre polynomial and the Christoffel-Darboux formula to evaluate Cauchy principal value integrals of oscillatory kind. Different numerical techniques like collocation and Galerkin's methods [4, 8], asymptotic method [9], generalized quadrature rule [5], and modified Clenshaw-Curtis method [18] have also been developed.

Motivated by the work of Okecha [12], the application of the collocation technique to provide solutions of the Fredholm integral equations of the form of Equation (1) is of concern here. The integral in Equation (1) is oscillatory and has a singularity of the Cauchy type. To deal with this pertinent problem, a method based on the Lagrange interpolation formula and on properties of orthogonal polynomials is presented here. The orthogonal polynomials that will be considered are the Legendre polynomials. Suppose q_{n-1} is the Lagrange interpolation polynomial of degree n-1 interpolating to u at the zeros $t_1, t_2, t_3, \cdots, t_n$, of the Legendre polynomial P_n of degree n. Then, by the Lagrange interpolation formula,

$$q_{n-1}(t) = \sum_{k=1}^{n} \frac{P_n(t)u(x_k)}{(t-x_k)P_n'(x_k)} + e_n(t)\,, \tag{2}$$

where

$$e_n(t) = \frac{u^{(n+1)}(\xi_t)}{(n+1)!} \prod_{j=1}^{n} (t-x_j), \quad \xi_t \in (-1,1) \tag{3}$$

is the error due to the interpolation formula.

2. THE APPROXIMATE SOLUTION METHOD

By the substitution of Equation (2) in Equation (1),

$$\sum_{k=1}^{n}\frac{u(x_k)}{P_n^{'}(x_k)}\int_{-1}^{1}\frac{P_n(t)}{(t-x_k)(t-x)}dt+E_n(x,v)=f(x), \tag{4}$$

is obtained and $E_n(x,v)=\int_{-1}^{1}\frac{e^{ivt}e_n(t)}{t-x}dt$ is the error due to the quadrature rule. Subsequently, a bound for $E_n(x.v)$ is obtained. Applying the Christoffel-Darboux formula [1] to Equation (4) gives

$$\sum_{k=1}^{n}\frac{u(t_k)\rho_n\phi_{n+1}}{P_n^{'}(t_k)\phi_n P_{n+1}(t_k)}\sum_{m=0}^{n-1}\frac{P_m(t_k)\mathcal{Z}_m(x,v)}{\rho_m}=-f(x), \tag{5}$$

where

$$\mathcal{Z}_m(x,v)=\int_{-1}^{1}\frac{e^{ivt}P_m(t)}{t-x}dt, \tag{6}$$

$\rho_n=\int_a^b w(t)P_n^2(t)dt$, $P_n(t)=\phi_n t^n+\cdots+\phi_0$ and ϕ_n is the coefficient of the term t^n in $P_n(t)$. However,

$$P_n^{'}(x)=\frac{nxP_n(x)-nP_{n-1}(x)}{x^2-1},\qquad x\neq\pm 1. \tag{7}$$

Thus, from Equation (5) gives

$$\sum_{k=1}^{n}\frac{u(t_k)(t_k^2-1)}{nP_{n-1}(t_k)(n+1)P_{n+1}(t_k)}\sum_{m=0}^{n-1}(2m+1)P_m(t_k)\mathcal{Z}_m(x,v)=f(x). \tag{8}$$

The Legendre polynomials $P_n(x)$ satisfy the recurrence relation

$$(1+l)P_{l+1}(x)-(2l+1)xP_l(x)+lP_{l-1}(x)=0 \tag{9}$$

and from Equations (6) and (9), it can be obtained that [12]

$$(1+l)\mathcal{Z}_{l+1}(x,v)=(2l+1)x\mathcal{Z}_l(x,v)-l\mathcal{Z}_{l-1}(x,v)+(2l+1)\tilde{\mathcal{Z}}_l(v), \tag{10}$$

where

$$\tilde{\mathcal{Z}}_l(v)=\int_{-1}^{1}e^{ivt}P_l(t)dt \tag{11}$$

and $\tilde{Z}(v)$ can be defined as [14]

$$Re[\tilde{Z}(v)]=\int_{-1}^{1}cos(vt)P_l(t)dt=2(-1)^k j_{2k}(v),\ l=2k,\ k=0,1,\cdots \tag{12a}$$

$$Im[\tilde{Z}(v)]=\int_{-1}^{1}sin(vt)P_l(t)dt=2(-1)^k j_{2k+1}(v),\ l=2k+1,\ k=0,1,\cdots \tag{12b}$$

$j_k(x)$ are the spherical Bessel functions of the first kind which can be evaluated as in [1 (Eq. 10.5)].

Let $x_j,\ j=1,\cdots,n$ defined as

$$x_j=-1+\frac{2}{n+2}(j+1) \tag{13}$$

be the collocation points. By placing these points in Equation (8),

$$\sum_{k=1}^{n}\frac{u(t_k)(t_k^2-1)}{nP_{n-1}(t_k)(n+1)P_{n+1}(t_k)}\sum_{m=0}^{n-1}(2m+1)P_m(t_k)\mathcal{Z}_m(x_j,v)$$
$$=f(x_j),\ j=1,2,\cdots,n \tag{14}$$

is obtained and which can be written in matrix form as

$$A\,\mathbf{u} = \mathbf{c}\,, \tag{15}$$

where

$$A = \frac{u(t_k)(t_k^2-1)}{nP_{n-1}(t_k)(n+1)P_{n+1}(t_k)}\sum_{m=0}^{n-1}(2m+1)P_m(t_k)\mathcal{Z}_m(x_j,v),$$

$$\mathbf{c}^T = [f(x_1),\cdots,f(x_n)], \qquad \mathbf{u}^T = [u(x_1),\cdots,u(x_n)]$$

2.1. Techniques in Evaluating $\mathcal{Z}_n(x,v)$

According to Abramowitz and Stegun [1], Legendre polynomial $P_l(x)$ satisfy the recurrence relation [12]

$$P_{l+1}(x) = (A_l + B_l x)P_l(x) - C_l P_{l-1}(x), \;\; l = 0,1,\cdots \tag{16}$$

with $B_l > 0,\; C_l > 0,\; P_0 = 1,\; P_1(x) = A_0 + B_0 x,\; P_{-1} = 0.$ By making use of Equations (11) and (16), it can be written that [12]

$$\mathcal{Z}_{l+1}(x,v) = (A_l + B_l x)\mathcal{Z}_l(x,v) - C_l\mathcal{Z}_{l-1}(x,v) + B_l\tilde{\mathcal{Z}}_l(v). \tag{17}$$

The starting value

$$\mathcal{Z}_0(x,v) = \int_{-1}^{1}\frac{e^{ivt}}{t-x}dt \tag{18}$$

and with the help of Equation (17), it is obtained that

$$\mathcal{Z}_1(x,v) = (A_0 + xB_0)\mathcal{Z}_0(x,v) + 2B_0\frac{sinv}{v}, \tag{19}$$

where $A_0,\; B_0$ are obtained from the coefficients in $P_1(x) = A_0 + B_0 x$. From Okecha [12],

$$Re[Z_0(x,v)] = \int_{-1}^{1}\frac{cosvt}{t-x}dx = cos(vx)Ci(w_1) - sin(vx)Si(w_1) + sin(vx)Si(w_2) - cos(vx)Ci(w_2) \tag{20a}$$

$$Im[Z_0(x,v)] = \int_{-1}^{1}\frac{sinvt}{t-x}dt = cos(vx)Si(w_1) + sin(vx)Ci(w_1) - cos(vx)Si(w_2) - sin(vx)Ci(w_2) \tag{20b}$$,

where

$$w_1 = v(1-x),\;\; w_2 = -v(1+x) \tag{21}$$

and Ci and Si are the cosine and sine integrals respectively. Furthermore, by applying Equation (19),

$$Re[\mathcal{Z}_1(x,v)] = \frac{2sinv}{v} + \; x\,Re[Z_0(x,v)] \tag{22a}$$

$$Im[\mathcal{Z}_1(x,v)] = \; x\,Im[Z_0(x,v)] \tag{22b}$$

2.2. Error Bound Analysis

An error bound based on the Lagrange interpolating polynomials shall be given but first consider the following lemma and theorem.

Lemma 1. Given any function $f(x)$ of bounded variation in $[a,b]$, there can be found a polynomial $Q_n(x)$, of degree n, such that

$$|f(x) - Q_n(x)| < \epsilon, \tag{23}$$

whenever $n \to \infty, \epsilon \to 0$ (Jackson's Theorem) [11].

Theorem 1. Let f be a function in $C^{n+1}[-1,1]$ and let p_n be a polynomial of degree $\leq n$ that interpolates the function f at $(n+1)$ distinct points $x_0, x_1, x_2, \cdots, x_n \in [-1,1]$. Then, for each $x \in [-1,1]$ there exists a point $\xi_x \in [-1,1]$ such that ([15]):

$$f(x) - p_n(x) = \prod_{i=0}^{n}(x - x_i)\frac{f^{(n+1)}(\xi(x))}{(n+1)!}, \qquad (24)$$

Let $g(x)$ be the exact solution of Equation (1) and $p_n(x)$ be the interpolation polynomial of g. Assume that g is sufficiently smooth, then g as $g = p_n + e_n$ can be written, where e_n is the error term expressed as

$$e_n(x) = \prod_{i=0}^{n}(x - x_i)\frac{g^{(n+1)}(\xi(x))}{(n+1)!} \qquad (25)$$

If $u_n(x)$ is the Lagrange polynomial series solution of Equation (1), then $u_n(x)$ satisfies Equation (1) on the nodes and so $u_n(x)$ and $p_n(x)$ are the solutions of $Au = c$ and $A\tilde{u} = c + \Delta c$, where

$$\Delta c \ = \int_{-1}^{1}\frac{e^{ivt}e_n(t)}{t - x_i}dt \qquad (26)$$

Theorem 2. Assume that $u(x)$ and $g(x)$ are Lagrange polynomial series solution and the exact solution of Equation (1) respectively, and let $p_n(x)$ denote the interpolation polynomial of $g(x)$. If $A, u, \tilde{u}, c$ and Δc are defined as above, and $g(x)$ is sufficiently smooth, then

$$|g(x) - u_n(x)| \leq \epsilon \ + \ N\zeta, \qquad (27)$$

where $max_{0<i<n}|u(x_i) - \tilde{u}(x_i)| \leq \ N$, [15]

Proof: By adding and subtracting $p_n(x)$ on the left hand side of Equation (27),

$$|g(x) - u_n(x)| \leq |g(x) - p_n(x)| + |u_n(x) - p_n(x)|$$
$$= |e_n(x)| + |u_n(x) - \ p_n(x)|.$$

is obtained. By using Equation (2) and Lemma 1,

$$|g(x) - u_n(x)| \leq \epsilon + \left|\sum_{i=1}^{n} l_i(x)\,(u_n(t) -_i \tilde{u}_n(t_i))\right|$$
$$\leq \epsilon + \left|\sum_{i=1}^{n} l_i(x)\right| |(u_n(t_i) - \tilde{u}_n(t_i))|$$
$$\leq \epsilon + N\left|\sum_{i=1}^{n} l_i(x)\right|$$
$$= \ \epsilon \ + \ N\left|\sum_{i=1}^{n} l_i(x)\right|$$
$$\leq \epsilon \ + \ N\zeta,$$

is obtained where that the upper bound of $|\sum_{i=1}^{n} l_i(x)| = \left|\sum_{i=1}^{n}\frac{P_n(x)}{(x-x_i)P_n^{'}(x_i)}\right|$ is ζ.

3. NUMERICAL EXAMPLE

Consider the integral equation

$$\int_{-1}^{1}\frac{\sin(12t)}{t - x}u(t)dt = f(x), \qquad -1 < x < 1\,, \qquad (28)$$

where it is chosen that

$$f(x) = \int_{-1}^{1} e^t\,\frac{\sin(12t)}{t - x}dt$$

so that the exact solution will be $u(x) = e^x$. The interpolation points are chosen to be the zeros of the Legendre polynomial, $P_6(x)$ of degree 6

$$t_1 = 0.238619186083197, \qquad t_2 = -0.238619186083197$$
$$t_3 = 0.661209386466265, \qquad t_4 = -0.661209386466265$$
$$t_5 = 0.932469514203152, \qquad t_6 = -0.932469514203152$$

Furthermore, the collocation points are chosen to be

$$x_i = -1 + \tfrac{1}{4}(i+1), \quad i = 1,2,3,4,5,6 \tag{29}$$

Equation (14) is used and n is set equal to 6 to obtain

$$\sum_{k=1}^{6} \frac{u(t_k)(t_k^2-1)}{42P_5(t_k)P_7(t_k)} \sum_{m=0}^{5} (2m+1)P_m(t_k)\mathcal{Z}_m(x_i,12)$$
$$= \sum_{k=1}^{6} \frac{e^{t_k}(t_k^2-1)}{42P_5(t_k)P_7(t_k)} \sum_{m=0}^{5} (2m+1)P_m(t_k)\,\mathcal{Z}_m(x_i,12) \tag{30}$$

$\mathcal{Z}_m(x_i, 12)$ is evaluated at the collocation points defined in relation (29) by using Equations (16), (17) and (18). By making use of Equations (12a) and (12b)

$$\tilde{\mathcal{Z}}_1(12) = 2j_1(12), \qquad \tilde{\mathcal{Z}}_3(12) = -2j_3(12), \qquad \tilde{\mathcal{Z}}_5(12) = 2j_5(12)$$

is obtained. The spherical Bessel functions of the first kind, $j_k(x)$ can be evaluated as follows [1 (Eqn. 10.1.11)]

$$j_0(z) = \frac{\sin\, z}{z}, \qquad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$
$$j_2(z) = \left(-\frac{1}{z} + \frac{3}{z^3}\right)\sin z \; - \frac{3}{z^2}\cos z \tag{31}$$

The spherical Bessel functions of the first kind satisfy the following recurrence relation [2]

$$j_{n+1}(z) = \frac{2n+1}{z} j_n(z) - j_{n-1}(z), \qquad n \in \mathcal{Z} \tag{32}$$

With the help of Equation (31), the recurrence relation (32), and Matlab software, the different values of spherical Bessel functions are obtained. The Si and Ci in the evaluation of $\mathcal{Z}_m(x_i, 12)$ are evaluated from a truncated infinite series defined as

$$Si(z) = \sum_{n1}^{50} \frac{(-1)^{n-1} z^{2(n-1)+1}}{(2(n-1)+1)(2(n-1)+1)!} \tag{33a}$$

$$Ci(z) = \gamma + \ln|z| + \sum_{n=1}^{50} \frac{(-1)^n z^{2n}}{(2n)(2n)!} \;, \tag{33b}$$

where $\gamma = 0.5772156649$ is the Euler's constant. By solving the Equation (30), the results in Table 1 are obtained.

Table 1: Approximation for $\int_{-1}^{1} \frac{\sin(12t)}{t-x} u(t)dt = \int_{-1}^{1} e^{t} \frac{\sin(12t)}{t-x} dt$

t_k	**Approx. (u)**	**Exact(u)**	**Abs. Error**
0.238619186083197	1.269495716853893	1.269495003157037	0.0000007136968593
-0.238619186083197	0.787712023380540	0.787714798020595	0.000002774640055
0.661209386466265	1.937132192317572	1.937133661565611	0.000001469248039
-0.661209386466265	0.516219270254926	0.516226639307785	0.000007369052859
0.932469514203152	2.540784657700185	2.540775918748306	0.000008738951879
-0.932469514203152	0.393534762970032	0.393580556483172	0.000032926782852

From the absolute errors shown on Table 1, it can be seen that the presented method is accurate and efficient and can be improved by increasing n.

4. CONCLUSION

Motivated by the work of Okecha [12], an algorithm was developed to solve singular Fredholm integral equations of the first kind with oscillatory trigonometric kernel and a test example used was derived from example (c) of Okecha [12]. The results obtained shows the convergence of the solution and this can be improved by increasing n.

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COMPETING INTERESTS

The author has declared that no competing interests exist.

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