Original research paper

The cocycle for the non-autonomous stochastic damped wave equations with white noises

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Abstract: This paper is devoted to the cocycle of solutions of the non-autonomous stochastic damped wave equations with multiplicative white noises defined on unbounded domains. And we obtain the existence of a pullback absorbing set of the cocycle in a certain parameter region.

Keywords: stochastic damped wave equations, cocycle, pullback absorbing set

1 Introduction

In this paper, we study the asymptotic behavior of solutions for the following non-autonomous stochastic damped wave equation with multiplicative white noises defined on the unbounded domain $\mathbb{R}^n$:

$$du_t + \alpha du + (\beta u + f(u) - \Delta u)dt = g(x,t)dt + \varepsilon u \circ d\omega, \tag{1.1}$$

with initial conditions

$$u(x,\tau) = u_\tau(x), \quad u_t(x,\tau) = u_\tau(x), \tag{1.2}$$

where $x \in \mathbb{R}^n$ with $1 \leq n \leq 3$, $t > \tau, \tau \in \mathbb{R}, x \in \mathbb{R}^n$, α and β are positive constants, ε is a constant, g is a time-dependent driving force and $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, and ω is a two-sided real-valued Wiener process on a probability space. The stochastic equation (1.1) is understood in the sense of Stratonovich's integration.

Stochastic damped wave equations have been used as models to study the phenomena of a stochastic resonance in physics, where g is a time-dependent input signal and ω is a Wiener process that is used to test the impact of stochastic fluctuations on g ([1]-[3]). Especially, if $\varepsilon = 0$, Eq. (1.1) is a deterministic wave equation, whose longtime behaviors have been studied by many experts, including global attractors, uniform attractors and pullback attractors, see e.g., [4]-[5] and the references therein. And when the function g does not depend on time, then equation (1.1) becomes an autonomous stochastic wave equation.

The equation (1.1) is a non-autonomous equation that the external force term g is time-dependent, and assuming that the external force term $g(x,t)$ satisfies:

$$\int_{-\infty}^{0} e^{\delta s}\|g(\cdot,\tau+s)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{1.3}$$

We remark that the technical hypothesis (1.3) is mainly for the existence of a pullback absorbing set.

In comparison with the results recently published in [6]-[7], the novelty of this work are in two aspects: (i) An Ornstein-Uhlenbeck (O-U) process is introduced to convert the system to a deterministic one with random parameters. (ii) The weakened assumptions (3.2) on the nonlinear term $f(u)$. (iii) The meaningful non-autonomous external force term $g(x,t)$.

This paper is organized as follows. In Section 2 we recall some basic concepts and results related to non-autonomous random dynamical systems. In Section 3 we formulate the problem and make assumptions to define a continuous cocycle generated by the stochastic wave equation (1.1). In Section 4, we conduct uniform estimate to prove the pullback absorbing property for the cocycle.

2 Preliminaries

Let $(\Omega,\mathcal{F},P)$ be a probability space, and $(X,\|\cdot\|_X)$ be a separable Banach space whose Borel σ-algebra is denoted by $\mathscr{B}(X)$.

Defintion 2.1 Let a mapping $\theta_t:\mathbb{R}\times\Omega\to\Omega$ be $(\mathscr{B}(\mathbb{R})\times\mathcal{F},\mathcal{F})$-measurable such that θ_0 is the identity on Ω, $\theta_{t+s}=\theta_t\circ\theta_s$ for all $t,s\in\mathbb{R}$, and $P\theta_t=P$ for all $t\in\mathbb{R}$. A mapping $\Phi:\mathbb{R}^+\times\Omega\times X\to X$ is called a random dynamical system on X over $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$, if for all $\omega\in\Omega$ and $t,s\in\mathbb{R}^+$ the following conditions are satisfied:

(i) $\Phi(t,\omega,\cdot):\mathbb{R}^+\times\Omega\times X\to X$ is a $(\mathscr{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathscr{B}(X),\mathscr{B}(X))$-measurable mapping;

(ii) $\Phi(0,\omega,\cdot)$ is the identity on X;

(iii) $\Phi(t+s,\omega,\cdot)=\Phi(t,\theta_s\omega,\cdot)\circ\Phi(s,\omega,\cdot)$;

(iv) $\Phi(t,\omega,\cdot):X\to X$ is continuous.

Defintion 2.2 Let Φ be a random dynamical system on a Banach space X over $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$.

(1) A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $\{\theta_t\}_{t\in\mathbb{R}}$ if for P-a.e. $\omega\in\Omega$,

$$\lim_{t\to\infty} e^{-\zeta t} d(B(\theta_{-t}\omega))=0 \quad \text{for all } \zeta>0,$$

where $d(B)=\sup_{x\in B}\|x\|_X$.

(2) Let $\mathcal{D}$ be a collection of random subsets of X. The parametric dynamical system Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X, if for any P-a.e. $\omega \in \Omega$ and any sequences $t_n \to \infty$, $x_n \in B(\theta_{-t_n}\omega)$ with $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, the sequence $\{\Phi(t_n, \theta_{-t_n}\omega, x_n)\}$ has a convergent subsequence in X.

(3) Let $\mathcal{D}$ be a collection of random subsets of X and $K = \{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then K is called a random absorbing set for Φ in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \quad \text{for all } t \geq t_B(\omega).$$

In this paper, we will take $\mathcal{D}$ to be the universe of all tempered random subsets of the product Hilbert space $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and prove that the cocycle generated by the stochastic wave equation (1.1) on $\mathbb{R}^n$ has a pullback absorbing set.

3 The cocycle for the stochastic damped wave equation

In this section, we define a continous cocycle for problem (1.1)-(1.2). Let $\xi = u_t + \delta u$, where δ is a positive number to be determined, then (1.1)-(1.2) can be rewritten as the equivalent system

$$\begin{cases} u_t + \delta u = \xi, \\ \xi_t + (\alpha - \delta)\xi + (\delta^2 - \alpha\delta)u - \Delta u + f(u) = g(x,t) + \varepsilon u \circ \frac{d\omega}{dt}, \\ u(x,\tau) = u_0(x), \quad \xi(x,\tau) = \xi_0 = u_1(x) + \delta u_0(x). \end{cases} \tag{3.1}$$

There exists a non-negative constant $c_1 \geq 0$ such that

$$|f(u_1) - f(u_2)| \leq c_1|u_1 - u_2|, \quad f(0) = 0, \quad \forall u_1, u_2 \in \mathbb{R}. \tag{3.2}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space as in Section 2. Define $\{\theta_t\}_{t\in\mathbb{R}}$ on Ω by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$, then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system defined by [8].

To define a cocycle for problem (3.1), we need to convert the system to a deterministic one with random parameters. Now we introduce an Ornstein-Uhlenbeck process given by the Brownian motion. Put

$$z(\theta_t\omega) := -\alpha \int_{-\infty}^{0} e^{\alpha s}(\theta_t\omega)(s)ds, \quad \omega \in \Omega, \quad t \in \mathbb{R}, \tag{3.3}$$

and solves the Itô equation

$$dz + \alpha z dt = d\omega(t). \tag{3.4}$$

From [1], it is known that the random variable $|z(\omega)|$ is tempered, and there is a θ_t-invariant set $\widetilde{\Omega} \subseteq \Omega$ of $\mathbb{P}$ measure such that $|z(\theta_t\omega)|$ is continuous in t for every $\omega \in \widetilde{\Omega}$. For convenience, we write $\widetilde{\Omega}$ as Ω.

Let v be a new variable given by $v(x,t) = \xi(x,t) - \varepsilon u(x,t)z(\theta_t\omega)$. By (3.1), we have

$$\begin{cases} u_t = v + \varepsilon u z(\theta_t\omega) - \delta u, \\ v_t + (\alpha - \delta)v + (\delta^2 - \alpha\delta + A)u + \varepsilon(v - 2\delta u + \varepsilon u z(\theta_t\omega))z(\theta_t\omega) + f(u) = g(x,t), \\ u(x,\tau) = u_0(x), \quad v(x,\tau) = v_0(x), \end{cases} \tag{3.5}$$

where $A = -\Delta, v_0 = u_1 + \delta u_0 - \varepsilon z(\theta_\tau\omega)u_0$.

Let $E = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, endowed with the usual norm

$$\|Y\|_{H^1 \times L^2} = (\|v\|^2 + \|u\|^2 + \|\nabla u\|^2)^{\frac{1}{2}}, \quad for \;\; Y = (u,v)^{\mathcal{T}} \in E, \tag{3.6}$$

where $\|\cdot\|$ denotes the usual norm in $L^2(\mathbb{R}^n)$ and $\mathcal{T}$ stands for the transposition.

The well-posedness of the deterministic problem (3.5) in $E = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ can be established by standard methods as in [8], [9]. One may show that under conditions (3.2), for every $\omega \in \Omega, \tau \in \mathbb{R}$ and $(u_0, v_0) \in E$, problem (3.5) has a unique solution $(u(\cdot,\tau,\omega,u_0), v(\cdot,\tau,\omega,v_0)) \in C([\tau,\infty), E)$ with $(u(\tau,\tau,\omega,u_0), v(\tau,\tau,\omega,v_0)) = (u_0,v_0)$. In addition, for $t \geq \tau$, $(u(t,\tau,\omega,u_0), v(t,\tau,\omega,v_0))$ is $(\mathcal{F}, \mathcal{B}(H^1(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable and continuous in (u_0,v_0) with respect to the norm of E.

Hence, the solution mapping can define a continuous cocycle for (3.1). Let Φ be a mapping, $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ given by

$$\Phi(t,\tau,\omega,(u_0,v_0)) = (u(t+\tau,\tau,\theta_{-\tau}\omega,u_0), v(t+\tau,\tau,\theta_{-\tau}\omega,v_0)) \tag{3.7}$$

for every $(t,\tau,\omega,(u_0,v_0)) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E$, where $v(t+\tau,\tau,\theta_{-\tau}\omega,v_0) = \xi(t+\tau,\tau,\theta_{-\tau}\omega,\xi_0) - \varepsilon z(\theta_t\omega)u(t+\tau,\tau,\theta_{-\tau}\omega,u_0)$ with $v_0 = \xi_0 - \varepsilon z(\omega)u_0$. Then Φ is a continuous cocycle over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ on E. And $\forall t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$, we have

$$\begin{aligned} &\Phi(t,\tau - t,\theta_{-t}\omega,(u_0,v_0)) = (u(\tau,\tau - t,\theta_{-\tau}\omega,u_0), v(\tau,\tau - t,\theta_{-\tau}\omega,v_0)) \\ &= (u(\tau,\tau - t,\theta_{-\tau}\omega,u_0), \xi(\tau,\tau - t,\theta_{-\tau}\omega,\xi_0) - \varepsilon z(\omega)u(\tau,\tau - t,\theta_{-\tau}\omega,u_0)). \end{aligned} \tag{3.8}$$

When deriving uniform estimates on solutions, we need the following condition on g in (1.1):

$$\int_{-\infty}^{0} e^{\delta s}\|g(\cdot,\tau + s)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}, \tag{3.9}$$

and

$$\lim_{k\to\infty}\int_{-\infty}^{0} e^{\delta s}\int_{|x|\geq k}\|g(x,\tau+s)\|^2 dxds = 0. \tag{3.10}$$

The condition (3.9) shows that $g(\cdot,t)$ is not bounded in $L^2(\mathbb{R})$ when $t\to\pm\infty$.

Let B be a bounded nonempty subset of E, and denote by $\|B\| = \sup_{\varphi\in B}\|\varphi\|_E$. Suppose $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ be a family of bounded nonempty subsets of E satisfying, for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\lim_{s\to-\infty} e^{\delta s}\|D(\tau+s,\theta_s\omega)\|^2 = 0. \tag{3.11}$$

Denote by $\mathcal{D}$ the collection of all families of bounded nonempty subsets of E,

$$\mathcal{D} = \{D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}: D \text{ satisfies (3.11)}\}. \tag{3.12}$$

It is evident that $\mathcal{D}$ is neighborhood-closed.

4 Pullback absorbing set

In this section, we derive uniform estimates on the solutions of the stochastic damped wave equations (3.1) defined on $\mathbb{R}^n$ when $t\to\infty$. These estimates are necessary for proving the existence of pullback absorbing sets of the system.

We define a new norm $\|\cdot\|_E$ by

$$\|Y\|_E = (\|v\|^2 + (\delta^2-\alpha\delta)\|u\|^2 + \|\nabla u\|^2)^{\frac{1}{2}}, \tag{4.1}$$

for $Y=(u,v)^{\mathcal{T}}\in E$. It is easy to check that $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H^1\times L^2}$ in (3.6).

Lemma 4.1 *Assume that $\alpha-3\delta>0$, (3.2) and (3.9) hold. Let $\tau\in\mathbb{R},\omega\in\Omega$, $D=\{D(\tau,\omega),\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$. Then there exists $T=T(\tau,\omega,D)>0$, for all $t\geq T$, the solution of problem (3.5) satisfies*

$$Y(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega)) \leq R(\tau,\omega),$$

and $R(\tau,\omega)$ is given by

$$R(\tau,\omega) = M\int_{-\infty}^{0}\exp\{2\int_0^s[\delta-|\varepsilon||z(\theta_r\omega)|-\beta_1(\frac{1}{2}\varepsilon^2|z(\theta_r\omega)|^2+\beta_2|\varepsilon||z(\theta_r\omega)|)]dr\}\|g(\cdot,s+\tau)\|^2 ds, \tag{4.2}$$

where M is a positive constant independent of τ,ω,D and ε.

Proof. Taking the inner product of the second equation of (3.5) with v in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 &= (\delta-\alpha-\varepsilon z(\theta_t\omega))\|v\|^2-(\delta^2-\alpha\delta)(u,v)-(Au,v)\\ &+ \big(\varepsilon z(\theta_t\omega)(2\delta-\varepsilon z(\theta_t\omega))u,v\big)+(g(x,t),v)-(f(u),v).\end{aligned} \tag{4.3}$$

By the first equation of (3.5), we have

$$v=u_t-\varepsilon uz(\theta_t\omega)+\delta u, \tag{4.4}$$

then substituting the above v into the second and third terms on the right-hand side of (4.1), we find that

$$\begin{aligned}(u,v) &= (u,u_t+\delta u-\varepsilon z(\theta_t\omega)u)\\ &= \frac{1}{2}\frac{d}{dt}\|u\|^2+\delta\|u\|^2-\varepsilon z(\theta_t\omega)\|u\|^2\\ &\geq \frac{1}{2}\frac{d}{dt}\|u\|^2+\delta\|u\|^2-|\varepsilon|\cdot|z(\theta_t\omega)|\cdot\|u\|^2,\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}-(Au,v) &= -(\nabla u,\nabla v)\\ &= -(\nabla u,\nabla u_t+\delta\nabla u-\varepsilon z(\theta_t\omega)\nabla u)\\ &= -\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2-\delta\|\nabla u\|^2+\varepsilon z(\theta_t\omega)\|\nabla u\|^2\\ &\leq -\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2-\delta\|\nabla u\|^2+|\varepsilon|\cdot|z(\theta_t\omega)|\cdot\|\nabla u\|^2.\end{aligned} \tag{4.6}$$

Using Cauchy-Schwartz inequality and Young inequality, we have

$$\begin{aligned}\big(\varepsilon z(\theta_t\omega)(2\delta-\varepsilon z(\theta_t\omega))u,v\big) &= (2\delta\varepsilon z(\theta_t\omega)-\varepsilon^2z^2(\theta_t\omega))(u,v)\\ &\leq (2\delta|\varepsilon|\cdot|z(\theta_t\omega)|+\varepsilon^2\cdot|z(\theta_t\omega)|^2)\|u\|\cdot\|v\|\\ &\leq (\delta|\varepsilon|\cdot|z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2\cdot|z(\theta_t\omega)|^2)(\|u\|^2+\|v\|^2),\end{aligned} \tag{4.7}$$

and

$$(g,v)\leq\|g\|\cdot\|v\|\leq\frac{\|g\|^2}{2(\alpha-\delta)}+\frac{\alpha-\delta}{2}\|v\|^2, \tag{4.8}$$

and by (3.2),

$$\begin{aligned}&-(f(u),v)\leq c_1(u,u_t+\delta u-\varepsilon z(\theta_t\omega)u)\\ \leq\ & c_1\frac{d}{dt}\|u\|^2+c_1\delta|u|^2+|\varepsilon|\cdot|z(\theta_t\omega)||u|^2.\end{aligned} \tag{4.9}$$

By (4.5)-(4.9), it follows from (4.3) that

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|v\|^2-(\delta-\alpha-\varepsilon z(\theta_t\omega))\|v\|^2+\frac{1}{2}(c_1+\delta^2-\alpha\delta)\frac{d}{dt}\|u\|^2+\delta(c_1+\delta^2-\alpha\delta)\|u\|^2 \\
& -|\varepsilon||z(\theta_t\omega)|(c_1+\delta^2-\alpha\delta)\|u\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2-(-\delta+|\varepsilon||z(\theta_t\omega)|)\|\nabla u\|^2 \\
\leq\ & (\delta|\varepsilon|\cdot|z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2\cdot|z(\theta_t\omega)|^2)(\|u\|^2+\|v\|^2)+\frac{\alpha-\delta}{2}\|v\|^2+\frac{\|g\|^2}{2(\alpha-\delta)}. \qquad (4.10)
\end{aligned}
$$

Then

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(\|v\|^2+(c_1+\delta^2-\alpha\delta)\|u\|^2+\|\nabla u\|^2)+\delta(\|v\|^2+(c_1+\delta^2-\alpha\delta)\|u\|^2+\|\nabla u\|^2) \\
\leq\ & (\delta|\varepsilon|\cdot|z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2\cdot|z(\theta_t\omega)|^2)(\|u\|^2+\|v\|^2)+\frac{3\delta-\alpha}{2}\|v\|^2+\frac{\|g\|^2}{2(\alpha-\delta)} \\
& +|\varepsilon||z(\theta_t\omega)|(\|v\|^2+(c_1+\delta^2-\alpha\delta)\|u\|^2+\|\nabla u\|^2). \qquad (4.11)
\end{aligned}
$$

From (4.11), we have

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(\|v\|^2+(c_1+\delta^2-\alpha\delta)\|u\|^2+\|\nabla u\|^2) \\
\leq\ & -[\delta-|\varepsilon|\cdot|z(\theta_t\omega)|-\beta_1(\frac{1}{2}\varepsilon^2\cdot|z(\theta_t\omega)|^2+\beta_2|\varepsilon||z(\theta_t\omega)|)](\|v\|^2+(c_1+\delta^2-\alpha\delta)\|u\|^2+\|\nabla u\|^2) \\
& +\frac{\|g\|^2}{2(\alpha-\delta)}, \qquad (4.12)
\end{aligned}
$$

where $\beta_1=1+\frac{1}{c_1+\delta^2-\alpha\delta}$, $\beta_2=\frac{3\delta+\alpha}{2}$.

Denote

$$
\Gamma(t,\omega)=\delta-|\varepsilon|\cdot|z(\theta_t\omega)|-\beta_1(\frac{1}{2}\varepsilon^2\cdot|z(\theta_t\omega)|^2+\beta_2|\varepsilon||z(\theta_t\omega)|). \qquad (4.13)
$$

Using Gronwall inequality to integrate (4.12) over $(\tau-t,\tau)$ with $t\geq 0$, we get

$$
\begin{aligned}
& \|v(\tau,\tau-t,\omega,v_0)\|^2+(c_1+\delta^2-\alpha\delta)\|u(\tau,\tau-t,\omega,u_0)\|^2+\|\nabla u(\tau,\tau-t,\omega,u_0)\|^2 \\
\leq\ & (\|v_0\|^2+(c_1+\delta^2-\alpha\delta)\|u_0\|^2+\|\nabla u_0\|^2)e^{2\int_\tau^{\tau-t}\Gamma(s,\omega)ds} \\
& +c\int_\tau^{\tau-t}e^{2\int_\tau^s\Gamma(r,\omega)dr}\|g(\cdot,s)\|^2ds. \qquad (4.14)
\end{aligned}
$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.14), we obtain, for every $t\in\mathbb{R}^+,\tau\in\mathbb{R},\omega\in\Omega$,

$$
\begin{aligned}
& \|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(c_1+\delta^2-\alpha\delta)\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2 \\
\leq\ & (\|v_0\|^2+(c_1+\delta^2-\alpha\delta)\|u_0\|^2+\|\nabla u_0\|^2)e^{2\int_\tau^{\tau-t}\Gamma(s-\tau,\omega)ds} \\
& +c\int_\tau^{\tau-t}e^{2\int_\tau^s\Gamma(r-\tau,\omega)dr}\|g(\cdot,s)\|^2ds. \qquad (4.15)
\end{aligned}
$$

then

$$
\begin{aligned}
& \|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(c_1+\delta^2-\alpha\delta)\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2 \\
\leq\ & (\|v_0\|^2+(c_1+\delta^2-\alpha\delta)\|u_0\|^2+\|\nabla u_0\|^2)e^{2\int_0^{-t}\Gamma(s,\omega)ds} \\
& +c\int_{-t}^{0} e^{2\int_0^s \Gamma(r,\omega)dr}\|g(\cdot,s+\tau)\|^2 ds. \qquad (4.16)
\end{aligned}
$$

Since $|z(\theta_t\omega)|$ is stationary and ergodic (see [10]), we get from (3.3) and the ergodic theorem that

$$
\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}|z(\theta_r\omega)|dr &= \mathbf{E}(|z(\theta_r\omega)|)=\frac{1}{\sqrt{\pi\delta}}, \\
\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}|z(\theta_r\omega)|^2dr &= \mathbf{E}(|z(\theta_r\omega)|^2)=\frac{1}{2\delta}. \qquad (4.17)
\end{aligned}
$$

By (4.16), there exists $T_1(\omega)>0$ such that for all $t\geq T_1(\omega)$,

$$
\begin{aligned}
\int_{-t}^{0}|z(\theta_r\omega)|dr &= \frac{2}{\sqrt{\pi\delta}}t, \\
\int_{-t}^{0}|z(\theta_r\omega)|^2dr &= \frac{1}{\delta}t. \qquad (4.18)
\end{aligned}
$$

Let ε satisfy

$$
|\varepsilon|<\frac{2\sqrt{\delta}(\beta_1\beta_2+1)+\sqrt{4\delta(\beta_1\beta_2+1)^2+\pi\beta_1\delta^2}}{\beta_1\sqrt{\pi}}, \qquad (4.19)
$$

We have

$$
e^{2\int_0^s\Gamma(r,\omega)dr}\leq e^{2(\frac{\delta}{2})s}=e^{\delta s}, \quad \forall s\leq -T_1. \qquad (4.20)
$$

Since $|z(\theta_s\omega)|$ is tempered, by (3.9) and (4.17), we have the following integral is convergent,

$$
R_1^2(\tau,\omega) \quad = \quad 2c\int_{-\infty}^{0} e^{2\int_0^s\Gamma(r,\omega)dr}(\|g(\cdot,s+\tau)\|^2)ds. \qquad (4.21)
$$

Since $D\in\mathcal{D}$ and $(u_0,v_0)\in D(\tau-t,\theta_{-t}\omega)$, for all $t\geq T_1$, we get from (4.18)-(4.20),

$$
\begin{aligned}
& (\|v_0\|^2+(c_1+\delta^2-\alpha\delta)\|u_0\|^2+\|\nabla u_0)\|^2 e^{2\int_0^{-t}\Gamma(s,\omega)ds} \\
\leq\ & ce^{-\delta t}(\|v_0\|^2+\|u_0\|^2+\|\nabla u_0\|) \\
\leq\ & ce^{-\delta t}(\|D(\tau-t,\theta_{-t}\omega\|^2)\to 0, \quad as \quad t\to+\infty. \qquad (4.22)
\end{aligned}
$$

From (4.1), (4.16), (4.21) and (4.22), there exists $T_2=T_2(\tau,\omega,D)\geq T_1$ such that for all $t\geq T_2$,

$$
\|Y(\tau,\tau-t,\theta_{-\tau}\omega,Y_0(\theta_{-\tau}\omega))\|_E^2\leq R_1^2(\tau,\omega). \qquad (4.23)
$$

So, the proof is completed. □

Moreover, under all the previous assumptions for the cocycle Φ governed by (3.7), we have the following corollary.

Corollary 4.1. Suppose that the external force term $g : \mathbb{R} \to L^2(\mathbb{R})$ is $\gamma - periodic$, then the cocycle Φ governed by (3.7) has a pullback absorbing set in E.

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