Original Research Article

On the Analytical Approximation of the Nonlinear Cubic Oscillator by an Iteration Method

Abstract

A modified approximate analytic solution of the cubic nonlinear oscillator "$\ddot{x}+x^3=0$" has been obtained based on an iteration procedure. Here we have used the truncated Fourier series in each iterative step. The approximate frequencies obtained by this technique show a good agreement with the exact frequency. The percentage of error between exact frequency and our fifth approximate frequency is as low as 0.009%. The calculation with this technique is very easy. This easily-calculated modified technique accelerates the rapid convergence, reduces the error and increases the validity range of the solution.

Keywords: Iteration procedure; Cubic oscillator; Nonlinearity; Nonlinear oscillations.

AMS subject classification: 34A34, 34B99.

1. Introduction:

Most nonlinear phenomena are models of our real-life problems. Nonlinear evolution of equations is widely used as models to describe complex physical phenomena in various fields of science, especially in fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, vibrations, heat transfer and so on. Nonlinear systems are classified differently and 'nonlinear cubic oscillator' is one of them and has its own merit. In this situation Perturbation method, Homotopy method, Homotopy Perturbation method, Harmonic Balance method, Rational Harmonic Balance method, Parameter Expansion method, Iteration method, etc are used to find approximate solutions to nonlinear problems.

The perturbation method is the most widely used method in which the nonlinear term is small. The method of Lindstedt-Poincare (LP) [1-3], Homotopy method [4-7], Homotopy perturbation method [8] and Differential Transform method [9-11] are the most important among all perturbation methods. An important aspect of various perturbation methods is their relationship with each other. Among them, those by Krylov and Bogoliubov [2] are certainly to be found most active. In most treatments of nonlinear oscillations by perturbation methods only periodic oscillations are treated, transients are not considered. They have introduced a new perturbation method to discuss transients.

Harmonic balance (HB) method is another technique for finding the periodic solutions of a nonlinear system. If a periodic solution does not exist of an oscillator, it may be sought in the form of Fourier series and its coefficients are determined by requiring the series to satisfy the equation of motion. HB method which is originated by Mickens [12] and farther work has been done by Mickens [13-15], Lim & Wu [16], Hu [17], Hu & Tang [18], Wu *et al.* [4], Gottlieb [8], Alam *et al.* [19], Haque *et al.* [20], Hosen [21] and so on for solving the strong nonlinear problems. However, in order to avoid solving an infinite system of algebraic equations, it is better to approximate the solution by a suitable finite sum of the trigonometric function. This is the main task of the harmonic balance method. Thus approximate solutions of an oscillator are obtained by harmonic balance method using a suitable truncation Fourier series. The method is capable of determining an analytic approximate solution to the nonlinear oscillator valid even for the case where the nonlinear terms are not small i.e., no particular parameter needs to exist.

The parameter expansion methodology was introduced in a paper by Senator & Bapat [22]. Subsequently, it was extended in a publication of Mickens [23]. However, the full generalization of this concept was done by He [24]. Recently this method was used by Mickens [31] in his book Truly Nonlinear Oscillation and before that also by Xu [25], Zengin *et al*. [26] etc.

Rational harmonic balance approximation technique [27-29] is a useful alternative procedure for calculating a second-order nonlinear dynamical systems. This technique was introduced by Mickens [27] and has been extended in its applications by Beléndez *et al*. [29]. A major advantage of rational approximation is that it gives an implicit inclusion of all the harmonics contributing to the periodic solutions.

Recently, some authors use an iteration procedure [30-35] which is valid for both small and large amplitude of oscillation, to attain the approximate frequency and the harmonious periodic solution of such nonlinear problems. Besides this, the method of Matko & Šafarič [36], Matko [37], Matko & Milanović [38] are used to find an approximate solution in the case of large amplitude of oscillations.

The iterative technique is also used as a technique for calculating approximate periodic solutions and corresponding frequencies of truly nonlinear oscillators for both small and large amplitude of oscillations. The method was originated by R.E. Mickens [30]. Latter, Xu & Cang [39] provided a general basis for iteration method to calculate the approximate periodic solutions of various nonlinear oscillatory successfully. Further, Mickens used the iterative technique to calculate a higher-order approximation to the periodic solutions of a conservative oscillator. Here, the iteration technique for determining the approximate solution of a cubic nonlinear oscillator is presented. In this method only linear inhomogeneous differential equations are required to be solved at each stage of the calculation. It is an important matter for higher order iteration of the solution. The obtained results are compared with those by Mickens Parameter Expansion method [31], Mickens HB method [31] and Mickens Iteration method [31].

2. Methodology

Let us suppose that the nonlinear oscillator

$$\ddot{x} + f(\ddot{x}, x) = 0, \; x(0) = A, \; \dot{x}(0) = 0, \qquad (1)$$

Where over dots denote differentiation with respect to time, t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ to both sides of Eq. (1), we obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\ddot{x}, x) \equiv G(x, \ddot{x}). \qquad (2)$$

Now, we formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \ddot{x}_k); \;\; k = 0,1,2,3,............ \qquad (3)$$

Together with initial condition

$$x_0(t) = A\cos(\Omega_0 t) \qquad (4)$$

Hence x_{k+1} satisfies the initial conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \qquad (5)$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$.

The above procedure gives the sequence of solutions: $x_0(t), \; x_1(t), \; x_2(t), \cdots$.

The method can proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order, usually the second.

At this point, the following observations should be noted:

(a) The solution for $x_{k+1}(t)$ depends on having the solutions for k less than $(k+1)$

(b) The linear differential equation for $x_{k+1}(t)$ allows the determination of Ω_k by the requirement that secular terms be absent. Therefore, the angular frequency, "Ω" appearing on the right-hand side of Eq. (5) in the function $x_k(t)$, is Ω_k.

3. Solution procedure

Let us consider the cubic nonlinear oscillator

$$\ddot{x} + x^3 = 0 \tag{6}$$

Now adding $\Omega^2 x$ to both sides of Equation (6), we obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x^3 \tag{7}$$

Now the iteration scheme is according to Eq. (3)

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - x_k^3 \tag{8}$$

The initial condition is rewritten as

$$x_0(t) = A\cos\theta \tag{9}$$

where $\theta = \Omega_0 t.$ For $k = 0.$ the Eq. (8) becomes

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A cos\theta - A^3 cos^3\theta \tag{10}$$

Now expanding $cos^3\theta$ in a Fourier Cosine series, the Eq. (10) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = (\Omega_0{}^2 - 0.75A^2)A\cos\theta - 0.25A^3\cos 3\theta \tag{11}$$

To check secular terms in the solution, we have to remove $Cos\theta$ from the right-hand side of Eq. (11).

Thus we have

$$\Omega_0 = 0.8660254037844386A \tag{12}$$

Then solving Eq. (11) and satisfying the initial condition $x_1(0) = A$, we obtain

$$x_1(t) = 0.958333295A\cos\theta + 0.041666705A\cos 3\theta \tag{13}$$

This is the first approximate solution of Eq. (6) and the related Ω_1 is to be determined.

The value of Ω_1 will be obtained from the solution of

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - x_1^3 \tag{14}$$

Substituting $x_1(t)$ from Eq. (13) into the right hand side of Eq. (14), we obtain

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 = &\Omega_1^2 (0.95833295\,A\cos\theta + 0.041666705\,A\cos 3\theta) \\ &-(0.6912976924435A^3\cos\theta + 0.2774884478877286A^3\cos 3\theta \\ &+ 0.02994794302083226A^3\cos 5\theta)\end{aligned} \tag{15}$$

Again avoiding secular terms in the solution of Eq. (15), now we obtain

$$\Omega_1 = 0.8493257129433129A \tag{16}$$

Then solving Eq. (15) and satisfying initial condition, we obtain the second approximate solution,

$$x_2(t) = 0.955393886A\cos\theta + 0.04287627A\cos 3\theta + 0.0017298439A\cos 5\theta \tag{17}$$

This is the second approximate solution of Eq. (6)

In similar way, the third and fourth approximate solutions are

$$x_3(t) = 0.955116283A\cos\theta + 0.043038747A\cos 3\theta + 0.00184497A\cos 5\theta \tag{18}$$

$$x_4(t) = 0.9550932806A\cos\theta + 0.043050742A\cos 3\theta + 0.001855971403A\cos 5\theta \tag{19}$$

Whereas the frequencies Ω_2, Ω_3 and Ω_4 are

$$\Omega_2 = 0.8474560185405289A \tag{20}$$

$$\Omega_3 = 0.8473021830725166A \tag{21}$$

$$\Omega_4 = 0.8472887677067594A \tag{22}$$

Thus $\Omega_0, \Omega_1, \Omega_2, \Omega_3, \Omega_4$ respectively obtained by Eqs. (12), (16), (20), (21), (22) represent the approximation of frequencies of oscillator (6).

4. Results and Discussion

An Iteration method is developed based on that by Mickens [30] to solve 'cubic nonlinear oscillator'. In this section, we express the accuracy of the modified technique of iteration method by comparing with the existing results from different methods and with the exact frequency of the oscillator. To show the accuracy, we have calculated the percentage errors (denoted by Er (%)) by the definitions.

$$Er = \left|100\{\Omega_e(A) - \Omega_i(A)\}/\Omega_e(A)\right| \quad ; i = 0, 1, 2, 3, \ldots\ldots, \tag{23}$$

where Ω_i represents the approximate frequencies obtained by the adopted method and Ω_e represents the corresponding exact frequency of the oscillator.

Herein we have calculated the first, second, third, fourth and fifth approximate frequencies which are denoted by Ω_0, Ω_1, Ω_2, Ω_3 and Ω_4 respectively. A comparison among the existing results showed by Mickens [31] with the obtained results is given in Table 1.

It is noted that Mickens [31] found only first approximate frequency by Parameter Expansion, and the second approximate frequencies by harmonic balance method. Mickens [31] also presented only the second approximate frequencies by iteration method.

Table 1: Comparison of the approximate frequencies obtained by the presented technique and other existing results with exact frequency Ω_e [31] of cubic nonlinear oscillator:

Exact Frequency Ω_e	$0.847213A$				
Amplitude A	First Approximate Frequency Ω_0 Er(%)	Second Approximate Frequency Ω_1 Er(%)	Third Approximate Frequency Ω_2 Er(%)	Fourth Approximate Frequency Ω_3 Er(%)	Fifth Approximate Frequency Ω_4 Er(%)
Mickens Parameter Expansion Method [31]	$0.866025A$ 2.2	_	_	_	_
Mickens HB Method [31]	$0.866025A$ 2.2	$0.848875A$ 0.2	_	_	_
Mickens Iteration Method [31]	$0.866025A$ 2.2	$0.849326A$ 0.2	_	_	_
Adopted Method	$0.866025A$ 2.2	$0.849326A$ 0.25	$0.847456A$ 0.03	$0.847302A$ 0.01	$0.847289A$ 0.009

5. Convergence and Consistency Analysis

The basic idea of iteration methods is to construct a sequence of solutions x_k (as well as frequencies Ω_k) that has a convergence property

$$x_e = \lim_{k \to \infty} x_k \quad \text{Or,} \quad \Omega_e = \lim_{k \to \infty} \Omega_k \tag{24}$$

Here x_e is the exact solution of the given nonlinear oscillator.

In the present method, it has been shown that the solution yield the less error in each iterative step compared to the previous iterative step and finally $|\Omega_4 - \Omega_e| = |0.847289 - 0.847213| < \varepsilon$, where ε is a small positive number and A is chosen to be unity. From this, it is clear that the adopted method is convergent.

An iterative method of the form represented by Eq. (3) with initial guesses given in Eq. (4) and Eq. (5) is said to be consistent if

$$\lim_{k \to \infty} |x_k - x_e| = 0 \quad \text{Or,} \quad \lim_{k \to \infty} |\Omega_k - \Omega_e| = 0. \tag{25}$$

In the present analysis we see that

$$\lim_{k \to \infty} |\Omega_k - \Omega_e| = 0, \text{ as } |\Omega_4 - \Omega_e| = 0. \tag{26}$$

Thus the consistency of the method is achieved.

6. Conclusion

An iteration method has been used to solve nonlinear oscillations of conservative single-degree of freedom systems with odd nonlinearity. The method is a powerful and effective mathematical tool in solving nonlinear differential equations of mathematical physics, applied mathematics, and engineering. The iteration procedure can be carried on if solutions of a higher degree of accuracy are required. In this paper, the method has been employed for analytic treatment of the cubic nonlinear differential equation. The adopted method is convergent and obtained solutions are consistent. Already it has been shown in the Table 1 that, Mickens Parameter Expansion method [31], Mickens HB method [31] and Mickens Iteration method [31] are not suitable for higher order approximation because of complexity of calculations and simplifications. But in our method it is very easy to calculate higher order approximations and for these reason the obtained result is closure to exact result with minimum error. Therefore we conclude that the performance of this method is reliable, simple and gives many new solutions.

7. References

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