

# Linear Summing Formulas of Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers

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**Abstract.** In this paper, we present linear summation formulas for generalized Pentanacci numbers and generalized Gaussian Pentanacci numbers. Also, as special cases, we give linear summation formulas of Pentanacci and Pentanacci-Lucas numbers; Gaussian Pentanacci and Gaussian Pentanacci-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery.

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## 1. Introduction and Preliminaries

In this work, we investigate linear summation formulas of generalized Pentanacci numbers and generalized Gaussian Pentanacci numbers. First, in this section, we present some background about generalized Pentanacci numbers.

There have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these type of sequences are the sequences of Pentanacci and Pentanacci-Lucas which are special case of generalized Pentanacci numbers. A generalized Pentanacci sequence  $\{V_n\}_{n \geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4)\}_{n \geq 0}$  is defined by the fifth-order recurrence relations

$$(1.1) \quad V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5},$$

with the initial values  $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4$  not all being zero.

The sequence  $\{V_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} - V_{-(n-4)} + V_{-(n-5)}$$

for  $n = 1, 2, 3, \dots$ . Therefore, recurrence (1.1) holds for all integer  $n$ . Pentanacci sequence has been studied by many authors, see for example [4], [5], [7], [10].

The first few generalized Pentanacci numbers with positive subscript and negative subscript are given in the following Table 1:

Table 1. A few generalized Pentanacci numbers

$n$	$V_n$	$V_{-n}$
0	$c_0$	$c_0$
1	$c_1$	$-c_0 - c_1 - c_2 - c_3 + c_4$
2	$c_2$	$2c_3 - c_4$
3	$c_3$	$2c_2 - c_3$
4	$c_4$	$2c_1 - c_2$
5	$c_0 + c_1 + c_2 + c_3 + c_4$	$2c_0 - c_1$
6	$c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4$	$-3c_0 - 2c_1 - 2c_2 - 2c_3 + 2c_4$
7	$2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4$	$c_0 + c_1 + c_2 + 5c_3 - 3c_4$
8	$4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4$	$4c_2 - 4c_3 + c_4$
9	$8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4$	$4c_1 - 4c_2 + c_3$
10	$16c_0 + 24c_1 + 28c_2 + 30c_3 + 31c_4$	$4c_0 - 4c_1 + c_2$

We consider two special cases of  $\{V_n\}_{n \geq 0}$ . Pentanacci sequence  $\{P_n\}_{n \geq 0}$  and Pentanacci-Lucas sequence  $\{Q_n\}_{n \geq 0}$  are defined by the fifth-order recurrence relations

$$(1.2) \quad P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, \quad P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4$$

and

$$(1.3) \quad Q_n = Q_{n-1} + Q_{n-2} + Q_{n-3} + Q_{n-4} + Q_{n-5}, \quad Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15$$

respectively. Note that  $P_n$  is the sequence A001591 in [8] and  $Q_n$  is the sequence A074048 in [8].

Next, we present the first few values of the Pentanacci and Pentanacci-Lucas numbers with positive and negative subscripts in the following Table 2:

Table 2. A few Pentanacci and Pentanacci-Lucas Numbers

$n$	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
$P_n$	2	0	0	0	-1	1	0	0	0	0	1	1	2	4	8	16	31	61	120
$Q_n$	-1	-1	-1	-7	9	-1	-1	-1	-1	5	1	3	7	15	31	57	113	223	439

## 2. Linear Sums of Generalized Pentanacci Numbers

Some summing formulas of the Pell and Pell-Lucas numbers are well known and given in [2, 3], see also [1]. For linear sums of Tribonacci and Tetranacci numbers, see [6] and [9, 11], respectively. The following Theorem present some summation formulas of generalized Pentanacci numbers.

**THEOREM 2.1.** *For  $n \geq 0$ , we have the following linear sum identities:*

- (a):  $\sum_{k=0}^n V_k = \frac{1}{4}(V_{n+4} - V_{n+2} - 2V_{n+1} + V_n - V_4 + V_2 + 2V_1 + 3V_0)$
- (b):  $\sum_{k=0}^n V_{2k+1} = \frac{1}{8}(3V_{2n+2} + 4V_{2n+1} + V_{2n} + 2V_{2n-1} - V_{2n-2} - 3V_4 + 4V_3 - V_2 + 6V_1 + V_0)$
- (c):  $\sum_{k=0}^n V_{2k} = \frac{1}{8}(-V_{2n+2} + 4V_{2n+1} + 5V_{2n} + 2V_{2n-1} + 3V_{2n-2} + V_4 - 4V_3 + 3V_2 - 2V_1 + 5V_0)$
- (d):  $\sum_{k=0}^n V_{3k} = \frac{1}{4}(-V_{3n+3} + 2V_{3n+2} + V_{3n+1} + 2V_{3n} + V_{3n-1} - V_4 + 2V_3 - V_2 + 3V_0)$
- (e):  $\sum_{k=0}^n V_{3k+1} = \frac{1}{4}(V_{3n+3} + V_{3n+1} - V_{3n-1} + V_4 - 2V_3 - V_2 + 2V_1 - V_0)$
- (f):  $\sum_{k=0}^n V_{3k+2} = \frac{1}{4}(V_{3n+3} + 2V_{3n+2} + V_{3n+1} + V_{3n-1} - V_4 + 3V_2 + V_0)$
- (g):  $\sum_{k=0}^n V_{4k} = \frac{1}{16}(-5V_{4n+4} + 4V_{4n+3} + 9V_{4n+2} + 10V_{4n+1} + 7V_{4n} + 5V_4 - 4V_3 - 9V_2 - 10V_1 + 9V_0)$
- (h):  $\sum_{k=0}^n V_{4k+1} = \frac{1}{16}(-V_{4n+4} + 4V_{4n+3} + 5V_{4n+2} + 2V_{4n+1} - 5V_{4n} + V_4 - 4V_3 - 5V_2 + 14V_1 + 5V_0)$
- (i):  $\sum_{k=0}^n V_{4k+2} = \frac{1}{16}(3V_{4n+4} + 4V_{4n+3} + V_{4n+2} - 6V_{4n+1} - V_{4n} - 3V_4 - 4V_3 + 15V_2 + 6V_1 + V_0)$
- (j):  $\sum_{k=0}^n V_{4k+3} = \frac{1}{16}(7V_{4n+4} + 4V_{4n+3} - 3V_{4n+2} + 2V_{4n+1} + 3V_{4n} - 7V_4 + 12V_3 + 3V_2 - 2V_1 - 3V_0)$
- (k):  $\sum_{k=0}^n V_{5k} = \frac{1}{4}(-3V_{5n+5} + 4V_{5n+4} + 3V_{5n+3} + 2V_{5n+2} + V_{5n+1} - V_4 + V_2 + 2V_1 + 7V_0)$
- (l):  $\sum_{k=0}^n V_{5k+1} = \frac{1}{4}(V_{5n+5} - V_{5n+3} - 2V_{5n+2} - 3V_{5n+1} - V_4 + V_2 + 6V_1 - V_0)$
- (m):  $\sum_{k=0}^n V_{5k+2} = \frac{1}{4}(V_{5n+5} - V_{5n+3} - 2V_{5n+2} + V_{5n+1} - V_4 + 5V_2 - 2V_1 - V_0)$
- (n):  $\sum_{k=0}^n V_{5k+3} = \frac{1}{4}(V_{5n+5} - V_{5n+3} + 2V_{5n+2} + V_{5n+1} - V_4 + 4V_3 - 3V_2 - 2V_1 - V_0)$
- (o):  $\sum_{k=0}^n V_{5k+4} = \frac{1}{4}(V_{5n+5} + 3V_{5n+3} + 2V_{5n+2} + V_{5n+1} + 3V_4 - 4V_3 - 3V_2 - 2V_1 - V_0)$ .

*Proof.*

(a): Using the recurrence relation

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5}$$

i.e.

$$V_n - V_{n-1} = V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} = V_{n-5} + V_{n-4} + V_{n-3} + V_{n-2}$$

we obtain

$$\begin{aligned}
V_5 - V_4 &= V_0 + V_1 + V_2 + V_3 \\
V_6 - V_5 &= V_1 + V_2 + V_3 + V_4 \\
V_7 - V_6 &= V_2 + V_3 + V_4 + V_5 \\
V_8 - V_7 &= V_3 + V_4 + V_5 + V_6 \\
V_9 - V_8 &= V_4 + V_5 + V_6 + V_7 \\
&\vdots \\
V_n - V_{n-1} &= V_{n-5} + V_{n-4} + V_{n-3} + V_{n-2} \\
V_{n+1} - V_n &= V_{n-4} + V_{n-3} + V_{n-2} + V_{n-1} \\
V_{n+2} - V_{n+1} &= V_{n-3} + V_{n-2} + V_{n-1} + V_n \\
V_{n+3} - V_{n+2} &= V_{n-2} + V_{n-1} + V_n + V_{n+1} \\
V_{n+4} - V_{n+3} &= V_{n-1} + V_n + V_{n+1} + V_{n+2} \\
V_{n+5} - V_{n+4} &= V_n + V_{n+1} + V_{n+2} + V_{n+3}
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
V_{n+5} - V_4 &= \sum_{k=0}^n V_k + \left( V_{n+1} - V_0 + \sum_{k=0}^n V_k \right) + \left( V_{n+2} + V_{n+1} - V_1 - V_0 + \sum_{k=0}^n V_k \right) \\
&\quad + \left( V_{n+3} + V_{n+2} + V_{n+1} - V_2 - V_1 - V_0 + \sum_{k=0}^n V_k \right)
\end{aligned}$$

or

$$4 \sum_{k=0}^n V_k = V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} - V_4 + V_2 + 2V_1 + 3V_0$$

which maybe reduced easily to (a) by using (1.1) and dividing both sides by 4. Note that

$$\begin{aligned}
V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} &= (V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} + V_n) - V_{n+3} - 2V_{n+2} - 3V_{n+1} \\
&= V_{n+4} - V_{n+2} - 2V_{n+1} + V_n
\end{aligned}$$

**(b),(c):** We write the following obvious equations;

$$\begin{aligned}
 V_3 &= V_4 - V_2 - V_1 - V_0 - V_{-1} \\
 V_5 &= V_6 - V_4 - V_3 - V_2 - V_1 \\
 V_7 &= V_8 - V_6 - V_5 - V_4 - V_3 \\
 V_9 &= V_{10} - V_8 - V_7 - V_6 - V_5 \\
 V_{11} &= V_{12} - V_{10} - V_9 - V_8 - V_7 \\
 V_{13} &= V_{14} - V_{12} - V_{11} - V_{10} - V_9 \\
 V_{15} &= V_{16} - V_{14} - V_{13} - V_{12} - V_{11} \\
 &\vdots \\
 V_{2n-1} &= V_{2n} - V_{2n-2} - V_{2n-3} - V_{2n-4} - V_{2n-5} \\
 V_{2n+1} &= V_{2n+2} - V_{2n} - V_{2n-1} - V_{2n-2} - V_{2n-3}.
 \end{aligned}$$

Now, adding these equations we have

$$\begin{aligned}
 -V_1 + \sum_{k=0}^n V_{2k+1} &= \left( -V_0 - V_2 + V_{2n+2} + \sum_{k=0}^n V_{2k} \right) + \left( V_0 - \sum_{k=0}^n V_{2k} \right) \\
 &\quad + \left( V_{2n+1} - \sum_{k=0}^n V_{2k+1} \right) + \left( V_{2n} - \sum_{k=0}^n V_{2k} \right) + \left( V_{2n+1} + V_{2n-1} - V_{-1} - \sum_{k=0}^n V_{2k+1} \right)
 \end{aligned}$$

or

$$3 \sum_{k=0}^n V_{2k+1} = V_1 - V_{-1} - V_2 + V_{2n+2} + 2V_{2n+1} + V_{2n} + V_{2n-1} - \sum_{k=0}^n V_{2k}.$$

or using  $V_{-1} = V_4 - V_3 - V_2 - V_1 - V_0$ ,

$$3 \sum_{k=0}^n V_{2k+1} = -V_4 + V_3 + 2V_1 + V_0 + V_{2n+2} + 2V_{2n+1} + V_{2n} + V_{2n-1} - \sum_{k=0}^n V_{2k}.$$

Note that

$$V_1 - V_{-1} - V_2 = V_1 - (V_4 - V_3 - V_2 - V_1 - V_0) - V_2 = -V_4 + V_3 + 2V_1 + V_0.$$

Similarly, we write the following obvious equations;

$$\begin{aligned}
V_2 &= V_3 - V_1 - V_0 - V_{-1} - V_{-2} \\
V_4 &= V_5 - V_3 - V_2 - V_1 - V_0 \\
V_6 &= V_7 - V_5 - V_4 - V_3 - V_2 \\
V_8 &= V_9 - V_7 - V_6 - V_5 - V_4 \\
V_{10} &= V_{11} - V_9 - V_8 - V_7 - V_6 \\
V_{12} &= V_{13} - V_{11} - V_{10} - V_9 - V_8 \\
V_{14} &= V_{15} - V_{13} - V_{12} - V_{11} - V_{10} \\
&\vdots \\
V_{2n-2} &= V_{2n-1} - V_{2n-3} - V_{2n-4} - V_{2n-5} - V_{2n-6} \\
V_{2n} &= V_{2n+1} - V_{2n-1} - V_{2n-2} - V_{2n-3} - V_{2n-4}.
\end{aligned}$$

Now, adding these equations, we have

$$\begin{aligned}
-V_0 + \sum_{k=0}^n V_{2k} &= \left( -V_1 + \sum_{k=0}^n V_{2k+1} \right) + \left( V_{2n+1} - \sum_{k=0}^n V_{2k+1} \right) + \left( V_{2n} - \sum_{k=0}^n V_{2k} \right) \\
&\quad + \left( V_{2n+1} + V_{2n-1} - V_{-1} - \sum_{k=0}^n V_{2k+1} \right) + \left( V_{2n-2} + V_{2n} - V_{-2} - \sum_{k=0}^n V_{2k} \right)
\end{aligned}$$

or

$$3 \sum_{k=0}^n V_{2k} = (-V_{-2} - V_{-1} + V_0 - V_1) + 2V_{2n+1} + 2V_{2n} + V_{2n-1} + V_{2n-2} - \sum_{k=0}^n V_{2k+1}$$

or using  $V_{-2} = V_3 - V_2 - V_1 - V_0 - V_{-1}$ ,

$$3 \sum_{k=0}^n V_{2k} = -V_3 + V_2 + 2V_0 + 2V_{2n+1} + 2V_{2n} + V_{2n-1} + V_{2n-2} - \sum_{k=0}^n V_{2k+1}.$$

Note that

$$-V_{-2} - V_{-1} + V_0 - V_1 = -(V_3 - V_2 - V_1 - V_0 - V_{-1}) - V_{-1} + V_0 - V_1 = -V_3 + V_2 + 2V_0.$$

Solving the following system

$$\begin{aligned}
3 \sum_{k=0}^n V_{2k+1} &= -V_4 + V_3 + 2V_1 + V_0 + V_{2n+2} + 2V_{2n+1} + V_{2n} + V_{2n-1} - \sum_{k=0}^n V_{2k}, \\
3 \sum_{k=0}^n V_{2k} &= -V_3 + V_2 + 2V_0 + 2V_{2n+1} + 2V_{2n} + V_{2n-1} + V_{2n-2} - \sum_{k=0}^n V_{2k+1},
\end{aligned}$$

we find that

$$\begin{aligned}\sum_{k=0}^n V_{2k+1} &= \frac{1}{8}(3V_{2n+2} + 4V_{2n+1} + V_{2n} + 2V_{2n-1} - V_{2n-2} + V_0 + 6V_1 - V_2 + 4V_3 - 3V_4) \\ \sum_{k=0}^n V_{2k} &= \frac{1}{8}(-V_{2n+2} + 4V_{2n+1} + 5V_{2n} + 2V_{2n-1} + 3V_{2n-2} + V_4 - 4V_3 + 3V_2 - 2V_1 + 5V_0).\end{aligned}$$

**(d),(e),(f):** Using the recurrence relation

$$V_k = V_{k-1} + V_{k-2} + V_{k-3} + V_{k-4} + V_{k-5}$$

i.e.

$$V_{k-1} = V_k - V_{k-2} - V_{k-3} - V_{k-4} - V_{k-5}$$

we write the obvious equations

$$\begin{aligned}V_0 &= V_1 - V_{-1} - V_{-2} - V_{-3} - V_{-4} \\ V_3 &= V_4 - V_2 - V_1 - V_0 - V_{-1} \\ V_6 &= V_7 - V_5 - V_4 - V_3 - V_2 \\ V_9 &= V_{10} - V_8 - V_7 - V_6 - V_5 \\ V_{12} &= V_{13} - V_{11} - V_{10} - V_9 - V_8 \\ V_{15} &= V_{16} - V_{14} - V_{13} - V_{12} - V_{11} \\ V_{18} &= V_{19} - V_{17} - V_{16} - V_{15} - V_{14} \\ V_{21} &= V_{22} - V_{20} - V_{19} - V_{18} - V_{17} \\ V_{24} &= V_{25} - V_{23} - V_{22} - V_{21} - V_{20} \\ V_{27} &= V_{28} - V_{26} - V_{25} - V_{24} - V_{23} \\ &\vdots \\ V_{3n-6} &= V_{3n-5} - V_{3n-7} - V_{3n-8} - V_{3n-9} - V_{3n-10} \\ V_{3n-3} &= V_{3n-2} - V_{3n-4} - V_{3n-5} - V_{3n-6} - V_{3n-7} \\ V_{3n} &= V_{3n+1} - V_{3n-1} - V_{3n-2} - V_{3n-3} - V_{3n-4}\end{aligned}$$

Now, adding these equations, we have

$$\begin{aligned}\sum_{k=0}^n V_{3k} &= \left( \sum_{k=0}^n V_{3k+1} \right) + \left( - \sum_{k=0}^n V_{3k+2} - V_{-1} + V_{3n+2} \right) + \left( - \sum_{k=0}^n V_{3k+1} - V_{-2} + V_{3n+1} \right) \\ &\quad + \left( - \sum_{k=0}^n V_{3k} - V_{-3} + V_{3n} \right) + \left( - \sum_{k=0}^n V_{3k+2} - V_{-4} - V_{-1} + V_{3n-1} + V_{3n+2} \right) \\ &\Rightarrow \\ 2 \sum_{k=0}^n V_{3k} &= 2V_{3n+2} + V_{3n+1} + V_{3n} + V_{3n-1} - V_{-4} - V_{-3} - V_{-2} - 2V_{-1} - 2 \sum_{k=0}^n V_{3k+2}\end{aligned}$$

Similarly, we write the obvious equations

$$\begin{aligned}
V_{-1} &= V_0 - V_{-2} - V_{-3} - V_{-4} - V_{-5} \\
V_2 &= V_3 - V_1 - V_0 - V_{-1} - V_{-2} \\
V_5 &= V_6 - V_4 - V_3 - V_2 - V_1 \\
V_8 &= V_9 - V_7 - V_6 - V_5 - V_4 \\
V_{11} &= V_{12} - V_{10} - V_9 - V_8 - V_7 \\
V_{14} &= V_{15} - V_{13} - V_{12} - V_{11} - V_{10} \\
V_{17} &= V_{18} - V_{16} - V_{15} - V_{14} - V_{13} \\
V_{20} &= V_{21} - V_{19} - V_{18} - V_{17} - V_{16} \\
V_{23} &= V_{24} - V_{22} - V_{21} - V_{20} - V_{19} \\
V_{26} &= V_{27} - V_{25} - V_{24} - V_{23} - V_{22} \\
&\vdots \\
V_{3n-4} &= V_{3n-3} - V_{3n-5} - V_{3n-6} - V_{3n-7} - V_{3n-8} \\
V_{3n-1} &= V_{3n} - V_{3n-2} - V_{3n-3} - V_{3n-4} - V_{3n-5} \\
V_{3n+2} &= V_{3n+3} - V_{3n+1} - V_{3n} - V_{3n-1} - V_{3n-2}
\end{aligned}$$

Now, adding these equations, we obtain

$$\begin{aligned}
V_{-1} + \sum_{k=0}^n V_{3k+2} &= \left( V_{3n+3} + \sum_{k=0}^n V_{3k} \right) + \left( -V_{-2} - \sum_{k=0}^n V_{3k+1} \right) + \left( -V_{-3} - \sum_{k=0}^n V_{3k} \right) \\
&\quad + \left( V_{3n+2} - V_{-1} - V_{-4} - \sum_{k=0}^n V_{3k+2} \right) + \left( V_{3n+1} - V_{-2} - V_{-5} - \sum_{k=0}^n V_{3k+1} \right) \\
&\Rightarrow \\
2 \sum_{k=0}^n V_{3k+2} &= -V_{-5} - V_{-4} - V_{-3} - 2V_{-2} - 2V_{-1} + V_{3n+3} + V_{3n+2} + V_{3n+1} - 2 \sum_{k=0}^n V_{3k+1}.
\end{aligned}$$



Similarly, we write the obvious equations

$$\begin{aligned}
 V_{-2} &= V_{-1} - V_{-3} - V_{-4} - V_{-5} - V_{-6} \\
 V_1 &= V_2 - V_0 - V_{-1} - V_{-2} - V_{-3} \\
 V_4 &= V_5 - V_3 - V_2 - V_1 - V_0 \\
 V_7 &= V_8 - V_6 - V_5 - V_4 - V_3 \\
 V_{10} &= V_{11} - V_9 - V_8 - V_7 - V_6 \\
 V_{13} &= V_{14} - V_{12} - V_{11} - V_{10} - V_9 \\
 V_{16} &= V_{17} - V_{15} - V_{14} - V_{13} - V_{12} \\
 V_{19} &= V_{20} - V_{18} - V_{17} - V_{16} - V_{15} \\
 V_{22} &= V_{23} - V_{21} - V_{20} - V_{19} - V_{18} \\
 V_{25} &= V_{26} - V_{24} - V_{23} - V_{22} - V_{21} \\
 &\vdots \\
 V_{3n-5} &= V_{3n-4} - V_{3n-6} - V_{3n-7} - V_{3n-8} - V_{3n-9} \\
 V_{3n-2} &= V_{3n-1} - V_{3n-3} - V_{3n-4} - V_{3n-5} - V_{3n-6} \\
 V_{3n+1} &= V_{3n+2} - V_{3n} - V_{3n-1} - V_{3n-2} - V_{3n-3}
 \end{aligned}$$

Now, adding these equations, we obtain

$$\begin{aligned}
 V_{-2} + \sum_{k=0}^n V_{3k+1} &= \left( V_{-1} + \sum_{k=0}^n V_{3k+2} \right) + \left( -V_{-3} - \sum_{k=0}^n V_{3k} \right) + \left( V_{3n+2} - V_{-4} - V_{-1} - \sum_{k=0}^n V_{3k+2} \right) \\
 &\quad + \left( V_{3n+1} - V_{-5} - V_{-2} - \sum_{k=0}^n V_{3k+1} \right) + \left( V_{3n} - V_{-6} - V_{-3} - \sum_{k=0}^n V_{3k} \right) \\
 &\Rightarrow \\
 2 \sum_{k=0}^n V_{3k+1} &= -2V_{-2} - V_{-6} - V_{-5} - V_{-4} - 2V_{-3} + V_{3n+2} + V_{3n+1} + V_{3n} - 2 \sum_{k=0}^n V_{3k}.
 \end{aligned}$$

Solving the following system

$$\begin{aligned}
 2 \sum_{k=0}^n V_{3k} &= 2V_{3n+2} + V_{3n+1} + V_{3n} + V_{3n-1} - V_{-4} - V_{-3} - V_{-2} - 2V_{-1} - 2 \sum_{k=0}^n V_{3k+2} \\
 2 \sum_{k=0}^n V_{3k+2} &= -V_{-5} - V_{-4} - V_{-3} - 2V_{-2} - 2V_{-1} + V_{3n+3} + V_{3n+2} + V_{3n+1} - 2 \sum_{k=0}^n V_{3k+1} \\
 2 \sum_{k=0}^n V_{3k+1} &= -2V_{-2} - V_{-6} - V_{-5} - V_{-4} - 2V_{-3} + V_{3n+2} + V_{3n+1} + V_{3n} - 2 \sum_{k=0}^n V_{3k}
 \end{aligned}$$

we find

$$\begin{aligned}\sum_{k=0}^n V_{3k} &= \frac{1}{4}(-V_{3n+3} + 2V_{3n+2} + V_{3n+1} + 2V_{3n} + V_{3n-1} - V_4 + 2V_3 - V_2 + 3V_0) \\ \sum_{k=0}^n V_{3k+1} &= \frac{1}{4}(V_{3n+3} + V_{3n+1} - V_{3n-1} + V_4 - 2V_3 - V_2 + 2V_1 - V_0) \\ \sum_{k=0}^n V_{3k+2} &= \frac{1}{4}(V_{3n+3} + 2V_{3n+2} + V_{3n+1} + V_{3n-1} - V_4 + 3V_2 + V_0).\end{aligned}$$

**(g),(h),(i),(j):** As in the cases (d),(e),(f), solving the following system

$$\begin{aligned}2\sum_{k=0}^n V_{4k} &= V_{4n+3} + V_{4n+2} + V_{4n+1} + V_{4n} - V_1 + V_0 - \sum_{k=0}^n V_{4k+2} - \sum_{k=0}^n V_{4k+3} \\ 2\sum_{k=0}^n V_{4k+1} &= V_{4n+3} + V_{4n+2} + V_{4n+1} - V_2 + V_1 + V_0 - \sum_{k=0}^n V_{4k+3} - \sum_{k=0}^n V_{4k} \\ 2\sum_{k=0}^n V_{4k+2} &= V_{4n+3} + V_{4n+2} - V_3 + V_2 + V_1 + V_0 - \sum_{k=0}^n V_{4k+1} - \sum_{k=0}^n V_{4k} \\ 2\sum_{k=0}^n V_{4k+3} &= V_{4n+4} + V_{4n+3} - V_4 + V_3 + V_2 + V_1 - \sum_{k=0}^n V_{4k+2} - \sum_{k=0}^n V_{4k+1}\end{aligned}$$

we find

$$\begin{aligned}\sum_{k=0}^n V_{4k} &= \frac{1}{16}(-5V_{4n+4} + 4V_{4n+3} + 9V_{4n+2} + 10V_{4n+1} + 7V_{4n} + 5V_4 - 4V_3 - 9V_2 - 10V_1 + 9V_0) \\ \sum_{k=0}^n V_{4k+1} &= \frac{1}{16}(-V_{4n+4} + 4V_{4n+3} + 5V_{4n+2} + 2V_{4n+1} - 5V_{4n} + V_4 - 4V_3 - 5V_2 + 14V_1 + 5V_0) \\ \sum_{k=0}^n V_{4k+2} &= \frac{1}{16}(3V_{4n+4} + 4V_{4n+3} + V_{4n+2} - 6V_{4n+1} - V_{4n} - 3V_4 - 4V_3 + 15V_2 + 6V_1 + V_0) \\ \sum_{k=0}^n V_{4k+3} &= \frac{1}{16}(7V_{4n+4} + 4V_{4n+3} - 3V_{4n+2} + 2V_{4n+1} + 3V_{4n} - 7V_4 + 12V_3 + 3V_2 - 2V_1 - 3V_0)\end{aligned}$$

**(k),(l),(m),(n),(o):** As in the cases (d),(e),(f), solving the following system

$$\begin{aligned} \sum_{k=0}^n V_{5k} &= V_{5n+4} + V_{5n+3} + V_{5n+2} + V_{5n+1} - V_1 + V_0 - \sum_{k=0}^n V_{5k+4} - \sum_{k=0}^n V_{5k+3} - \sum_{k=0}^n V_{5k+2} \\ \sum_{k=0}^n V_{5k+1} &= V_{5n+4} + V_{5n+3} + V_{5n+2} - V_2 + V_1 + V_0 - \sum_{k=0}^n V_{5k+4} - \sum_{k=0}^n V_{5k+3} - \sum_{k=0}^n V_{5k} \\ \sum_{k=0}^n V_{5k+2} &= V_{5n+4} + V_{5n+3} - V_3 + V_2 + V_1 + V_0 - \sum_{k=0}^n V_{5k+4} - \sum_{k=0}^n V_{5k+1} - \sum_{k=0}^n V_{5k} \\ \sum_{k=0}^n V_{5k+3} &= V_{5n+4} - V_4 + V_3 + V_2 + V_1 + V_0 - \sum_{k=0}^n V_{5k+2} - \sum_{k=0}^n V_{5k+1} - \sum_{k=0}^n V_{5k} \\ \sum_{k=0}^n V_{5k+4} &= \sum_{k=0}^n V_{5k+5} - \sum_{k=0}^n V_{5k+3} - \sum_{k=0}^n V_{5k+2} - \sum_{k=0}^n V_{5k+1} - \sum_{k=0}^n V_{5k} \\ \sum_{k=0}^n V_{5k+4} &= V_{5n+5} - V_0 + \sum_{k=0}^n V_{5k} - \sum_{k=0}^n V_{5k+3} - \sum_{k=0}^n V_{5k+2} - \sum_{k=0}^n V_{5k+1} - \sum_{k=0}^n V_{5k} \end{aligned}$$

we find

$$\begin{aligned} \sum_{k=0}^n V_{5k} &= \frac{1}{4}(4V_{5n+4} + 3V_{5n+3} + 2V_{5n+2} + V_{5n+1} - 3V_{5n+5} + 7V_0 - V_4 + V_2 + 2V_1) \\ \sum_{k=0}^n V_{5k+1} &= \frac{1}{4}(V_{5n+5} - V_{5n+3} - 2V_{5n+2} - 3V_{5n+1} - V_4 + V_2 + 6V_1 - V_0) \\ \sum_{k=0}^n V_{5k+2} &= \frac{1}{4}(V_{5n+5} - V_{5n+3} - 2V_{5n+2} + V_{5n+1} - V_4 + 5V_2 - 2V_1 - V_0) \\ \sum_{k=0}^n V_{5k+3} &= \frac{1}{4}(V_{5n+5} - V_{5n+3} + 2V_{5n+2} + V_{5n+1} - V_4 + 4V_3 - 3V_2 - 2V_1 - V_0) \\ \sum_{k=0}^n V_{5k+4} &= \frac{1}{4}(V_{5n+5} + 3V_{5n+3} + 2V_{5n+2} + V_{5n+1} - 4V_3 - 3V_2 + 3V_4 - 2V_1 - V_0). \end{aligned}$$

For a different proof of (a),(b),(c) see [10]. As special cases of above Theorem, we have the following two Corollaries. First one present some summation formulas of Pentanacci numbers.

**COROLLARY 2.2.** For  $n \geq 0$ , we have the following formulas:

$$\begin{aligned} \text{(a): } \sum_{k=0}^n P_k &= \frac{1}{4}(P_{n+4} - P_{n+2} - 2P_{n+1} + P_n - 1) \\ \text{(b): } \sum_{k=0}^n P_{2k+1} &= \frac{1}{8}(3P_{2n+2} + 4P_{2n+1} + P_{2n} + 2P_{2n-1} - P_{2n-2} + 1) \\ \text{(c): } \sum_{k=0}^n P_{2k} &= \frac{1}{8}(-P_{2n+2} + 4P_{2n+1} + 5P_{2n} + 2P_{2n-1} + 3P_{2n-2} - 3) \\ \text{(d): } \sum_{k=0}^n P_{3k} &= \frac{1}{4}(-P_{3n+3} + 2P_{3n+2} + P_{3n+1} + 2P_{3n} + P_{3n-1} - 1) \\ \text{(e): } \sum_{k=0}^n P_{3k+1} &= \frac{1}{4}(P_{3n+3} + P_{3n+1} - P_{3n-1} + P_4 - 3) \\ \text{(f): } \sum_{k=0}^n P_{3k+2} &= \frac{1}{4}(P_{3n+3} + 2P_{3n+2} + P_{3n+1} + P_{3n-1} - 1) \\ \text{(g): } \sum_{k=0}^n P_{4k} &= \frac{1}{16}(-5P_{4n+4} + 4P_{4n+3} + 9P_{4n+2} + 10P_{4n+1} + 7P_{4n} - 7) \\ \text{(h): } \sum_{k=0}^n P_{4k+1} &= \frac{1}{16}(-P_{4n+4} + 4P_{4n+3} + 5P_{4n+2} + 2P_{4n+1} - 5P_{4n} + 5) \\ \text{(i): } \sum_{k=0}^n P_{4k+2} &= \frac{1}{16}(3P_{4n+4} + 4P_{4n+3} + P_{4n+2} - 6P_{4n+1} - P_{4n} + 1) \end{aligned}$$

$$\begin{aligned}
\text{(j): } \sum_{k=0}^n P_{4k+3} &= \frac{1}{16} (7P_{4n+4} + 4P_{4n+3} - 3P_{4n+2} + 2P_{4n+1} + 3P_{4n} - 3) \\
\text{(k): } \sum_{k=0}^n P_{5k} &= \frac{1}{4} (-3P_{5n+5} + 4P_{5n+4} + 3P_{5n+3} + 2P_{5n+2} + P_{5n+1} - 1) \\
\text{(l): } \sum_{k=0}^n P_{5k+1} &= \frac{1}{4} (P_{5n+5} - P_{5n+3} - 2P_{5n+2} - 3P_{5n+1} + 3) \\
\text{(m): } \sum_{k=0}^n P_{5k+2} &= \frac{1}{4} (P_{5n+5} - P_{5n+3} - 2P_{5n+2} + P_{5n+1} - 1) \\
\text{(n): } \sum_{k=0}^n P_{5k+3} &= \frac{1}{4} (P_{5n+5} - P_{5n+3} + 2P_{5n+2} + P_{5n+1} - 1) \\
\text{(o): } \sum_{k=0}^n P_{5k+4} &= \frac{1}{4} (P_{5n+5} + 3P_{5n+3} + 2P_{5n+2} + P_{5n+1} - 1)
\end{aligned}$$

Next Corollary gives some summation formulas of Pentanacci-Lucas numbers.

**COROLLARY 2.3.** *For  $n \geq 0$ , we have the following formulas:*

$$\begin{aligned}
\text{(a): } \sum_{k=0}^n Q_k &= \frac{1}{4} (Q_{n+4} - Q_{n+2} - 2Q_{n+1} + Q_n + 5) \\
\text{(b): } \sum_{k=0}^n Q_{2k+1} &= \frac{1}{8} (3Q_{2n+2} + 4Q_{2n+1} + Q_{2n} + 2Q_{2n-1} - Q_{2n-2} - 9) \\
\text{(c): } \sum_{k=0}^n Q_{2k} &= \frac{1}{8} (-Q_{2n+2} + 4Q_{2n+1} + 5Q_{2n} + 2Q_{2n-1} + 3Q_{2n-2} + 19) \\
\text{(d): } \sum_{k=0}^n Q_{3k} &= \frac{1}{4} (-Q_{3n+3} + 2Q_{3n+2} + Q_{3n+1} + 2Q_{3n} + Q_{3n-1} + 11) \\
\text{(e): } \sum_{k=0}^n Q_{3k+1} &= \frac{1}{4} (Q_{3n+3} + Q_{3n+1} - Q_{3n-1} - 5) \\
\text{(f): } \sum_{k=0}^n Q_{3k+2} &= \frac{1}{4} (Q_{3n+3} + 2Q_{3n+2} + Q_{3n+1} + Q_{3n-1} - 1) \\
\text{(g): } \sum_{k=0}^n Q_{4k} &= \frac{1}{16} (-5Q_{4n+4} + 4Q_{4n+3} + 9Q_{4n+2} + 10Q_{4n+1} + 7Q_{4n} + 55) \\
\text{(h): } \sum_{k=0}^n Q_{4k+1} &= \frac{1}{16} (-Q_{4n+4} + 4Q_{4n+3} + 5Q_{4n+2} + 2Q_{4n+1} - 5Q_{4n} + 11) \\
\text{(i): } \sum_{k=0}^n Q_{4k+2} &= \frac{1}{16} (3Q_{4n+4} + 4Q_{4n+3} + Q_{4n+2} - 6Q_{4n+1} - Q_{4n} - 17) \\
\text{(j): } \sum_{k=0}^n Q_{4k+3} &= \frac{1}{16} (7Q_{4n+4} + 4Q_{4n+3} - 3Q_{4n+2} + 2Q_{4n+1} + 3Q_{4n} - 29) \\
\text{(k): } \sum_{k=0}^n Q_{5k} &= \frac{1}{4} (-3Q_{5n+5} + 4Q_{5n+4} + 3Q_{5n+3} + 2Q_{5n+2} + Q_{5n+1} + 25) \\
\text{(l): } \sum_{k=0}^n Q_{5k+1} &= \frac{1}{4} (Q_{5n+5} - Q_{5n+3} - 2Q_{5n+2} - 3Q_{5n+1} - 11) \\
\text{(m): } \sum_{k=0}^n Q_{5k+2} &= \frac{1}{4} (Q_{5n+5} - Q_{5n+3} - 2Q_{5n+2} + Q_{5n+1} - 7) \\
\text{(n): } \sum_{k=0}^n Q_{5k+3} &= \frac{1}{4} (Q_{5n+5} - Q_{5n+3} + 2Q_{5n+2} + Q_{5n+1} - 3) \\
\text{(o): } \sum_{k=0}^n Q_{5k+4} &= \frac{1}{4} (Q_{5n+5} + 3Q_{5n+3} + 2Q_{5n+2} + Q_{5n+1} + 1)
\end{aligned}$$

### 3. Linear Sums of Generalized Gaussian Pentanacci Numbers

A Gaussian integer  $z$  is a complex number whose real and imaginary parts are both integers, i.e.,  $z = a + ib$ ,  $a, b \in \mathbb{Z}$ . If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers. Gaussian generalized Pentanacci numbers  $\{GV_n\}_{n \geq 0} = \{GV_n(GV_0, GV_1, GV_2, GV_3, GV_4)\}_{n \geq 0}$  are defined by

$$(3.1) \quad GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3} + GV_{n-4} + GV_{n-5},$$

with the initial conditions

$$\begin{aligned} GV_0 &= c_0 + (-c_0 - c_1 - c_2 - c_3 + c_4)i, GV_1 = c_1 + c_0i, GV_2 = c_2 + c_1i, \\ GV_3 &= c_3 + c_2i, GV_4 = c_4 + c_3i \end{aligned}$$

not all being zero. The sequences  $\{GV_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$GV_{-n} = -GV_{-(n-1)} - GV_{-(n-2)} - GV_{-(n-3)} - GV_{-(n-4)} + GV_{-(n-5)}$$

for  $n = 1, 2, 3, \dots$ . Therefore, recurrence (3.1) hold for all integer  $n$ . Note that for  $n \geq 0$

$$(3.2) \quad GV_n = V_n + iV_{n-1}$$

and

$$GV_{-n} = V_{-n} + iV_{-n-1}$$

We consider two special cases of  $GV_n : GV_n(0, 1, 1 + i, 2 + i, 4 + 2i) = GP_n$  is the sequence of Gaussian Pentanacci numbers and  $GV_n(5 - i, 1 + 5i, 3 + i, 7 + 3i, 15 + 7i) = GQ_n$  is the sequence of Gaussian Pentanacci-Lucas numbers. We formally define them as follows:

Gaussian Pentanacci numbers are defined by

$$(3.3) \quad GP_n = GP_{n-1} + GP_{n-2} + GP_{n-3} + GP_{n-4} + GP_{n-5},$$

with the initial conditions

$$GP_0 = 0, GP_1 = 1, GP_2 = 1 + i, GP_3 = 2 + i, GP_4 = 4 + 2i$$

and Gaussian Pentanacci-Lucas numbers are defined by

$$(3.4) \quad GQ_n = GQ_{n-1} + GQ_{n-2} + GQ_{n-3} + GQ_{n-4} + GQ_{n-5}$$

with the initial conditions

$$GQ_0 = 5 - i, GQ_1 = 1 + 5i, GQ_2 = 3 + i, GQ_3 = 7 + 3i, GQ_4 = 15 + 7i.$$

Note that for  $n \geq 0$

$$GP_n = M_n + iM_{n-1}, GQ_n = R_n + iR_{n-1}$$

and

$$GP_{-n} = M_{-n} + iM_{-n-1}, GQ_{-n} = R_{-n} + iR_{-n-1}.$$

The following Theorem present some summation formulas of Gaussian generalized Pentanacci numbers.

**THEOREM 3.1.** *For  $n \geq 0$  we have the following formulas:*

$$(a): \sum_{k=0}^n GV_k = \frac{1}{4}(GV_{n+4} - GV_{n+2} - 2GV_{n+1} + GV_n - GV_4 + GV_2 + 2GV_1 + 3GV_0)$$

$$(b): \sum_{k=0}^n GV_{2k+1} = \frac{1}{8}(3GV_{2n+2} + 4GV_{2n+1} + GV_{2n} + 2GV_{2n-1} - GV_{2n-2} - 3GV_4 + 4GV_3 - GV_2 + 6GV_1 + GV_0)$$

- (c):  $\sum_{k=0}^n GV_{2k} = \frac{1}{8}(-GV_{2n+2} + 4GV_{2n+1} + 5GV_{2n} + 2GV_{2n-1} + 3GV_{2n-2} + GV_4 - 4GV_3 + 3GV_2 - 2GV_1 + 5GV_0)$
- (d):  $\sum_{k=0}^n GV_{3k} = \frac{1}{4}(-GV_{3n+3} + 2GV_{3n+2} + GV_{3n+1} + 2GV_{3n} + GV_{3n-1} - GV_4 + 2GV_3 - GV_2 + 3GV_0)$
- (e):  $\sum_{k=0}^n GV_{3k+1} = \frac{1}{4}(GV_{3n+3} + GV_{3n+1} - GV_{3n-1} + GV_4 - 2GV_3 - GV_2 + 2GV_1 - GV_0)$
- (f):  $\sum_{k=0}^n GV_{3k+2} = \frac{1}{4}(GV_{3n+3} + 2GV_{3n+2} + GV_{3n+1} + GV_{3n-1} - GV_4 + 3GV_2 + GV_0)$
- (g):  $\sum_{k=0}^n GV_{4k} = \frac{1}{16}(-5GV_{4n+4} + 4GV_{4n+3} + 9GV_{4n+2} + 10GV_{4n+1} + 7GV_{4n} + 5GV_4 - 4GV_3 - 9GV_2 - 10GV_1 + 9GV_0)$
- (h):  $\sum_{k=0}^n GV_{4k+1} = \frac{1}{16}(-GV_{4n+4} + 4GV_{4n+3} + 5GV_{4n+2} + 2GV_{4n+1} - 5GV_{4n} + GV_4 - 4GV_3 - 5GV_2 + 14GV_1 + 5GV_0)$
- (i):  $\sum_{k=0}^n GV_{4k+2} = \frac{1}{16}(3GV_{4n+4} + 4GV_{4n+3} + GV_{4n+2} - 6GV_{4n+1} - GV_{4n} - 3GV_4 - 4GV_3 + 15GV_2 + 6GV_1 + GV_0)$
- (j):  $\sum_{k=0}^n GV_{4k+3} = \frac{1}{16}(7GV_{4n+4} + 4GV_{4n+3} - 3GV_{4n+2} + 2GV_{4n+1} + 3GV_{4n} - 7GV_4 + 12GV_3 + 3GV_2 - 2GV_1 - 3GV_0)$
- (k):  $\sum_{k=0}^n GV_{5k} = \frac{1}{4}(-3GV_{5n+5} + 4GV_{5n+4} + 3GV_{5n+3} + 2GV_{5n+2} + GV_{5n+1} - GV_4 + GV_2 + 2GV_1 + 7GV_0)$
- (l):  $\sum_{k=0}^n GV_{5k+1} = \frac{1}{4}(GV_{5n+5} - GV_{5n+3} - 2GV_{5n+2} - 3GV_{5n+1} - GV_4 + GV_2 + 6GV_1 - GV_0)$
- (m):  $\sum_{k=0}^n GV_{5k+2} = \frac{1}{4}(GV_{5n+5} - GV_{5n+3} - 2GV_{5n+2} + GV_{5n+1} - GV_4 + 5GV_2 - 2GV_1 - GV_0)$
- (n):  $\sum_{k=0}^n GV_{5k+3} = \frac{1}{4}(GV_{5n+5} - GV_{5n+3} + 2GV_{5n+2} + GV_{5n+1} - GV_4 + 4GV_3 - 3GV_2 - 2GV_1 - GV_0)$
- (o):  $\sum_{k=0}^n GV_{5k+4} = \frac{1}{4}(GV_{5n+5} + 3GV_{5n+3} + 2GV_{5n+2} + GV_{5n+1} + 3GV_4 - 4GV_3 - 3GV_2 - 2GV_1 - GV_0)$ .

*Proof.* (a)-(o) can be proved exactly as in the proof of Theorem 2.1.

As special cases of the above Theorem, we have the following two Corollaries. First one present summation formulas of Gaussian Pentanacci numbers.

**COROLLARY 3.2.** *For  $n \geq 0$  we have the following formulas:*

- (a):  $\sum_{k=0}^n GP_k = \frac{1}{4}(GP_{n+4} - GP_{n+2} - 2GP_{n+1} + GP_n - 1 - i)$
- (b):  $\sum_{k=0}^n GP_{2k+1} = \frac{1}{8}(3GP_{2n+2} + 4GP_{2n+1} + GP_{2n} + 2GP_{2n-1} - GP_{2n-2} + 1 - 3i)$
- (c):  $\sum_{k=0}^n GP_{2k} = \frac{1}{8}(-GP_{2n+2} + 4GP_{2n+1} + 5GP_{2n} + 2GP_{2n-1} + 3GP_{2n-2} - 3 + i)$
- (d):  $\sum_{k=0}^n GP_{3k} = \frac{1}{4}(-GP_{3n+3} + 2GP_{3n+2} + GP_{3n+1} + 2GP_{3n} + GP_{3n-1} - 1 - i)$
- (e):  $\sum_{k=0}^n GP_{3k+1} = \frac{1}{4}(GP_{3n+3} + GP_{3n+1} - GP_{3n-1} + 1 - i)$
- (f):  $\sum_{k=0}^n GP_{3k+2} = \frac{1}{4}(GP_{3n+3} + 2GP_{3n+2} + GP_{3n+1} + GP_{3n-1} - 1 + i)$
- (g):  $\sum_{k=0}^n GP_{4k} = \frac{1}{16}(-5GP_{4n+4} + 4GP_{4n+3} + 9GP_{4n+2} + 10GP_{4n+1} + 7GP_{4n} - 7 - 3i)$
- (h):  $\sum_{k=0}^n GP_{4k+1} = \frac{1}{16}(-GP_{4n+4} + 4GP_{4n+3} + 5GP_{4n+2} + 2GP_{4n+1} - 5GP_{4n} + 5 - 7i)$
- (i):  $\sum_{k=0}^n GP_{4k+2} = \frac{1}{16}(3GP_{4n+4} + 4GP_{4n+3} + GP_{4n+2} - 6GP_{4n+1} - GP_{4n} + 1 + 5i)$
- (j):  $\sum_{k=0}^n GP_{4k+3} = \frac{1}{16}(7GP_{4n+4} + 4GP_{4n+3} - 3GP_{4n+2} + 2GP_{4n+1} + 3GP_{4n} - 3 + i)$

$$\begin{aligned}
 \text{(k): } \sum_{k=0}^n GP_{5k} &= \frac{1}{4} (-3GP_{5n+5} + 4GP_{5n+4} + 3GP_{5n+3} + 2GP_{5n+2} + GP_{5n+1} - 1 - i) \\
 \text{(l): } \sum_{k=0}^n GP_{5k+1} &= \frac{1}{4} (GP_{5n+5} - GP_{5n+3} - 2GP_{5n+2} - 3GP_{5n+1} + 3 - i) \\
 \text{(m): } \sum_{k=0}^n GP_{5k+2} &= \frac{1}{4} (GP_{5n+5} - GP_{5n+3} - 2GP_{5n+2} + GP_{5n+1} - 1 + 3i) \\
 \text{(n): } \sum_{k=0}^n GP_{5k+3} &= \frac{1}{4} (GP_{5n+5} - GP_{5n+3} + 2GP_{5n+2} + GP_{5n+1} - 1 - i) \\
 \text{(o): } \sum_{k=0}^n GP_{5k+4} &= \frac{1}{4} (GP_{5n+5} + 3GP_{5n+3} + 2GP_{5n+2} + GP_{5n+1} - 1 - i)
 \end{aligned}$$

Second Corollary gives summation formulas of Gaussian Pentanacci-Lucas numbers.

COROLLARY 3.3. For  $n \geq 0$  we have the following formulas:

$$\begin{aligned}
 \text{(a): } \sum_{k=0}^n GQ_k &= \frac{1}{4} (GQ_{n+4} - GQ_{n+2} - 2GQ_{n+1} + GQ_n + 5 + i) \\
 \text{(b): } \sum_{k=0}^n GQ_{2k+1} &= \frac{1}{8} (3GQ_{2n+2} + 4GQ_{2n+1} + GQ_{2n} + 2GQ_{2n-1} - GQ_{2n-2} - 9 + 19i) \\
 \text{(c): } \sum_{k=0}^n GQ_{2k} &= \frac{1}{8} (-GQ_{2n+2} + 4GQ_{2n+1} + 5GQ_{2n} + 2GQ_{2n-1} + 3GQ_{2n-2} + 19 - 17i) \\
 \text{(d): } \sum_{k=0}^n GQ_{3k} &= \frac{1}{4} (-GQ_{3n+3} + 2GQ_{3n+2} + GQ_{3n+1} + 2GQ_{3n} + GQ_{3n-1} + 11 - 5i) \\
 \text{(e): } \sum_{k=0}^n GQ_{3k+1} &= \frac{1}{4} (GQ_{3n+3} + GQ_{3n+1} - GQ_{3n-1} - 5 + 11i) \\
 \text{(f): } \sum_{k=0}^n GQ_{3k+2} &= \frac{1}{4} (GQ_{3n+3} + 2GQ_{3n+2} + GQ_{3n+1} + GQ_{3n-1} - 1 - 5i) \\
 \text{(g): } \sum_{k=0}^n GQ_{4k} &= \frac{1}{16} (-5GQ_{4n+4} + 4GQ_{4n+3} + 9GQ_{4n+2} + 10GQ_{4n+1} + 7GQ_{4n} + 55 - 45i) \\
 \text{(h): } \sum_{k=0}^n GQ_{4k+1} &= \frac{1}{16} (-GQ_{4n+4} + 4GQ_{4n+3} + 5GQ_{4n+2} + 2GQ_{4n+1} - 5GQ_{4n} + 11 + 55i) \\
 \text{(i): } \sum_{k=0}^n GQ_{4k+2} &= \frac{1}{16} (3GQ_{4n+4} + 4GQ_{4n+3} + GQ_{4n+2} - 6GQ_{4n+1} - GQ_{4n} - 17 + 11i) \\
 \text{(j): } \sum_{k=0}^n GQ_{4k+3} &= \frac{1}{16} (7GQ_{4n+4} + 4GQ_{4n+3} - 3GQ_{4n+2} + 2GQ_{4n+1} + 3GQ_{4n} - 29 - 17i) \\
 \text{(k): } \sum_{k=0}^n GQ_{5k} &= \frac{1}{4} (-3GQ_{5n+5} + 4GQ_{5n+4} + 3GQ_{5n+3} + 2GQ_{5n+2} + GQ_{5n+1} + 25 - 3i) \\
 \text{(l): } \sum_{k=0}^n GQ_{5k+1} &= \frac{1}{4} (GQ_{5n+5} - GQ_{5n+3} - 2GQ_{5n+2} - 3GQ_{5n+1} - 11 + 25i) \\
 \text{(m): } \sum_{k=0}^n GQ_{5k+2} &= \frac{1}{4} (GQ_{5n+5} - GQ_{5n+3} - 2GQ_{5n+2} + GQ_{5n+1} - 7 - 11i) \\
 \text{(n): } \sum_{k=0}^n GQ_{5k+3} &= \frac{1}{4} (GQ_{5n+5} - GQ_{5n+3} + 2GQ_{5n+2} + GQ_{5n+1} - 3 - 7i) \\
 \text{(o): } \sum_{k=0}^n GQ_{5k+4} &= \frac{1}{4} (GQ_{5n+5} + 3GQ_{5n+3} + 2GQ_{5n+2} + GQ_{5n+1} + 1 - 3i)
 \end{aligned}$$

#### 4. Conclusion

- In section 2, linear summation formulas have been presented for generalized Pentanacci numbers. As special cases, linear summation formulas of Pentanacci and Pentanacci-Lucas numbers have been given.
- In section 3, linear summation formulas have been presented for generalized Gaussian Pentanacci numbers. As special cases, linear summation formulas of Gaussian Pentanacci and Gaussian Pentanacci-Lucas numbers have been given.

In this work, a number of linear sum identities were discovered and proved. We have written them in terms of the generalized Pentanacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Pentanacci and Pentanacci-Lucas sequences. All the listed identities may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

It is our intention to continue the study and explore some linear summation properties of some type of number sequences, such as Hexanacci and Hexanacci-Lucas numbers.

### Competing Interests

Author have declared that no competing interests exist.

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### References

- [1] Gökbaş, H., Köse, H., Some sum formulas for products of Pell and Pell-Lucas numbers, *Int. J. Adv. Appl. Math. and Mech.* 4(4), 1-4, 2017.
- [2] Koshy, T., *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience Publication, New York, 2001.
- [3] Koshy, T., *Pell and Pell-Lucas Numbers with Applications*, Springer, New York, 2014.
- [4] Melham, R. S., Some Analogs of the Identity  $F_n^2 + F_{n+1}^2 = F_{2n+1}^2$ , *Fibonacci Quarterly*, 305-311, 1999.
- [5] Natividad, L. R., On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order, *International Journal of Mathematics and Computing*, 3 (2), 2013.
- [6] Parpar, T., k'ncü Mertebeden Rekürans Bağıntısının Özellikleri ve Bazı Uygulamaları, Selçuk Üniversitesi, Fen Bilimleri Enstitüsü, Yüksek Lisans Tezi, 2011.
- [7] Rathore, G.P.S., Sikhwal, O., Choudhary, R., Formula for finding nth Term of Fibonacci-Like Sequence of Higher Order, *International Journal of Mathematics And its Applications*, 4 (2-D), 75-80, 2016.
- [8] Sloane, N.J.A., The on-line encyclopedia of integer sequences, <http://oeis.org/>
- [9] Soykan, Y., Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers, arXiv:1809.07809v1 [math.NT] 20 Sep 2018.
- [10] Soykan Y., On Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers, Preprints 2019, 2019060110 (doi: 10.20944/preprints201906.0110.v1).
- [11] Waddill, M. E., The Tetranacci Sequence and Generalizations, *The Fibonacci Quarterly*, 9-20, 1992.