QUANTUM ENERGY OF A PARTICLE IN A FINITE-POTENTIAL WELL BASED UPON GOLDEN METRIC TENSOR

Abstract

In our previous work titled "Riemannian Quantum Theory of a Particle in a Finite-Potential Well", we constructed the Riemannian Laplacian operator and used it to obtain the Riemannian Schrodinger equation for a particle in a finite-potential well. In this work we solved the golden Riemannian Schrodinger equation analytically to obtain the particle energy. The solution resulted to two expressions for the energy of a particle in a finite-potential well. One of the expressions is for the odd energy levels while the other is for the even energy levels.

Keywords: Energy, Finite-potential, Quantum Theory, Particle, Schrodinger equation.

1. Introduction

The origin of quantum physics occupies a time period in history that covers a quarter of a century. Classical or Newtonian mechanics was available in the powerful formulations of Lagrange and Hamilton by the year 1900. Thus, classical electromagnetic theory was embodied in the differential equations of Maxwell. Defects were, however, made clear by the failure of the classical theories to explain some experimental results, notably, the frequency dependence of the intensity of radiation emitted by a blackbody, the photoelectric effect and the stability and size of atoms [2].

Quantum Physics came to existence in 1900 when a famous pronouncement was put forward by Planck to unfold and illustrate the meaning of the observed properties of the radiation ejected by a blackbody [3]. This phenomenon posed an unsolved problem to theoretical physicists for several decades.

Principles of thermodynamics and electromagnetism had been applied to the problem but, these classical methods had failed to give a sensible explanation of the experimental results [11; 1].

The quantum hypothesis of Planck and the subsequent interpretation of the idea by Einstein in 1905 gave electromagnetic radiation discrete properties; somewhat similar to those of a particle. The quantum theory made provision for radiation to have both wave and particle aspects in a complementary form of coexistences. The theory was extended when matter was found to have wave characteristics as well as particle properties by de Broglie in 1923 [9]. These notions continued to evolve until 1925 when the formal apparatus of quantum theory came into being.

The discovery of the wave like behavior of an electron created the need for a wave theory describing the behavior of a particle on the atomic scale. This theory was proposed by Schrodinger in the year 1926, two years after De Broglie formulated the idea of a particle wave nature [8]. Schrodinger reasoned that if an electron behaves as a wave, then it should be possible to mathematically describe the electrons behavior in space time coordinate as a wave. The Schrodinger proposed theory; yielded the fundamental equation of quantum mechanics known as the Schrodinger wave equation. This equation has the same central importance to quantum mechanics as Newton's law of motion has for classical mechanics [10].

2. Theoretical Analysis

2.1 Derivation of Riemannian Laplacian Operator in Spherical Polar Coordinate Based upon the Golden Metric Tensor

Consider a particle of mass, m in a finite-potential well of width, a and depth, V_o.

The Riemannian Laplacian operator [12; 6] is given by

$$\nabla_R^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left\{ \sqrt{g}.\ g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right\} \tag{1}$$

where $g_{\mu\nu} \equiv$ metric and $g =$ determinant of $g_{\mu\nu}$

The Golden Riemannian metric tensors in spherical polar coordinate [6; 7] are given by

$$g_{11} = \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{2}$$

$$g_{22} = r^2 \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{3}$$

$$g_{33} = r^2 \sin^2\theta \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{4}$$

$$g_{00} = -\left(1 + \frac{2}{c^2} f\right) \tag{5}$$

$$g_{\mu\nu} = 0; \text{otherwise} \tag{6}$$

and

$$g = r^4 \sin^2\theta \left(1 + \frac{2}{c^2} f\right)^{-2} \tag{7}$$

$$\sqrt{g} = r^2 sin\theta \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{8}$$

From equation (1) we have:

$$\nabla_R^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{\sqrt{g}.\, g^{11} \frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} \left\{\sqrt{g}.\, g^{22} \frac{\partial}{\partial x^2}\right\} +$$

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} \left\{\sqrt{g}.\; g^{33} \frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}} \frac{1}{\partial x^0} \left\{\sqrt{g}.\, g^{00} \frac{\partial}{\partial x^0}\right\} \qquad (9)$$

If we let

$$\alpha = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{\sqrt{g}.\, g^{11} \frac{\partial}{\partial x^1}\right\},$$

$$\beta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} \left\{\sqrt{g}.\, g^{22} \frac{\partial}{\partial x^2}\right\},$$

$$\gamma = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} \left\{\sqrt{g}.\; g^{33} \frac{\partial}{\partial x^3}\right\} \text{ and}$$

$$\xi = \frac{1}{\sqrt{g}} \frac{1}{\partial x^0} \left\{\sqrt{g}.\, g^{00} \frac{\partial}{\partial x^0}\right\}$$

Equation (9) reduces to

$$\nabla_R^2 = \alpha + \beta + \gamma + \xi$$

(10)

For $\alpha = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{\sqrt{g}.\, g^{11} \frac{\partial}{\partial x^1}\right\}$

(11)

To obtain α in spherical polar coordinate, we substitute equations (2) and (7) into equation (11) as follows:

$$\alpha = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{\sqrt{g}.\, g^{11} \frac{\partial}{\partial x^1}\right\} =$$

$$\frac{1}{r^2 sin\theta \left(1 + \frac{2}{c^2} f\right)^{-1}} \frac{\partial}{\partial r} \left\{r^2 sin\theta \left(1 + \frac{2}{c^2} f\right)^{-1} . \left(1 + \frac{2}{c^2} f\right) \frac{\partial}{\partial r}\right\}$$

$$= \frac{1}{r^2 sin\theta \left(1 + \frac{2}{c^2} f\right)^{-1}} \frac{\partial}{\partial r} \left\{r^2 sin\theta \frac{\partial}{\partial r}\right\}$$

$$= \frac{1}{r^2\left(1+\frac{2}{c^2}f\right)^{-1}} \frac{\partial}{\partial r} \left\{r^2 \frac{\partial}{\partial r}\right\}$$

$$\alpha = \frac{1}{r^2} \left(1 + \frac{2}{c^2} f\right) \frac{\partial}{\partial r} \left\{r^2 \frac{\partial}{\partial r}\right\}$$

(12)

For $\beta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} \left\{\sqrt{g}.g^{22} \frac{\partial}{\partial x^2}\right\}$

(13)

To obtain β in spherical polar coordinate, we substitute equations (3) and (7) into equation (13) as follows:

$$\beta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} \left\{\sqrt{g} \, . \, g^{22} \frac{\partial}{\partial x^2} \right\} =$$

$$\frac{1}{r^2 sin\theta\left(1 + \frac{2}{c^2} \mathrm{f}\right)^{-1}} \frac{\partial}{\partial\theta} \left\{r^2 sin\theta \left(1 + \frac{2}{c^2} \mathrm{f}\right)^{-1} . \left(1 + \frac{2}{c^2} f\right) \frac{1}{r^2} \frac{\partial}{\partial\theta}\right\}$$

$$\beta = \frac{1}{r^2 sin\theta} \left(1 + \frac{2}{c^2} f\right) \frac{\partial}{\partial\theta}\left\{sin\theta \frac{\partial}{\partial\theta}\right\}$$

(14)

For $\gamma = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} \left\{\sqrt{g}. \; g^{33} \frac{\partial}{\partial x^3}\right\}$

(15)

To obtain γ in spherical polar coordinate, we substitute equations (4) and (7) into equation (15) as follows:

$$\gamma = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} \left\{\sqrt{g} \, . \, g^{33} \frac{\partial}{\partial x^3}\right\} =$$

$$\frac{1}{r^2 sin\theta}\left(1+\frac{2}{c^2}f\right)\frac{\partial}{\partial\emptyset}\left\{r^2 sin\theta\left(1+\frac{2}{\mathrm{c}^2}\mathrm{f}\right)^{-1}.\left(1+\frac{2}{c^2}f\right)\frac{1}{r^2 sin^{2\theta}}\frac{\partial}{\partial\emptyset}\right\}$$

$$\gamma=\frac{1}{r^2 sin^2\theta}\left(1+\frac{2}{c^2}f\right)\frac{\partial}{\partial\emptyset}\left\{\frac{\partial}{\partial\emptyset}\right\}$$

(16)

For $\xi=\frac{1}{\sqrt{g}}\frac{1}{\partial x^0}\left\{\sqrt{g}.g^{00}\frac{\partial}{\partial x^0}\right\}$

(17)

To obtain γ in spherical polar coordinate, we substitute equations (5) and (7) into equation (17) as follows:

$$\xi=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}.g^{00}\frac{\partial}{\partial x^0}\right\}=$$

$$-\frac{1}{r^2 sin\theta}\left(1+\frac{2}{c^2}f\right)\frac{\partial}{\partial x^0}\left\{r^2 sin\theta\left(1+\frac{2}{\mathrm{c}^2}f\right)^{-1}.\left(1+\frac{2}{\mathrm{c}^2}f\right)^{-1}\frac{\partial}{\partial x^0}\right\}$$

$$\xi=-\left(1+\frac{2}{\mathrm{c}^2}\mathrm{f}\right)^{-1}\frac{\partial}{\partial x^0}\left\{\frac{\partial}{\partial x^0}\right\}$$

(18)

Substituting equations (12), (14), (16) and (18) into equation (10), we have thus:

$$\nabla_R^2=\frac{1}{r^2}\left(1+\frac{2}{c^2}f\right)\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}\right\}+\frac{1}{r^2 sin\theta}\left(1+\frac{2}{c^2}f\right)\frac{\partial}{\partial\theta}\left\{sin\theta\frac{\partial}{\partial\theta}\right\}$$

$$+\frac{1}{r^2 sin^2\theta}\left(1+\frac{2}{c^2}f\right)\frac{\partial}{\partial\emptyset}\left\{\frac{\partial}{\partial\emptyset}\right\}-\left(1+\frac{2}{\mathrm{c}^2}\mathrm{f}\right)^{-1}\frac{\partial}{\partial x^0}\left\{\frac{\partial}{\partial x^0}\right\}$$

(19)

Equation (19) is the golden Riemannian Laplacian operator in spherical polar coordinate. The well-known Laplacian operator is derived based on Euclidean geometry while equation (19) is derived based on the Riemannian geometry using the golden metric tensor. This equation is further applied to the Schrodinger equation in order to obtain the golden Riemannian Schrodinger equation.

2.2 Derivation of golden Riemannian Schrodinger equation in Spherical Polar Coordinate

Consider the well-known Schrodinger equation [4; 5] given by

$$E\psi = \mathrm{H}\psi = \frac{-\hbar^2\nabla^2}{2m}\psi + \mathrm{V}(\mathrm{r})\psi \tag{20}$$

where E is energy of the particle, H is Hamiltonian of the system, m is mass of the particle, $\hbar$ is normalized Planck's constant, ∇^2 is Euclidean Laplacian of the system, V is particle potential and ψ is wave function.

We replace the Euclidean Laplacian operator with the golden Riemannian Laplacian operator in equation (19); that is:

$$E\psi = \mathrm{H}\psi = \frac{-\hbar^2\nabla_R^2}{2m}\psi + \mathrm{V}(\mathrm{r})\psi \tag{21}$$

Substituting the expression for the Riemannian Laplacian operator, ∇_R^2 into equation (21), we obtain

$$\mathrm{H}\psi = -\frac{\hbar^2}{2m}\left\{\frac{1}{r^2}\left(1+\frac{2f}{c^2}\right)\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\left(1+\frac{2f}{c^2}\right)\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\left(1+\frac{2f}{c^2}\right)\frac{\partial}{\partial\emptyset}\left(\frac{\partial}{\partial\emptyset}\right) - \left(1+\frac{2}{c^2}\mathrm{f}\right)^{-1}\frac{\partial}{\partial x^0}\left(\frac{\partial}{\partial x^0}\right)\right\}\psi(r,t) + V\,\psi(r,t) \tag{22}$$

Expanding equation (22) and considering that $V = V_o$ which is the depth of the potential well, we obtain

$$i\hbar\left(\frac{\partial}{\partial\theta}\psi(r,\theta,\phi,x^o)\right) = -\frac{\hbar^2\eta}{mr}\left(\frac{\partial}{\partial r}\psi(r,\theta,\phi,x^o)\right) - \frac{\hbar^2\eta}{2m}\left(\frac{\partial^2}{\partial r^2}\psi(r,\theta,\phi,x^o)\right) -$$

$$\frac{\hbar^2\eta cos\theta}{2mr^2 sin\theta}\left(\frac{\partial}{\partial\theta}\psi(r,\theta,\phi,x^o)\right) - \frac{\hbar^2\eta}{2mr^2}\left(\frac{\partial^2}{\partial\theta^2}\psi(r,\theta,\phi,x^o)\right) -$$

$$\frac{\hbar^2\eta}{2mr^2 sin\theta^2}\left(\frac{\partial^2}{\partial\phi^2}\psi(r,\theta,\phi,x^o)\right) - \frac{\hbar^2}{2m\eta}\left(\frac{\partial^2}{\partial(x^o)^2}\psi(r,\theta,\phi,x^o)\right) + V_o\,\psi(\,r,t)$$

(23)

where $\eta = \left(1 + \frac{2}{c^2}\,f\right)$

(24)

Equation (23) is the golden Riemannian Schrodinger equation in spherical polar coordinates.

Using the method of separation of variables, we seek to express the wave function, ψ as

$$\psi = R(r)\Phi(\phi)\Theta(\theta)\exp\left(-\frac{iEt}{\hbar}\right)$$

(25)

Putting equation (25) into (23) yields

$$-\frac{R(r)\Phi(\phi)\Theta(\theta)E}{exp\frac{iEt}{\hbar}} = -\frac{\hbar^2\eta\left(\frac{d}{dr}R(r)\right)\Phi(\phi)\Theta(\theta)}{mrexp\frac{iEt}{\hbar}} - \frac{1}{2}\frac{\hbar^2\eta\left(\frac{d^2}{dr^2}R(r)\right)\Phi(\phi)\Theta(\theta)}{mexp\frac{iEt}{\hbar}} -$$

$$\frac{1}{2}\frac{\hbar^2\eta\cos(\theta)R(r)\Phi(\phi)\left(\frac{d}{d\theta}\Theta(\theta)\right)}{mr^2\sin(\theta)exp\frac{iEt}{\hbar}} - \frac{1}{2}\frac{\hbar^2\eta R(r)\Phi(\phi)\left(\frac{d^2}{d\theta^2}\Theta(\theta)\right)}{mr^2exp\frac{iEt}{\hbar}} -$$

$$\frac{1}{2}\frac{\hbar^2\eta R(r)\left(\frac{d^2}{d\phi^2}\Phi(\phi)\right)\Theta(\theta)}{mr^2\sin\theta^2 exp\frac{iEt}{\hbar}}-\frac{1}{2}\frac{R(r)\Phi(\phi)\Theta(\theta)i^2E^2}{m\eta exp\frac{iEt}{\hbar}}+\frac{V_oR(r)\Phi(\phi)\Theta(\theta)}{exp\frac{iEt}{\hbar}}$$

(26)

Dividing equation (26) by (25) and bringing the like terms together we have

$$E=-\frac{\hbar^2\eta\left(\frac{d}{dr}R(r)\right)}{R(r)mr}-\frac{1}{2}\frac{\hbar^2\eta\left(\frac{d^2}{dr^2}R(r)\right)}{R(r)m}-\frac{1}{2}\frac{\hbar^2\eta\cos(\theta)\left(\frac{d}{d\theta}\Theta(\theta)\right)}{\Theta(\theta)mr^2\sin(\theta)}-$$

$$\frac{1}{2}\frac{\hbar^2\eta\left(\frac{d^2}{d\theta^2}\Theta(\theta)\right)}{\Theta(\theta)mr^2}-\frac{1}{2}\frac{\hbar^2\eta\left(\frac{d^2}{d\phi^2}\Phi(\phi)\right)}{\Phi(\phi)mr^2\sin\theta^2}+\frac{1}{2}\frac{E^2}{m\eta}+V_o$$

(27)

Rearranging equation (27) we have

$$-\frac{1}{2}\frac{\hbar^2\eta\left(\frac{d^2}{dr^2}R(r)\right)}{R(r)m}-\frac{\hbar^2\eta\left(\frac{d}{dr}R(r)\right)}{R(r)mr}+\frac{1}{2}\frac{E^2}{m\eta}+V_o-E=$$

$$-\frac{1}{2}\frac{\hbar^2\eta\cos(\theta)\left(\frac{d}{d\theta}\Theta(\theta)\right)}{\Theta(\theta)mr^2\sin(\theta)}-\frac{1}{2}\frac{\hbar^2\eta\left(\frac{d^2}{d\theta^2}\Theta(\theta)\right)}{\Theta(\theta)mr^2}-\frac{1}{2}\frac{\hbar^2\eta\left(\frac{d^2}{d\phi^2}\Phi(\phi)\right)}{\Phi(\phi)mr^2\sin\theta^2}$$

(28)

Equating the left hand side of equation (28) to $-\lambda^2$ implies that

$$-\frac{1}{2}\frac{\hbar^2\eta\left(\frac{d^2}{dr^2}R(r)\right)}{R(r)m}-\frac{\hbar^2\eta\left(\frac{d}{dr}R(r)\right)}{R(r)mr}+\frac{1}{2}\frac{E^2}{m\eta}+V_o-E=-\lambda^2$$

(29)

Multiplying through equation (29) by $-\frac{2mR(r)}{\hbar^2\eta}$

$$\frac{d^2}{dr^2}R(r)+\frac{2\left(\frac{d}{dr}R(r)\right)}{r}-\frac{R(r)E^2}{\hbar^2\eta^2}-\frac{2mR(r)V_o}{\hbar^2\eta}+\frac{2mR(r)E}{\hbar^2\eta}=\frac{2mR(r)\lambda^2}{\hbar^2\eta}$$

(30)

Rearranging equation (30) we have

$$\frac{d^2}{dr^2}R(r)+\frac{2\left(\frac{d}{dr}R(r)\right)}{r}-\frac{R(r)E^2}{\hbar^2\eta^2}-\frac{2mR(r)V_o}{\hbar^2\eta}+\frac{2mR(r)E}{\hbar^2\eta}-\frac{2mR(r)\lambda^2}{\hbar^2\eta}=0 \quad (31)$$

Equation (31) becomes

$$\frac{d^2}{dr^2}R(r)+\frac{2}{r}\left(\frac{d}{dr}R(r)\right)-\frac{1}{\hbar^2\eta}\left(\frac{E^2}{\eta}+2mV_o-2mE+2m\lambda^2\right)R(r)=0 \quad (32)$$

From equation (32)

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{1}{\hbar^2\eta}\left(\frac{E^2}{\eta}+2mV_o-2mE+2m\lambda^2\right)R=0 \quad (33)$$

Let $R=a_o+a_1r+a_2r^2+a_3r^3+\cdots+a_kr^k$ (34)

Thus,

$$R = \sum_{k=0}^{\infty} a_k r^k \tag{35}$$

$$R' = \sum_{k=1}^{\infty} a_k r^{k-1} \tag{36}$$

$$R'' = \sum_{k=2}^{\infty} a_k r^{k-2} \tag{37}$$

Substituting equations (35) to (37) into (33) we have

$$\sum_{k=2}^{\infty} \mathrm{k}(\mathrm{k}-1) a_k r^{k-2} + 2r^{-1} \sum_{k=1}^{\infty} \mathrm{k} a_k r^{k-1} - \tau \sum_{k=0}^{\infty} a_k r^k = 0 \tag{38}$$

Where $\tau = \frac{1}{\hbar^2 \eta}\left(\frac{E^2}{\eta} + 2mV_o - 2mE + 2m\lambda^2\right)$ (39)

This implies that

$$\sum_{k=2}^{\infty} \mathrm{k}(\mathrm{k}-1) a_k r^{k-2} + \sum_{k=1}^{\infty} 2\mathrm{k} a_k r^{k-2} - \sum_{k=0}^{\infty} \tau a_k r^k = 0 \tag{40}$$

Shifting the first term of equation (40) yields

$$\sum_{k=0}^{\infty} (\mathrm{k}+2)(\mathrm{k}+1) a_k r^k + \sum_{k=0}^{\infty} 2(\mathrm{k}+2) a_{k+2} r^k - \sum_{k=0}^{\infty} \tau a_k r^k = 0 \tag{41}$$

$$\sum_{k=0}^{\infty} \{(\mathrm{k}+2)(\mathrm{k}+1) + 2(\mathrm{k}+2)\} a_{k+2} r^k - \sum_{k=0}^{\infty} \tau a_k r^k = 0 \tag{42}$$

$$\{(\mathrm{k}+2)(\mathrm{k}+1) + 2(\mathrm{k}+2)\} a_{k+2} - \tau a_k = 0 \tag{43}$$

It implies that

$$\{(\mathrm{k}+2)(\mathrm{k}+3)\}a_{k+2} - \tau a_k = 0 \quad (44)$$

and

$$a_{k+2} = \frac{\tau a_k}{(\mathrm{k}+2)(\mathrm{k}+3)} \quad ; \mathrm{k} = 0,1,2,3 \ldots \quad (45)$$

From equation (45) we have

$$a_2 = \frac{\tau a_o}{3!} \ ; k = 0 \quad (46)$$

$$a_3 = \frac{\tau a_1}{3\times 4} \ ; k = 1 \quad (47)$$

$$a_4 = \frac{\tau^2 a_o}{5!} \ ; k = 2 \quad (48)$$

$$a_5 = \frac{\tau^2 a_1}{6\times 5\times 4\times 3} \ ; k = 3 \quad (49)$$

$$a_6 = \frac{\tau^3 a_o}{7!} \quad ; k = 4 \tag{50}$$

$$a_7 = \frac{\tau^3 a_1}{8\times7\times6\times5\times4\times3} \quad ; k = 5 \tag{51}$$

Substituting equations (46) to (51) into (34) we have

$$R = a_o + a_1 r + \frac{\tau a_o}{3!} r^2 + \frac{\tau a_1}{3\times4} r^3 + \frac{\tau^2 a_o}{5!} r^4 + \frac{\tau^2 a_1}{6\times5\times4\times3} r^5 + \frac{\tau^3 a_o}{7!} r^6 + \frac{\tau^3 a_1}{8\times7\times6\times5\times4\times3} r^7 + \cdots \tag{52}$$

$$R = \left(a_o + \frac{\tau a_o}{3!} r^2 + \frac{\tau^2 a_o}{5!} r^4 + \frac{\tau^3 a_o}{7!} r^6\right) + \left(a_1 r + \frac{\tau a_1}{3\times4} r^3 + \frac{\tau^2 a_1}{6\times5\times4\times3} r^5 + \frac{\tau^3 a_1}{8\times7\times6\times5\times4\times3} r^7\right) + \ldots \tag{53}$$

Therefore,

$$R(r) = \frac{c_1}{r} \exp\left(-\sqrt{\tau}\right) r + \frac{c_2}{r\sqrt{\tau}} \exp\left(\sqrt{\tau}\right) r \tag{54}$$

Substituting for τ we have

$$R(r) = \frac{c_1}{r}\exp\left\{-\frac{1}{\hbar^2\eta}\left(\frac{E^2}{\eta} + 2mV_o - 2mE + 2m\lambda^2\right)\right\}^{\frac{1}{2}} r + \frac{c_2}{\left\{\frac{1}{\hbar^2\eta}\left(\frac{E^2}{\eta}+2mV_o-2mE+2m\lambda^2\right)\right\}^{\frac{1}{2}} r}$$

$$\exp\left\{\frac{1}{\hbar^2\eta}\left(\frac{E^2}{\eta} + 2mV_o - 2mE + 2m\lambda^2\right)\right\}^{\frac{1}{2}} r \qquad (55)$$

Solving equation (55) for E, we obtain

$$E = \frac{1}{r}\left\{m\eta r + \left(m^2\eta^2 r^2 + ln\left(\frac{R(r)r+\sqrt{R(r)^2r^2+c_1{}^2-c_2{}^2}}{c_1{}^2+c_2{}^2}\right)^2 \hbar^2\eta^2 - 2m\lambda^2\eta r^2 - \right.\right.$$

$$\left.\left. 2mV_o\eta r^2\right)^{\frac{1}{2}}\right\}$$

(56)

Also equating the right hand side of equation (28) to $-\lambda^2$ implies that

$$-\frac{\hbar^2\eta\cos\theta}{2\Theta(\theta)mr^2\sin\theta}\left(\frac{d}{d\theta}\Theta(\theta)\right) - \frac{\hbar^2\eta}{2\Theta(\theta)mr^2}\left(\frac{d^2}{d\theta^2}\Theta(\theta)\right) -$$

$$\frac{\hbar^2\eta}{2\Phi(\phi)mr^2\sin\theta^2}\left(\frac{d^2}{d\phi^2}\Phi(\phi)\right) = -\lambda^2 \qquad (57)$$

Multiplying through equation (57) by $-\frac{2mr^2}{\hbar^2\eta}$, we obtain

$$\frac{\cos(\theta)\left(\frac{d}{d\theta}\Theta(\theta)\right)}{\Theta(\theta)\sin(\theta)} + \frac{\frac{d^2}{d\theta^2}\Theta(\theta)}{\Theta(\theta)} + \frac{\left(\frac{d^2}{d\phi^2}\Phi(\phi)\right)}{\Phi(\phi)\sin\theta^2} = \frac{2mr^2\lambda^2}{\hbar^2\eta}$$

(58)

Rearranging we have

$$\frac{\cos(\theta)\left(\frac{d}{d\theta}\Theta(\theta)\right)}{\Theta(\theta)\sin(\theta)} + \frac{\frac{d^2}{d\theta^2}\Theta(\theta)}{\Theta(\theta)} + \frac{\left(\frac{d^2}{d\phi^2}\Phi(\phi)\right)}{\Phi(\phi)\sin\theta^2} - \frac{2mr^2\lambda^2}{\hbar^2\eta} = 0$$

(59)

Equivalently

$$\frac{\cos(\theta)\left(\frac{d}{d\theta}\Theta(\theta)\right)}{\Theta(\theta)\sin(\theta)} + \frac{\frac{d^2}{d\theta^2}\Theta(\theta)}{\Theta(\theta)} - \frac{2mr^2\lambda^2}{\hbar^2\eta} = -\frac{\left(\frac{d^2}{d\phi^2}\Phi(\phi)\right)}{\Phi(\phi)\sin\theta^2}$$

(60)

Equating the left hand side of equation (61) to $-k$ implies that

$$\frac{\cos(\theta)\left(\frac{d}{d\theta}\Theta(\theta)\right)}{\Theta(\theta)\sin(\theta)} + \frac{\frac{d^2}{d\theta^2}\Theta(\theta)}{\Theta(\theta)} - \frac{2mr^2\lambda^2}{\hbar^2\eta} = -k$$

(61)

Multiplying through equation (61) by $\Theta(\theta)$ gives

$$\frac{\cos(\theta)\left(\frac{d}{d\theta}\Theta(\theta)\right)}{\sin(\theta)} + \frac{d^2}{d\theta^2}\Theta(\theta) - \frac{2\Theta(\theta)mr^2\lambda^2}{\hbar^2\eta} = -\Theta(\theta)k$$

(62)

From equation (62) we have

$$\frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + \left(k - \frac{1}{\hbar^2\eta}(2mr^2\lambda^2)\right)\Theta = 0$$

(63)

Let $\varrho = k - \frac{1}{\hbar^2\eta}\left(2mr^2\lambda^2\right)$

(64)

Equation (64) becomes

$$\frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + \varrho\Theta = 0$$

(65)

Using same method of obtaining equation (56) we have

$$\Theta(\theta) = c_1\left\{1 - \frac{\varrho}{2!}\rho^2 - \frac{\varrho}{4!}(6-\varrho)\rho^4 - \frac{\varrho}{6!}(20-\varrho)(6-\varrho)\rho^6\right\} + c_2\left\{\rho + \right.$$

$$\left.\frac{1}{3!}(2-\varrho)\rho^3 + \frac{1}{5!}(12-\varrho)(2-\varrho)\rho^5 + \frac{1}{7!}(30-\varrho)(12-\varrho)(2-\varrho)\varrho\rho^7\right\}$$

(66)

Equating the right hand side of equation (60) to – k implies that

$$-\frac{\frac{d^2}{d\phi^2}\Phi(\phi)}{\Phi(\phi)\sin\theta^2} = -k \tag{67}$$

Multiplying through by $\Phi(\phi)\sin\theta^2$ we have

$$\frac{d^2}{d\phi^2}\Phi(\phi) - \Phi(\phi)(\sin\theta^2)k = 0 \tag{68}$$

From equation (68)

$$\frac{d^2\Phi}{d\phi^2} - \Phi\sin^2\theta k = 0 \tag{69}$$

This implies that

$$\frac{d^2\Phi}{d\phi^2} - k\sin^2\theta\Phi = 0 \tag{70}$$

The characteristic equation is given by

$$m^2 - k\sin^2\theta = 0 \tag{71}$$

and

$$m = \pm\sqrt{ksin^2\theta} = \pm\sqrt{k}sin\theta \tag{72}$$

Hence,

$$\Phi(\phi) = c_1 \exp\left(\sqrt{k}(sin\theta)\phi\right) + c_2 \exp\left(-\sqrt{k}(sin\theta)\phi\right) \quad (73)$$

Seeking the solution for equation (73) as

$$\frac{1}{r}\left[\left(\frac{-2m\left(-\lambda^2+E-V_o\right)\eta+E^2}{\hbar^2\eta^2}\right)^{1/2} r\right] = n\pi \quad (74)$$

$$\left(-\frac{1}{\hbar^2\eta^2} 2m(-\lambda^2 + E - V_o)\eta + E^2\right)^{\frac{1}{2}} - n\pi = 0 \quad (75)$$

Solving for E from equation (75) yields

$$\begin{bmatrix} E = \eta m + \sqrt{\eta^2\hbar^2 n\pi^2 + \eta^2 m^2 - 2\eta m\lambda^2 - 2V_o\eta m}, \\ E = \eta m - \sqrt{\eta^2\hbar^2 n\pi^2 + \eta^2 m^2 - 2\eta m\lambda^2 - 2V_o\eta m} \end{bmatrix} \quad (76)$$

From equation (76) we have two sets of values for the energy which are identified as

$$E_1 = \eta m + \sqrt{\eta^2\hbar^2 n\pi^2 + \eta^2 m^2 - 2\eta m\lambda^2 - 2V_o\eta m} \quad (77)$$

and

$$E_2 = \eta m - \sqrt{\eta^2\hbar^2 n\pi^2 + \eta^2 m^2 - 2\eta m\lambda^2 - 2V_o\eta m} \quad (78)$$

Substituting the expression for η from equation (24) into equations (77) and (78) we have

$$E_1 = \left(1 + \frac{2}{c^2} f\right) m +$$

$$\sqrt{\left(1+\frac{2}{c^2}f\right)^2 \hbar^2 n\pi^2 + \left(1+\frac{2}{c^2}f\right)^2 m^2 - 2\left(1+\frac{2}{c^2}f\right) m\lambda^2 - 2V_o\left(1+\frac{2}{c^2}f\right) m}$$

(79)

and

$$E_2 = \left(1+\frac{2}{c^2}f\right) m -$$

$$\sqrt{\left(1+\frac{2}{c^2}f\right)^2 \hbar^2 n\pi^2 + \left(1+\frac{2}{c^2}f\right)^2 m^2 - 2\left(1+\frac{2}{c^2}f\right) m\lambda^2 - 2V_o\left(1+\frac{2}{c^2}f\right) m}$$

(80)

Further simplification and expansion of equations (79) and (80) gives

$$E_{n\,(for\ odd\ n)} = m + \frac{2fm}{c^2} + \left(n\pi^2\hbar^2 - \frac{4n\pi^2\hbar^2 f}{c^2} + \frac{4n\pi^2\hbar^2 f^2}{c^4} + m^2 -\right.$$

$$\left.\frac{4m^2 f}{c^2} + \frac{4m^2 f^2}{c^4} - 2m\lambda^2 + \frac{4m\lambda^2 f}{c^2} - 2V_o m + \frac{4V_o m f}{c^2}\right)^{\frac{1}{2}}$$

(81)

and

$$E_{n\,(for\ even\ n)} = m + \frac{2fm}{c^2} - \left(n\pi^2\hbar^2 - \frac{4n\pi^2\hbar^2 f}{c^2} + \frac{4n\pi^2\hbar^2 f^2}{c^4} + m^2 - \frac{4m^2 f}{c^2} + \frac{4m^2 f^2}{c^4} - 2m\lambda^2 + \frac{4m\lambda^2 f}{c^2} - 2V_o m + \frac{4V_o mf}{c^2}\right)^{\frac{1}{2}} \quad (82)$$

where n is energy level of the particle in a finite potential well, m is the mass of the particle, c is speed of light, V_o is depth of the well, f is gravitational scalar potential, $\hbar$ is normalized Planck's constant π and λ are constants.

3. Discussion

Equation (81) and (82) are the solutions to the golden Riemannian Schrodinger equation. They represent the quantum energies of the particle in a finite-potential well. Equation (81) represents the energy at odd energy levels and equation (82) represents the energy at even energy levels.

This can also be applied to all entities of non-zero rest mass such as: infinite potential well, rectangular potential well, simple harmonic oscillator etc.

4. Remarks and Conclusion

We have in this article, shown how to formulated and constructed the Riemannian Laplacian operator and the golden Riemannian Schrodinger equation. We have solved the golden Riemannian Schrodinger equation analytically and obtained the expressions for the quantum energies for both odd and even states.

References

[1] Anchaver, R. S. (2003). *Introduction to Non-Relativistic Quantum Mechanics*. Nigeria, ISBN: 978-056-139-0, 1-3

[2] Brumel, R. (2005). Analytical Solution of the Finite Quantum Square-Well Problem. *Journal of Physics*. 38, 673-678

[3] Hewitt, P. G. (2002). *Conceptual Physics*. 9th Edition, World Student Series, Brad Lewis/Stone Publishers, 630-631

[4] Howusu, S.X.K. (2003a). *Riemannian Revolution in Physics and Mathematics*. Jos University Press Ltd, Jos, 1-200

[5] Howusu, S. X. K. (2003b). *The Natural Philosophy of Quantum Mechanics*. 2nd Edition, Jos University Press Ltd., Jos, ISBN: 978-166-073-2, 20-25

[6] Howusu, S. X. K. (2009). *The Metric Tensors for gravitational Fields and The Mathematical Principle of Riemannian Theoretical Physics*. 1st Edition, Jos University Press Ltd., Jos, ISBN: 978-166-639-0, 121-122

[7] Howusu, S. X. K. (2011). *The Golden Metric Tensor in Orthogonal Curvilinear Co-ordinates*. 1st Edition, Jos University Press Ltd., Jos, ISBN: 978-166-294-8, 5-6

[8] Jones, E. & Childers, R. (1983). *Contemporary College Physics*. 3rd Edition, McGraw Hill, New York, 889

[9] Luca, N. (2015). The Hydrogen Atom: A Review on the Birth of Modern Quantum Mechanics. 1-3

[10] Lumbi, W. L. & Ewa, I. I. (2013). General Relativistic Equation of Motion for a Photon Moving Round a Time Varying Spherical Distribution of Mass. *Advances in Natural Science*, 6(3), 23-25.

[11] Ronald, C. B. (2007). Inverse Quantum Mechanics of the Hydrogen Atom: A General Solution. *Adv. Studies Theor. Phys*. 1(8), 381-393

[12] Spiegel, M. R. (1974). *Theory and Problems of Vector Analysis and Introduction to Tensor Analysis*. 1st Edition, McGraw Hill, New York, 166-217