Competitive Reaction-Diffusion Systems: Traveling Waves and Numerical Solutions

Abstract

In this paper, we consider a competitive reaction-diffusion model to describe the existence of travelling wave solutions of two competing species. Moreover, the non-linear system is also studied by introducing different competitive-cooperative coefficients (constant and space dependent) which leads to the persistence and extinction of organisms in a biology. If the diffusion coefficients and other parameters are positive constant, it is seen that one species is in extinction by the other and coexistence is also possible under certain conditions on carrying capacity. The results are numerically investigated by using the Finite difference method (FDM).

Keywords: Nonlinear PDEs, Travelling wave solutions, Reaction-diffusion, Crank-Nicolson scheme.

***AMS Subject Classification*:** 92D25, 35K57 (primary), 35K61, 37N25.

1. Introduction

In nature there are two or more species compete for the same limited food source or in some way inhibit each other's growth. This type of interspecies interactions is known as mutual competitive suppression, or competition for a common resource [1]. Their dynamics is considerably very rich, and also of great importance for the functioning of ecosystems. To describe the dynamics of two competing populations, the basic 2-species Lotka–Volterra competition model with diffusion can be used [2], which has the following set of equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(1 - au - \gamma v) \\ \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + v(1 - bu - \delta v) \end{cases} \qquad (x,t) \in \mathbb{R} \times [0,\infty) \tag{1.1}$$

where, u and v are the density of the two interacting species, $1/a$, $1/\delta$ are carrying capacities, γ, b are competition coefficients and d_1, d_2 are diffusion coefficients, all non-negative. Here Δ is an operator which can also be written as $\Delta = \frac{\partial^2}{\partial x^2}$. Note that the competition model (1.1) is reaction-diffusion type and not a conservative system like its Lotka–Volterra predator–prey counterpart.

In modern mathematics, the theory of traveling wave solution of partial differential equation is applied to describe different phenomena in ecology [3], farming [4], forestry [5], cell culture [6] and other natural sciences [7]. In this paper, we will study the traveling wave solution of the competitive reaction-diffusion system (1.1). We evaluate an approximate transformation of the traveling wave equations into monotone form and we reduce the existence proof to application of well-defined theory about monotone traveling wave systems [8]. Let us now consider the system (1.1) as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{d}\dfrac{\partial^2 u}{\partial x^2} + u(1 - au - \gamma v) \\ \\ \dfrac{\partial v}{\partial t} = \mathrm{d}\dfrac{\partial^2 v}{\partial x^2} + v(1 - bu - \delta v) \end{cases} \qquad (x,t) \in \mathbb{R} \times [0, \infty) \tag{1.2}$$

For traveling wave solutions of the above systems, we will consider the following hypotheses:

$[A1]$ $\quad a < b$

$[A2]$ $\quad \gamma < \delta$

We will discuss the existence and uniqueness of the traveling wave solutions of the form $\left(u\left(\sqrt{\frac{1}{\mathrm{d}}}x + ct\right), v\left(\sqrt{\frac{1}{\mathrm{d}}}x + ct\right)\right)$ joining the equilibria $\left(0,\ \frac{1}{\delta}\right)$ and $\left(\frac{1}{a},\ 0\right)$ as $\sqrt{\frac{1}{\mathrm{d}}}x + ct$ moves from $-\infty$ to $+\infty$. It means, when the second species move from carrying capacity to extinction, first species move from extinction to carrying capacity. If the inequality of hypothesis in $[A1]$ is interchanged, the existence of traveling wave solutions activating from $(0, 0)$ to positive coexistence equilibrium which proved in [9]. However, if $[A2]$ is interchanged, [10] and [11] assured us the existence of traveling wave solutions activating from one equilibrium on one positive axis to the equilibrium on another positive axis. Generally, we are able to observe that in some papers [8, 9] and [10] are used to solve the existence of traveling wave solutions using dynamical system and ordinary differential equation methods. We get help for studying about traveling wave solutions on other interacting species in related papers [13, 14, 15] and [16]. We can also be found various types of boundary value problems including the system (1.2) in [17, 18, 19, 20] and [21]. These books are not related to traveling wave solutions. The novelty of this work is that we use an alternative method of upper–lower solutions to prove the existence of traveling wave solutions. Moreover, we make the resulting system into a monotone system by changing the variable in the second equation of system (1.2) with reversing order.

2. Existence of Travelling Wave Solution

In this section, we will show the existence of travelling wave solution and explore the system (1.2) which has of the form $\left(u\left(\sqrt{\frac{1}{\mathrm{d}}}x + ct\right),\ v\left(\sqrt{\frac{1}{\mathrm{d}}}x + ct\right)\right)$ adding the equilibria $\left(0,\ \frac{1}{\delta}\right)$ and $\left(\frac{1}{a},\ 0\right)$ as $\sqrt{\frac{1}{\mathrm{d}}}x + ct$ moves from $-\infty$ to $+\infty$.

Let us consider

$$t = \bar{t} \quad \text{and} \quad x = \sqrt{\mathrm{d}}\bar{x} \tag{2.1}$$

Equation (1.2) can be written as

$$\begin{cases} \dfrac{\partial u}{\partial \bar{t}} = \mathrm{d}\dfrac{1}{\left(\sqrt{\mathrm{d}}\right)^2}\dfrac{\partial^2 u}{\partial \bar{x}^2} + u(1 - au - \gamma v) \\ \\ \dfrac{\partial v}{\partial \bar{t}} = \mathrm{d}\dfrac{1}{\left(\sqrt{\mathrm{d}}\right)^2}\dfrac{\partial^2 v}{\partial \bar{x}^2} + v(1 - bu - \delta v) \end{cases}$$

where $\bar{x} \in \mathbb{R}$, $\bar{t} \in \mathbb{R}^+$. Now we can simplify above system such that

$$\begin{cases} \dfrac{\partial u}{\partial \bar{t}} = \dfrac{\partial^2 u}{\partial \bar{x}^2} + u(1 - au - \gamma v) \\ \\ \dfrac{\partial v}{\partial \bar{t}} = \dfrac{\partial^2 v}{\partial \bar{x}^2} + v(1 - bu - \delta v) \end{cases} \qquad \bar{x} \in \mathbb{R},\ \bar{t} \in \mathbb{R}^+ \tag{2.2}$$

Let

$$u = yM, \qquad v = wN, \tag{2.3}$$

where $y = \frac{1}{a}$ and w is a constant satisfying

$$\frac{1}{\delta} < w < \frac{1}{\gamma} \tag{2.4}$$

Then the system (2.2) becomes

$$\begin{cases} \dfrac{\partial (yM)}{\partial \bar{t}} = \dfrac{\partial^2 (yM)}{\partial \bar{x}^2} + yM(1 - ayM - \gamma wN) \\ \\ \dfrac{\partial (wN)}{\partial \bar{t}} = \dfrac{\partial^2 (wN)}{\partial \bar{x}^2} + wN(1 - byM - \delta wN) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial M}{\partial \bar{t}} = \dfrac{\partial^2 M}{\partial \bar{x}^2} + M(1 - M - \gamma wN) \\ \\ \dfrac{\partial N}{\partial \bar{t}} = \dfrac{\partial^2 N}{\partial \bar{x}^2} + N\left(1 - \dfrac{b}{a}M - \delta wN\right) \end{cases}$$

After rearranging the above system, we get

$$\begin{cases} \dfrac{\partial M}{\partial \bar{t}} = \dfrac{\partial^2 M}{\partial \bar{x}^2} + M(1 - M - zN) \\ \\ \dfrac{\partial N}{\partial \bar{t}} = \dfrac{\partial^2 N}{\partial \bar{x}^2} + N(\rho_1 - b_1 M - \rho_1(1 + \rho_2)N) \end{cases} \tag{2.5}$$

where

$$z = \gamma w, \qquad \rho_1 = 1, \qquad b_1 = \frac{b}{a}, \qquad \rho_2 = \delta w - 1, \tag{2.6}$$

Here from $[A1]$ and (2.4), we have

$$z \in (0,1), \qquad \rho_2 > 0 \tag{2.7}$$

We can make ρ_2 arbitrarily small by taking w close to $\frac{1}{\delta}$ in (2.4).

Theorem 2.1. [3] Let us consider the system (1.2) under $[A1]$ and $[A2]$. For transforming the system (1.2) into system (2.5), we use the change of variables (2.1) and (2.3) with w satisfying (2.4). The parameters in (2.5) are related to those in (1.2) by (2.6) and the parameters z, ρ_1, ρ_2 and b_1 satisfy the inequalities in (2.7).

If $(M(\bar{t},\bar{x}),\ N(\bar{t},\bar{x}))$ is a solution of (2.5) we can easily verify that

$$(u(t,x),v(t,x)) = \left(u\left(\bar{t},\sqrt{\mathrm{d}}\bar{x}\right),v\left(\bar{t},\sqrt{\mathrm{d}}\bar{x}\right)\right) = (yM(\bar{t},\bar{x}),wN(\bar{t},\bar{x})) \tag{2.8}$$

is a solution of (1.2), where y and w are introduced in (2.3), (2.4). Now we have to find for solution of system (2.5). Let us consider the transformation

$$\left(M(\bar{t},\bar{x}),N(\bar{t},\bar{x})\right) = \left(M(s^*),N(s^*)\right) \text{ where } s^* = \bar{x}+c\bar{t}$$

and it satisfies

$$\begin{cases} \lim\limits_{s\to-\infty}\left(M(s^*),N(s^*)\right) = \left(0,\dfrac{1}{1+\rho_2}\right) \\ \\ \lim\limits_{s\to+\infty}\left(M(s^*),N(s^*)\right) = (1,0) \end{cases} \tag{2.9}$$

Using this transformation, relating back to (2.5), we are now finding for solution of

$$\begin{cases} c\dfrac{\partial M}{\partial s^*} = \dfrac{\partial^2 M}{\partial {s^*}^2} + M(1-M-zN) \\ \\ c\dfrac{\partial N}{\partial s^*} = \dfrac{\partial^2 N}{\partial {s^*}^2} + N(\rho_1 - b_1M - \rho_1(1+\rho_2)N) \end{cases} \qquad s^* \in (-\infty,\infty) \tag{2.10}$$

Theorem 2.2. [3] System (1.2) has a travelling wave solution of the form

$$(u(t,x),v(t,x)) = \left(yM\left(\sqrt{\frac{1}{\mathrm{d}}}x+ct\right),wN\left(\sqrt{\frac{1}{\mathrm{d}}}x+ct\right)\right) \tag{2.11}$$

for any $c > 2$ under the hypotheses [A1], [A2] and newly [A3] such that

[A3] $\qquad b \leq 2a$.

Now, (M,N) is a function of one variable which is denoted by s^* satisfying (2.10) for $\mathrm{s}^* \in (-\infty,\infty)$ and (2.9) as $s^* \to \pm\infty$ and also $M(s^*)$ and $N(s^*)$ are positive monotonic functions for $s^* \in (-\infty,\infty)$. Remarkable thing is that

$$\begin{cases} \lim\limits_{t\to-\infty}\left(u(t,x),v(t,x)\right) = \left(0,\dfrac{1}{\delta}\right) \\ \\ \lim\limits_{t\to+\infty}\left(u(t,x),v(t,x)\right) = \left(\dfrac{1}{a},0\right) \end{cases} \tag{2.12}$$

Proof: The change of variables

$$\begin{gathered} u_1(s^*) = M(s^*) \\ u_2(s^*) = \frac{1}{1+\rho_2} - N(s^*) \end{gathered} \tag{2.13}$$

For $s^* \in (-\infty, \infty)$, turns (2.10) into

$$\begin{cases} c\dfrac{\partial u_1}{\partial s^*} = \dfrac{\partial^2 u_1}{\partial s^{*2}} + u_1\left(1 - u_1 - z\left(\dfrac{1}{1+\rho_2} - u_2\right)\right) \\ c\dfrac{\partial u_2}{\partial s^*} = \dfrac{\partial^2 u_2}{\partial s^{*2}} + \left(\dfrac{1}{1+\rho_2} - u_2\right)(b_1 u_1 - \rho_1(1+\rho_2)u_2) \end{cases}$$

$$= \begin{cases} c\dfrac{\partial u_1}{\partial s^*} = \dfrac{\partial^2 u_1}{\partial s^{*2}} + u_1\left(1 - u_1 - \dfrac{z}{1+\rho_2} + zu_2\right) \\ c\dfrac{\partial u_2}{\partial s^*} = \dfrac{\partial^2 u_2}{\partial s^{*2}} + \left(\dfrac{1}{1+\rho_2} - u_2\right)(b_1 u_1 - \rho_1(1+\rho_2)u_2) \end{cases}$$

$$= \begin{cases} -\dfrac{\partial^2 u_1}{\partial s^{*2}} + c\dfrac{\partial u_1}{\partial s^*} = u_1\left(\dfrac{1+\rho_2-z}{1+\rho_2} - u_1 + zu_2\right), \\ -\dfrac{\partial^2 u_2}{\partial s^{*2}} + c\dfrac{\partial u_2}{\partial s^*} = \left(\dfrac{1}{1+\rho_2} - u_2\right)(b_1 u_1 - \rho_1(1+\rho_2)u_2), \end{cases} \tag{2.14}$$

This equation is also monotone for following conditions such that

$$0 \le u_1, 0 \le u_2 \le \frac{1}{1+\rho_2}.$$

Now we have to construct a pair of coupled upper solutions for the system (2.14). Let us consider an increasing function $\bar{u}(s^*)$ satisfying the following Kolmogorov-Petrovskii-Piscunov (KPP) equation for $c > 2$ such that

$$-\frac{d^2\bar{u}}{ds^{*2}} + c\frac{d\bar{u}}{ds^*} = \bar{u}(1-\bar{u}) \tag{2.15}$$

For $s^* \in (-\infty, \infty)$ and also $\lim_{s^*\to-\infty} \bar{u}(s^*) = 0$ and $\lim_{s^*\to\infty} \bar{u}(s^*) = 1$. Let

$$\bar{u}_1(s^*) = \bar{u}(s^*), \qquad \bar{u}_2(s^*) = \frac{1}{1+\rho_2}\bar{u}(s^*), \tag{2.16}$$

For $0 \le u_2 \le \bar{u}_2(s^*)$, we can easily observe that

$$\begin{aligned} -\frac{d^2\bar{u}_1}{ds^{*2}} + c\frac{d\bar{u}_1}{ds^*} - \bar{u}_1\left(\frac{1+\rho_2-z}{1+\rho_2} - \bar{u}_1 + zu_2\right) \\ &= -\frac{d^2\bar{u}}{ds^{*2}} + c\frac{d\bar{u}}{ds^*} - \bar{u}\left(1 - \frac{z}{1+\rho_2} - \bar{u} + zu_2\right) \\ &= \bar{u}(1-\bar{u}) - \bar{u}\left(1 - \frac{z}{1+\rho_2} - \bar{u} + zu_2\right) \\ &= \bar{u}\left(1 - \bar{u} - 1 + \frac{z}{1+\rho_2} + \bar{u} - zu_2\right) \\ &= \bar{u}\left(\frac{z}{1+\rho_2} - zu_2\right) \\ &\ge \frac{z}{1+\rho_2}\bar{u}(1-\bar{u}) > 0 \end{aligned} \tag{2.17}$$

for all $s^* \in (-\infty, \infty)$. For $0 \le u_1 \le \bar{u}_1(s^*)$, we also can check that

$$
\begin{aligned}
&-\frac{d^2\bar{u}_2}{ds^{*2}}+c\frac{d\bar{u}_2}{ds^*}-\left(\frac{1}{1+\rho_2}-\bar{u}_2\right)(b_1u_1-\rho_1(1+\rho_2)\bar{u}_2)\\
&\quad=\frac{1}{1+\rho_2}\left(-\frac{d^2\bar{u}}{ds^{*2}}\right)+c\frac{1}{1+\rho_2}\frac{d\bar{u}}{ds^*}-\left(\frac{1}{1+\rho_2}-\frac{\bar{u}}{1+\rho_2}\right)\left(b_1u_1-\rho_1(1+\rho_2)\frac{\bar{u}}{(1+\rho_2)}\right)\\
&\quad=-\frac{1}{1+\rho_2}\frac{d^2\bar{u}}{ds^{*2}}+\frac{c}{1+\rho_2}\frac{d\bar{u}}{ds^*}-\frac{1}{1+\rho_2}(1-\bar{u})(b_1u_1-\rho_1\bar{u})\\
&\quad=\frac{1}{1+\rho_2}\left[-\frac{d^2\bar{u}}{ds^{*2}}+c\frac{d\bar{u}}{ds^*}-(1-\bar{u})(b_1u_1-\rho_1\bar{u})\right]\\
&\quad=\frac{1}{1+\rho_2}[\bar{u}(1-\bar{u})+(1-\bar{u})(\bar{u}\rho_1-b_1u_1)]\\
&\quad\geq\frac{1}{1+\rho_2}\bar{u}(1-\bar{u})(1+\rho_1-b_1)\\
&\quad\geq 0
\end{aligned}
$$

for all $s^*\in(-\infty,\infty)$. We can say that the inequalities are true cause

$$1+\rho_1-b_1=1+1-\frac{b}{a}=2-\frac{b}{a}\geq 0$$

by hypothesis $[A3]$. Consider a pair of functions denoted by $\bar{\eta}_1(s^*)$ and $\bar{\eta}_2(s^*)$ and defined by

$$\bar{\eta}_1(s^*)=\bar{u}_1(-s^*),\bar{\eta}_2(s^*)=\bar{u}_2(-s^*) \tag{2.18}$$

Now let us consider the monotone system

$$
\begin{cases}
\dfrac{\partial^2\eta_1}{\partial s^{*2}}+c\dfrac{\partial\eta_1}{\partial s^*}+\eta_1\left(\dfrac{1+\rho_2-z}{1+\rho_2}-\eta_1+z\eta_2\right)=0,\\
\dfrac{\partial^2\eta_2}{\partial s^{*2}}+c\dfrac{\partial\eta_2}{\partial s^*}+\left(\dfrac{1}{1+\rho_2}-\eta_2\right)(b_1\eta_1-\rho_1(1+\rho_2)\eta_2)=0,
\end{cases} \tag{2.19}
$$

For $s^*\in(-\infty,\infty)$, the problem reduces to

$$
\begin{cases}
\dfrac{\partial^2\bar{\eta}_1}{\partial s^{*2}}+c\dfrac{\partial\bar{\eta}_1}{\partial s^*}+\bar{\eta}_1\left(\dfrac{1+\rho_2-z}{1+\rho_2}-\bar{\eta}_1+z\eta_2\right)\leq 0,\\
\dfrac{\partial^2\bar{\eta}_2}{\partial s^{*2}}+c\dfrac{\partial\bar{\eta}_2}{\partial s^*}+\left(\dfrac{1}{1+\rho_2}-\bar{\eta}_2\right)(b_1\eta_1-\rho_1(1+\rho_2)\bar{\eta}_2)\leq 0,
\end{cases} \tag{2.20}
$$

For $s^*\in(-\infty,\infty)$, all$0\leq\eta_2\leq\bar{\eta}_2(s^*)$, $0\leq\eta_1\leq\bar{\eta}_1(s^*)$. In the region , $0\leq\eta_1\leq 1$ and $0\leq\eta_2\leq\frac{1}{1+\rho_2}$, the system (2.19) is monotone. When $\eta_1=\bar{\eta}_1(s^*)$is the first equation and $\eta_2=\bar{\eta}_2(s^*)$ is the second equation for all $s^*\in(-\infty,\infty)$, particularly (2.20) is absolutely true. Here let

$$f_1(\eta_1,\eta_2)=\eta_1\left(\frac{1+\rho_2-z}{1+\rho_2}-\eta_1+z\eta_2\right)$$

$$f_2(\eta_1,\eta_2)=\left(\frac{1}{1+\rho_2}-\eta_2\right)(b_1\eta_1-\rho_1(1+\rho_2)\eta_2)$$

Hence $f_i\left(S,\frac{Sb_1}{2\rho_1(1+\rho_2)}\right)>0$ for $i=1,2$ and $S>0$ is sufficiently small. Let V_1 be a class of vector valued functions $\vec{\eta}(s^*)\in C^2(-\infty,\infty)$ is monotonically decreasing and satisfying $\lim_{s^*\to\pm\infty}\vec{\eta}(s^*)=\overrightarrow{M_\pm}$ with $\vec{f}=(f_1,f_2),\overrightarrow{M_+}=(0,0)$ and $\overrightarrow{M_-}=\left(1,\frac{1}{1+\rho_2}\right)$.

We have $c \geq M^*$ for the existence of the function $(\bar{\eta}_1(s^*), \bar{\eta}_2(s^*))$ satisfying (2.20) where

$$M^* = max\left\{Inf_{\vec{\eta}\in V_1}\left\{Sup_{s^*,V_1}\frac{\frac{d^2\eta_{V_1}}{ds^{*2}}+f_{V_1}(\vec{\eta}(s^*))}{\frac{d\eta_{V_1}}{ds^*}}\right\},0\right\}.$$

Since the function $\vec{f}$ can be reduced at the top left corner of the rectangle $[\overrightarrow{M_+},\ \overrightarrow{M_-}] = [0,1] \times [0, \frac{1}{1+\rho_2}]$, then the system (2.14) has a solution which is a function denoted defined by

$$(\hat{u}_1(s^*), \hat{u}_2(s^*)) \coloneqq (\hat{\eta}_1(s^*), \hat{\eta}_2(s^*))$$

After setting $M(s^*) = \hat{u}_1(s^*)$ and $N(s^*) = \frac{1}{1+\rho_2} - \hat{u}_2(s^*)$ for $s^* \in (-\infty, \infty)$ as in (2.13), then $(u(t,x), v(t,x))$ as defined in (2.11) is a travelling wave solution of system (1.2) for $x \in (-\infty, \infty)$, $t > 0$, satisfying (2.12) as described in the statement of theorem 2.2.

3. Examples and Applications

3.1 Effects of competitive constant coefficients

We consider the following system with the boundary and initial conditions as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{d}\dfrac{\partial^2 u}{\partial x^2} + u(1 - 2u - v) \\ \dfrac{\partial v}{\partial t} = \mathrm{d}\dfrac{\partial^2 v}{\partial x^2} + v(1 - 3u - \dfrac{19}{10}v) \end{cases} \tag{3.1}$$

Where,$a = 2, \gamma = 1, b = 3, \delta = \frac{19}{10}$, with the domain $\sigma = (0,1)$ and homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0$$

Here $[A1]$ to $[A2]$ are readily satisfied and $[A3]$ is always true for $b \leq 2a$. Now, our goal is to solve these equations numerically by using Implicit Finite Difference Method such as Crank-Nicolson method [22]. First we discretize the above equations (see Appendix 1.) and after that constructing the algorithm in FORTRAN languages by code block software; we can have the solutions which are graphically presented by following figures.

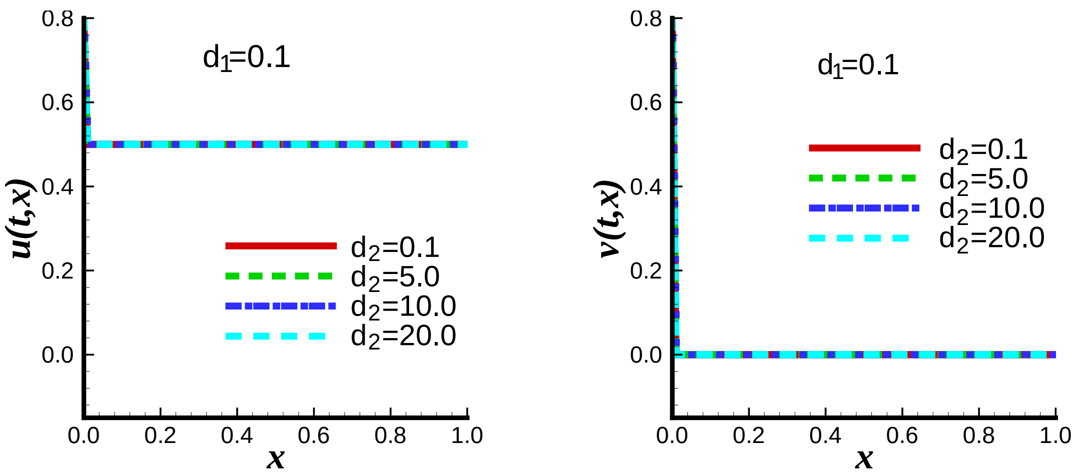

Figure-3.1: The illustration of the solutions $u(t,x)$ (left) and $v(t,x)$ (right) according to system of system of equation (3.1) for different diffusion coefficients at same time $t = 200$.

The behavior of diffusion coefficients and domain is reported in Figure-3.1. We take different values of diffusion coefficients at time $t = 200$ over domain x. Considering one diffusion coefficient is fixed such as $d_1 = 0.1$ and another one is replaced by different values such as $d_2 = 0.1, 5, 10, 20$ and we can observe from the above figure, the solutions are coinciding separately. That means the solutions $u(t,x)$ and $v(t,x)$ do not depend on different values of diffusion coefficient.

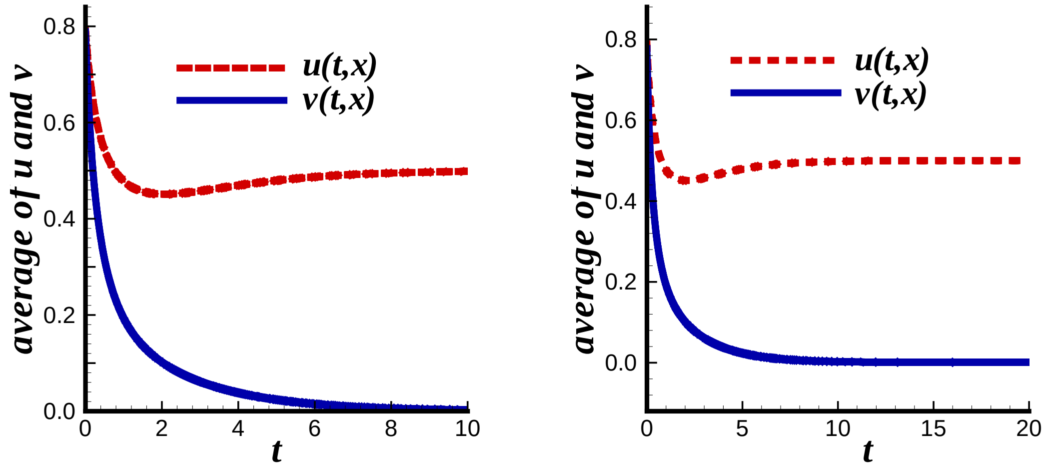

Figure-3.2: The graphical representation of average solutions at different times $t = 10,\ 20$ and taking same diffusion coefficient $d_1 = d_2 = 0.1$.

The above figures (Figure-3.2) represent us the nature of the average solutions verses time. By taking same diffusion coefficient $d_1 = d_2 = 0.1$ at different times $t = 10$ (left) and $t = 20$ (right), we observe that the values of these solutions are changed or decreased over different times. That is, average solutions are varies on time.

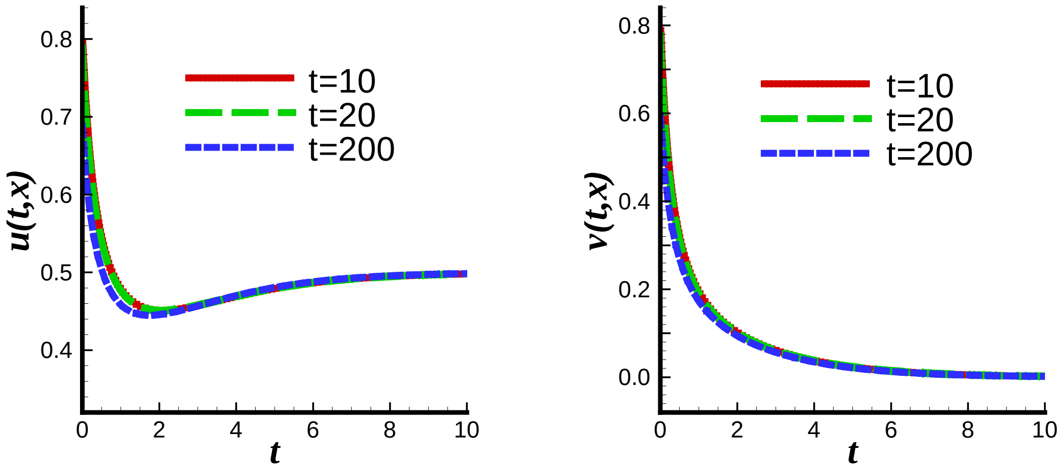

Figure-3.3: Comparison at different times $t = 10, t = 20, t = 200$ and corresponding average solutions of $u(t,x)$ (left) and $v(t,x)$ (right) for same diffusion coefficient $d = 0.1$ according to (3.1).

From Figure-3.2, we have known that the average solutions are time dependent. Those descriptions of Figure-3.2 are same as for the above figure-3.3. Here we represent the multiple plot of the average solutions at different times $t = 10$ (solid), $t = 20$ (longdashed) and $t = 200$ (dashed). So, generally we can say that solutions of the system (3.1) are diffusion coefficient independent but obviously depend on time.

3.2 Effects of spatially distribute competition coefficients

Let us consider a generalized form of (1.2), when competition coefficients are spatially distributed:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{d}\dfrac{\partial^2 u}{\partial x^2} + u(k(x) - a(x)u - \gamma(x)v) \\ \dfrac{\partial v}{\partial t} = \mathrm{d}\dfrac{\partial^2 v}{\partial x^2} + v(k(x) - b(x)u - \delta(x)v) \end{cases} \tag{3.2}$$

where $k(x)$ is the carrying capacity and $a(x),\ \gamma(x),\ b(x),\ \delta(x),\ k(x)$ are all functions of x and positive. Our next step is to establish some results using the equation (3.2). Rewrite the equations as

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{d}\dfrac{\partial^2 u}{\partial x^2} + u(1 - (1.1 + sin(\pi x))u - (2.0 + cos(\pi x))v) \\ \dfrac{\partial v}{\partial t} = \mathrm{d}\dfrac{\partial^2 v}{\partial x^2} + v(1 - (1.2 + sin(\pi x))u - (2.2 + cos(\pi x))v) \end{cases} \tag{3.3}$$

with $\sigma = (0,1)$, $k(x) = 1$, $a(x) = 1.1 + sin(\pi x) < b(x) = 1.2 + sin(\pi x)$, and $\gamma(x) = 2.0 + cos(\pi x) < \delta(x) = 2.2 + cos(\pi x)$.

Using same numerical strategy in section 3.1, we produce the following results:

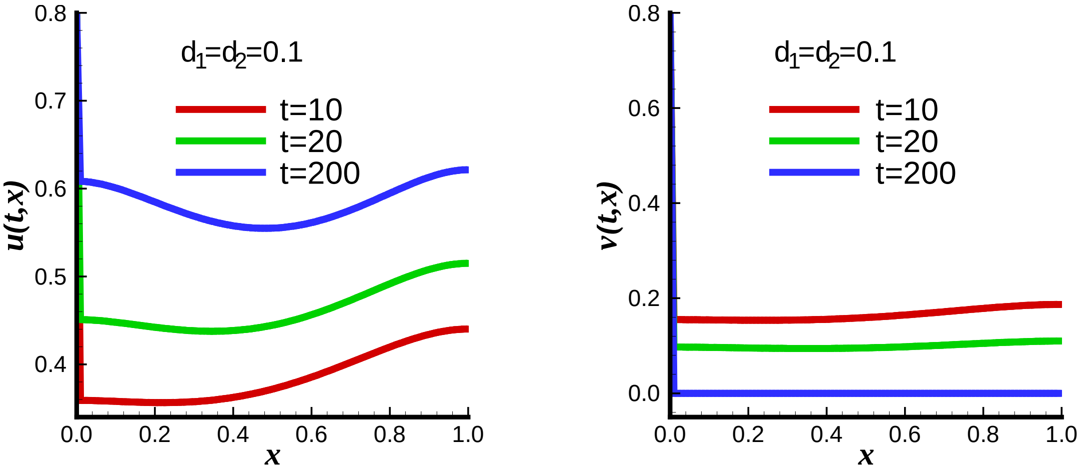

Figure-3.4: Solutions of (3.3) for same diffusion coefficient $d_1 = d_2 = 0.1$, initial value $u_0 = 0.8, v_0 = 0.8$ at increasing times $t = 10, t = 20$ and $t = 200$ over the domain.

We can see that $u(t,x)$ (left) and $v(t,x)$ (right) are the solutions of the system (3.3) which shows two species are decreasing over the domain for same diffusion coefficient at different times. It also satisfies the third hypothesis $b(x) < 2a(x)$. Now we consider (3.3) with different diffusion coefficient as $d_1 = 0.1$ and $d_2 = 0.5, 1.0, 5.0, 10.0, 20.0$ at same time. It is observed (Figure-3.5) that all the solutions for different diffusion coefficients coincide. That is, solutions do not depend on diffusions coefficients and they decrease. So, variation of diffusion coefficient does not effect on two species.

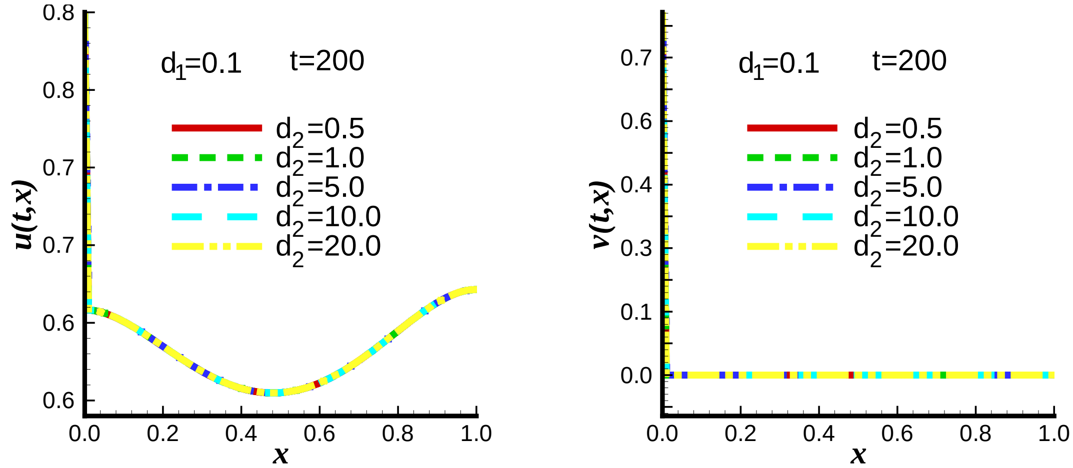

Figure-3.5: Solutions of (3.3) for d_1= 0.1, $\sigma = (0,1)$, $u_0 = v_0 = 0.8$ for various d_2= 0.5, 1.0, 5.0, 10.0 and 20.0 at time $t = 200$.

We can investigate the solutions for increasing of times from 10 to 200 using same diffusion coefficient of (3.3) which is shown in the following Figure-3.6. Average solutions are indicating by $u(t,x)$ (left) and $v(t,x)$ (right). When time varies, we observe that solutions are decreasing with $u_0 = v_0 = 0.8$ for same diffusion coefficient 0.1 over the domain.

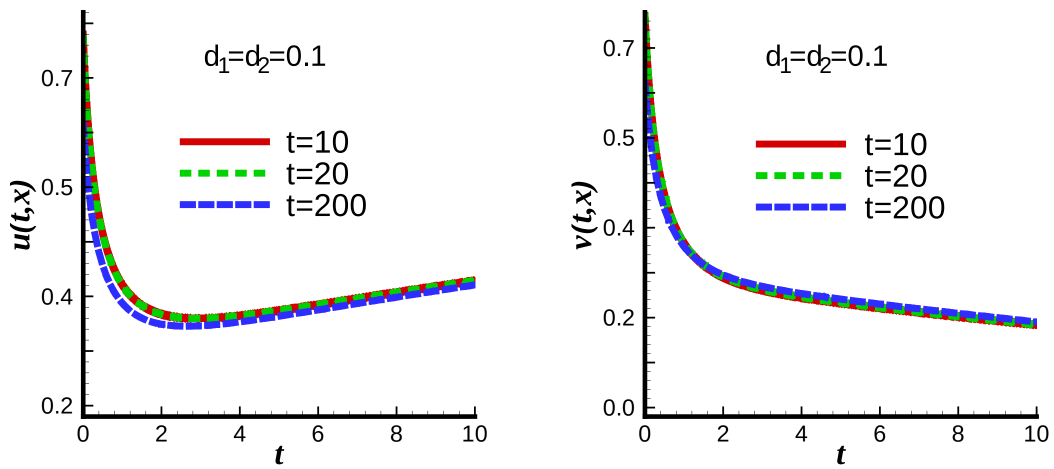

Figure-3.6: Average solutions of (3.3) for same diffusion coefficient 0.1 at different times $t = 10, t = 20$ and $t = 200$ respectively.

The following figure-3.7 establishes for larger $k(x) = 2.5 + \cos(\pi x)$ from all other parameters such that $a(x) = 1.1 + sin(\pi x)$, $b(x) = 1.2 + sin(\pi x)$, $\gamma(x) = 2.0 + cos(\pi x)$ and $\delta(x) = 2.2 + cos(\pi x)$ for same diffusion coefficient d_1= d_2= 0.5.

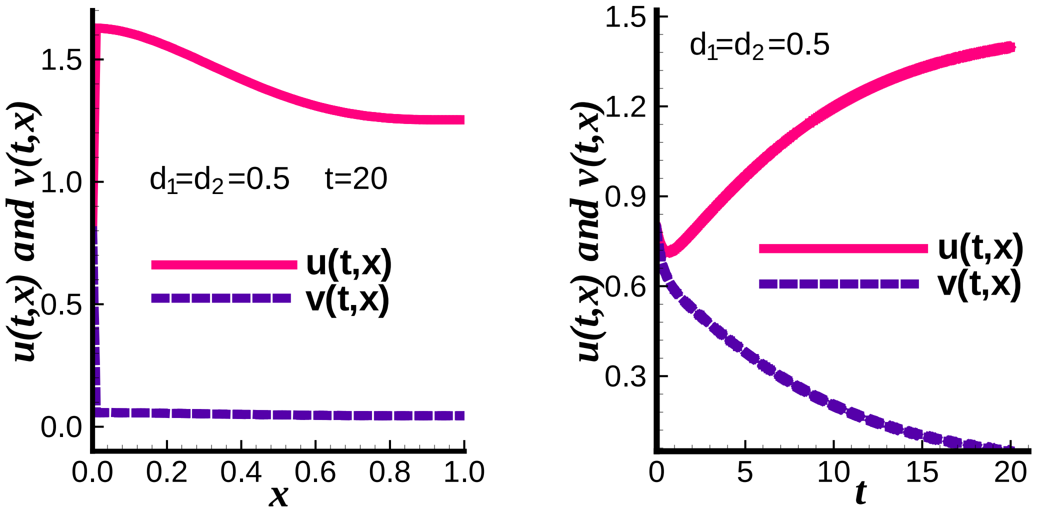

Figure-3.7: Solutions (left) and average solutions (right) of (3.3) using carrying capacity $k(x) = 2.5 + \cos(\pi x)$.

The above figures depict, if the carrying capacity is larger than all other parameters, then left illustration shows us $u(t,x)$ increases and $v(t,x)$ decreases. Similarly, right illustration of same figure shows us average solutions of u increases and v decreases at time $t = 20$ for same diffusion coefficient d_1=d_2=0.5 and same initial value 0.8. Next to test the difference between small and larger carrying capacity, we consider the carrying capacity $k(x) = 0.5 + \cos(\pi x)$ which is less than all parameters discussed earlier.

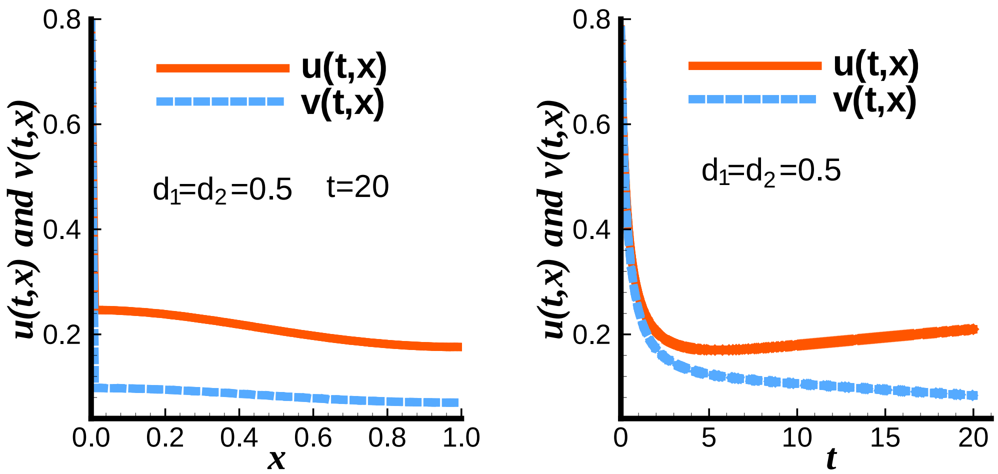

Figure-3.8: Solutions (left) and average solutions (right) of equation (3.3) using smaller carrying capacity $k(x) = 0.5 + \cos(\pi x)$.

Here $k(x) = 0.5 + \cos(\pi x)$ is small than parameters $a(x)$, $b(x)$, $\gamma(x)$, $\delta(x)$ and also from the carrying capacity used in figure-3.7. By considering same for diffusion coefficients, we get $u(t,x)$ and $v(t,x)$ are increases and decreases respectively in figure-3.7; on the other hand, both the solutions of species are decreasing at certain time over the domain in figure-3.8. It's obvious in nature that with smaller carrying capacity in the competitive species, there is a formidable chance for both the species step toward extinction.

4. Conclusion

We introduce an appropriate transformation of the travelling wave solutions using three hypotheses and the realistic significances of these hypotheses and models are presented us the interconnection between growth, competition, diffusion coefficient etc. of two species and for these reasons we observe that the travelling wave can exist. In this paper, we investigate the characteristic of competitive reaction-diffusion equations using two species. The selected equation based on KPP equation does not depend on the changes of diffusion coefficient over the domain and two species are decreasing at certain and increasing of time. We also construct two different form of governing equation and observe that one form shows us all species are becoming less for different times and these solutions are diffusion coefficient independent. One species is increasing and another one is decreasing over the domain if we take the larger carrying capacity for same or different diffusion coefficients and for small carrying capacity two are decreasing. The numerical results obtained by implicit finite difference method using Neumann boundary conditions.

Conflict of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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Appendix 1
Implicit Finite Difference Method

To discretized the equation (3.1), at first, we have to use the Taylor series in t to form the difference quotient

$$\frac{\partial u}{\partial t}(t_j, x_i) = \frac{u(t_j + \Delta t, x_i) - u(t_j, x_i)}{\Delta t} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j, x_i) \tag{1}$$

for some $\tau_j \in (t_j, t_{j+1})$ and $\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j, x_i)$ is the error term. Now using central-difference method to form the difference quotient by Taylor series in x, we have

$$\frac{\partial^2 u}{\partial x^2}(t_j, x_i) = \left[\frac{u(t_j, x_i + \Delta x) - 2u(t_j, x_i) + u(t_j, x_i - \Delta x)}{(\Delta x)^2}\right] + \left[\frac{u(t_j + \Delta t, x_i + \Delta x) - 2u(t_j, x_i) + u(t_j + \Delta t, x_i - \Delta x)}{(\Delta x)^2}\right] - \frac{(\Delta x)^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j, \gamma_i) \tag{2}$$

where $\gamma_i \in (x_{i-1}, x_{i+1})$ and (t_j, x_i) is the interior gridpoint and $\frac{(\Delta x)^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j, \gamma_i)$ is the error term.

Suppose that, $\Delta x = h,\ \Delta t = K$. Then (1) becomes

$$\frac{\partial u}{\partial t}(t_j, x_i) = \frac{u(t_j + K, x_i) - u(t_j, x_i)}{K} - \frac{K}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j, x_i) \tag{3}$$

and (2) becomes

$$\frac{\partial^2 u}{\partial x^2}(t_j, x_i) = \left[\frac{u(t_j, x_i + h) - 2u(t_j, x_i) + u(t_j, x_i - h)}{h^2}\right] + \left[\frac{u(t_j + K, x_i + h) - 2u(t_j, x_i) + u(t_j + K, x_i - h)}{h^2}\right] - \frac{h^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j, \gamma_i) \tag{4}$$

Putting (3) and (4) in first part of the system of equation (1.27) and ignoring the local truncation error of order $O(K^2 + h^2)$ consisting of $-\frac{K}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j, x_i)$ and $-\frac{h^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j, \gamma_i)$ and next discretizing the first part of system of equation (3.1) by Crank-Nicolson scheme, we have

$$\frac{u_i^{j+1} - u_i^j}{K} = \frac{\mathrm{d}}{2}\left[\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2}\right] + u_i^j\left(1 - 2u_i^j - v_i^j\right)$$

$$\Rightarrow 2u_i^{j+1} - 2u_i^j = \frac{\mathrm{d}K}{h^2}\left[u_{i+1}^j - 2u_i^j + u_{i-1}^j + u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}\right] + 2Ku_i^j\left(1 - 2u_i^j - v_i^j\right) \tag{5}$$

Now let us consider a new parameter R_u such that $R_u = \frac{\mathrm{d}K}{h^2}$ and then from (5), we obtain the discretized equation in the following form

$$-R_u u_{i-1}^{j+1} + (2 + 2R_u)u_i^{j+1} - R_u u_{i+1}^{j+1} = R_u\left[u_{i+1}^j + u_{i-1}^j\right] + (2 - 2R_u)u_i^j + 2Ku_i^j\left(1 - 2u_i^j - v_i^j\right) \tag{6}$$

Since equation (3.1) has Neumann boundary conditions. Using central difference formula into this boundary condition, we get

$$\frac{\partial u}{\partial x} = 0 \Rightarrow u_{i+1}^j = u_{i-1}^j$$

From the equation (6),

$$-R_u u_{i-1}^{j+1} + (2 + 2R_u)u_i^{j+1} - R_u u_{i+1}^{j+1} = 2R_u u_{i-1}^j + (2 - 2R_u)u_i^j + 2Ku_i^j\left(1 - 2u_i^j - v_i^j\right) \tag{7}$$

Now using similar procedure for second part of the system of equation (3.1), using Neumann boundary conditions we obtain

$$-R_v v_{i-1}^{j+1} + (2 + 2R_v)v_i^{j+1} - R_v v_{i+1}^{j+1} = 2R_v v_{i-1}^j + (2 - 2R_v)v_i^j + 2Kv_i^j\left(1 - 3u_i^j - \frac{19}{10}v_i^j\right) \tag{8}$$

Equations (7) and (8) are the discretized version of the equations in (3.1) to be used for further numerical implementations.