Original Research Article

A NEW SMOOTHING METHOD FOR TIME SERIES DATA IN THE PRESENCE OF AUTOCORRELATED ERROR

Abstract

Spline Smoothing is used to filter out noise or disturbance in an observation, its performance depends on the choice of smoothing parameters. There are many methods of estimating smoothing parameters; most popular among them are; Generalized Maximum Likelihood (GML), Generalized Cross-Validation (GCV), and Unbiased Risk (UBR), this methods tend to overfit smoothing parameters in the presence of autocorrelation error. A new Spline Smoothing estimation method is proposed and compare with three existing methods in order to eliminate the problem of over fitting associated with the presence of Autocorrelation in the error term. It is demonstrated through a simulation study performed by using a program written in R based on the predictive Mean Score Error criteria. The result indicated that the predictive mean square error (PMSE) of the four smoothing methods decreases as the smoothing parameters increases and decreases as the sample sizes increases. This study discovered that the proposed smoothing method is the best for time series observations with Autocorrelated error because it doesn't over fit and works well for large sample sizes. This study will help researchers overcome the problem of over fitting associated with applying Smoothing spline method time series observation.

Key words: Autocorrelation, Generalized Maximum Likelihood, Generalized Cross-Validation, Splines Smoothing, Time series and Unbiased Risks.

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1.0 Introduction

In non-parametric regression, smoothing is of great importance because it is used to filter out noise or disturbance in an observation; it is commonly used to estimate the mean function in a nonparametric regression model, it is also the most popular methods used for prediction in non-parametric regression models. The general spline smoothing model is given as:

$$y_i = f(X_i) + \varepsilon_i \tag{1}$$

Where; Y_i is the observation values of the response variable *y*, f is an unknown smoothing function, X_i is the observation values of the predictor variable *x* and ε_i is normally distributed random errors with zero mean and constant variance.

The main objective of this research is to estimate f (.) when $x_i = t_i$ but not necessarily equally spaced, with $t_1 < . . . < t_n$ (time) and ε_i is assumed to be correlated. Diggle and Hutchinson (1989). Therefore, this research shall consider the spline smoothing for non-parametric estimation of a regression function in a time-series context with the model;

$$y_i = f(t_i) + \varepsilon_{ti} \tag{2}$$

Where; Y_i = observation values of the response variable *y*, f = an unknown smoothing function, t_i is the time for i = 1 . . . n, e_{ti} = zero mean autocorrelated stationary process.

Smoothing spline arises as the solution to a nonparametric regression problem having the function f(x) with two continuous derivatives that minimizes the penalized sum of squares

$$S(f) = \sum_{i-1}^{n} \left(y_i - f(x_i)\right)^2 + \lambda \int_0^1 \left(f^{II}(x_i)\right)^2 dx \tag{3}$$

Where; λ denotes a smoothing parameter, the smoothing parameter λ represents the rate of exchange between residual error and roughness of the curve f, the parameter λ controls the trade-off between goodness-of-fit and the smoothness of the estimate. If λ is 0 then $f^{II}(x)$simply

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interpolates the data, if λ is very large, then $f^{II}(x)$ will be selected so that $f^{II}(x)$ is everywhere 0, which implies a globally linear least-squares fit to all data. Wahba et.al (1995). There is the need to tackle the problem associated with estimating the best spline smoothing methods for time series observation in the presence of correlational error. Diggle and Hutchinson (1989).

There are vast literatures on Spline Smoothing modeling of time series data in the presence autocorrelated error; Diggle and Hutchinson (1989), Yuedong (1998), Yuedong et. al. (2000), Opsomer, Yuedong and Yang (2001), Wahba et. al. (1995), Carew et. al (2002), Hall and Keilegom (2003), Francisco-Fernandez and Opsomer (2005), Hart and Lee (2005), Krivobokova and Kauermann (2007), Shen (2008), Kim, Park, Moon, and Kim (2009), Morton et.al. (2009), Wang, Meyer and Opsomer (2013), Adams, Ipinyomi and Yahaya (2017) Chen and Huang (2011).

The aim of this study is to propose a new smoothing method (PSM) by modifying two of the existing spline smoothing methods (i.e. the Generalized Cross Validation (GCV) and Unbiased Risk (UBR)) and compare it with three existing estimation methods namely; Generalized Maximum Likelihood (GML), Generalized Cross Validation (GCV) and Unbiased Risk (UBR) for time series observations in the presence of Autocorrelated error in order to eliminate the problem of over fitting associated with the presence of Autocorrelation in the error term. Section one presents the introduction to the study. Section two reviews the existing spline smoothing method and the proposed selection method, Section 3 presents the Monte Carlo simulation study, equation used for generating values in simulation and experimental design and data generation, section four compares the four methods via a simulation study, and finally, the result discussion and conclusion were presented in last section.

2.0: Parameter Estimation

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2.1: Generalized Cross-Validation (GCV) with Autocorrelation Structure

The term generalized cross-validation (GCV) was coined by Wahba (1977) and was applied by Hastie and Tibshirani, (1999), Aydin and Memmedli (2011). Diggle and Hutchinson (1989) and Wahba (1983) introduced the Autocorrelation structure in GCV, this is given as;

$$GCV(\lambda) = \frac{(y - \hat{g})^T V^{-1}(y - g)}{[trace(I - S_\lambda)]^2} \quad (4)$$

Where; (Sλ) = the ith diagonal element of smoother matrix, W = V-1 = [vij], the correlation structure, y = (yl, . . . ,yn)^T and f = (f(t1) ,. . . ,f(tn))T

2.2: Generalized Maximum Likelihood (GML) Estimation Method with Autocorrelation Structure

The Generalized Maximum Likelihood (GML) estimation method is an empirical Bayes type criteria developed by Wecker and Ansley (1983) and Wahba (1985) while Yuedong (1998) proposed the GML methods for correlated observations with one smoothing parameter given by;

$$GML(\lambda) = \frac{\lambda^{'}W(I - S_\lambda)}{[det^{+}W(I - S_\lambda)]^{\frac{1}{n-m}}} \quad (5)$$

Where; $det^{+}(I – S_\lambda)$ is the product of the n – m nonzero eigenvalues of $(I – S_\lambda)$, λ is Smoothing parameter, w is the correlation structure, S_λ is the diagonal element of smoother matrix, n is $n_1 + n_2$, Pairs of measurement/observations and m is number of zero eigenvalues.

2.3: Unbiased Risk (UBR) Estimation Method with Autocorrelation Structure

The UBR method or CP criterion was suggested by C.L. Mallows' (1973) and had been applied successfully by Craven and Wahba (1979), Gu (1992); Wahba, Wang, Gu (1995); Klein, and Klein (1995) and (Wang, 1998), but Yuedong (1998) provides UBR method with a known

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Autocorrelation structure for selecting smoothing parameters for spline estimates with non-Gaussian data. It is written as;

$$UBR(\lambda) = \frac{\frac{1}{n}\left\| W^{\frac{k}{2}}(I - S_{\lambda})\lambda \right\|^{2}}{\left[\frac{1}{n}trace\left(W^{k-1}(I - S_{\lambda})\right)\right]^{2}} \quad k = 0,1,2 \qquad (6)$$

Where; n is pairs of measurement/observations {xi,yi}, W is the correlation structure, λ is Smoothing parameters, S_{λ} = is the ith diagonal element of smoother matrix.

2.4 Proposed Smoothing Method (PSM) with Autocorrelation Structure

A Spline Smoothing model is defined as

$$y_i = f(x_i) + \varepsilon_i \qquad (7)$$

Where; Y is the variable of interest, X is vector of the predictor variable, f is Regression function and ε is error term. There is a number of option to consider when model (7) above is to be used in order to take care of non-linearity, they include; Data transformation, additive terms e.g. quadratic or cubic term and Spline smoothing. This study is interested in Spline Smoothing because it considers non-linearity based on the regression curve by introducing a kink or bends in the $\hat{Y}$, this kinks is produced by hinge function and the point of bend on the fit is called knots. Spline Smoothing is simpler to plot and easy to interpret when the relationship is between y and (x, x^2). The number of knots is denoted by λ , model (7) above can also take the form;

$$y_i = f_1(x_1) + f_2\left(x_1^{\,2}\right) + \varepsilon_i \text{ (Polynomial regression)} \qquad (8)$$

The main purpose of the conversional regression analysis is to minimize the residual Sum of Square (RSS), if RSS is used to compare regression models, the largest model would be chosen even though its not the best model. It is worthy to note that in Spline Smoothing, a method of

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selection known as Cross Validation (CV) was proposed by Wahba (1979). In place of RSS in the conventional simple regression analysis, the error term is therefore defined as;

$$\varepsilon = Y_i - \hat{Y}_i$$

$$RSS = \sum_{i=1}^{n} e_i^2 = \sum_{i-1}^{n} \left(Y_i - \hat{Y}_i\right)^2 \qquad (9)$$

Recall that;

$Y_i = f(x_i)$, for the observed and $\hat{Y}_i = f_\lambda(x_i)$ for the fitted value when a number of knots are introduced

Then;

$$Var = \tfrac{1}{n} \sum (Y_i - \hat{Y}_i)^2$$

Cross Validation method is defined in terms of variance, thus;

$$Var_{cv} = \tfrac{1}{n} \sum \left(\frac{y_{i\,-}\; f_\lambda(x_i)}{1 - ((S_\lambda)_{ii})} \right)^2 \qquad (10)$$

The main of this proposed selection method was to minimize the variance as much as possible in order to have a precise estimate of the parameter of interest,

Where;

S_λ is smoothen matrix, it is the squared diagonal matrix and its diagonal entries are denoted by

$$S_\lambda = x\left(x^I x + n\lambda \mathrm{I}\right)^{-1} x^T$$

And;

$$f_\lambda(x_i) = \begin{pmatrix} f_\lambda(x_1) \\ . \\ . \\ f_\lambda(x_n) \end{pmatrix} = S_\lambda y \qquad (11)$$

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Recall that;

$$diag[I - (S_\lambda)_{ii}] = [I - (S_\lambda)] \tag{12}$$

Where; I is an identity matrix and $diag[I - (S_\lambda)_{ii}]$is a squared matrix with diagonal entries $[1 - (s_\lambda)_{ii}]$.

Remember that $Y_i = f(x_i)$ and $\hat{Y}_i = f_\lambda(x_i) = S_\lambda(y)$, CV selection method is therefore given as;

$$CV = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{y_i - f_\lambda(x_i)}{1 - (S_\lambda)_{ii}}\right\}^2 \tag{13}$$

$$\begin{aligned} y_i - f_\lambda(x_i) &= y_i - S_\lambda y_i \\ &= (1 - S_\lambda) y_i \\ &= (I - S_\lambda) y_i \end{aligned} \tag{14}$$

Since the Euclidean distance makes use of the summation and trace of a matrix, a new spline smoothing selection method was proposed by Wahba (1979) called Generalized Cross Validation (GCV) defined as;

$$Var_{gcv} = \frac{\frac{1}{n}\left\|(I - S_\lambda) y\right\|^2}{\left[\frac{1}{n} Trace\ (1 - S_\lambda)\right]} \tag{15}$$

GCV uses additives operation by considering Euclidean distance and trace of a matrix

$$Trace\ (1 - S_\lambda) = \sum_{i=1}^{n}(1 - (S_\lambda) ii) \tag{16}$$

Using Multiplicative operations, another Spline Smoothing selection method was proposed by Wahba (1976) called Generalized Maximum Likelihood (GML) defined as

$$Var_{gml} = \frac{y^I (1 - S_\lambda) y}{\det (1 - S_\lambda)^{\frac{1}{n-m}}} \tag{17}$$

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Where; M is number of zero eigenvalues, n – m = non-zero eigenvalues of $(1 - S_\lambda)$ for correlated error terms such as $\gamma(\varepsilon) = \sigma^2 w^{-1}$. Where; W = the correlation structure

GML becomes modified as

$$Var_{GCV} = \frac{Y^{I} W (I - S_\lambda) Y}{\det [W (I - S_\lambda)]^{\frac{1}{n-m}}} \tag{18}$$

To extend GCV, Unbiased Risk method was proposed with correlation structure;

$$UBR(\lambda) = \frac{\frac{1}{n}\left\| W^{\frac{k}{2}} (I - S_\lambda)\lambda \right\|^2}{\left[\frac{1}{n} trace\left(W^{k-1}(I - S_\lambda)\right)\right]^2} \tag{19}$$

And from equation (15), GCV method for estimating spline smoothing (λ) in the presence of autocorrelation structure was given by,

$$GCV(\lambda) = \frac{(y - \hat{g})^T V^{-1} (y - g)}{[tr\{1 - S_\lambda\}]^2} \tag{20}$$

A new Spline Smoothing estimation method is proposed to allow for the presence of correlation structure when UBR (19) and GCV (20) methods were modified when k is set as 1, as seen below;

Combining equations (19) and (20) and substituting k = 1, $PSM(\lambda) = \frac{GCV(\lambda)}{UBR(\lambda)}\ when\ k = 1$

$$PSM(\lambda) = \frac{\dfrac{\left(y - \hat{f}\right)^T V^{-1} (y - f)}{[trace(I - S_\lambda)]^2}}{\dfrac{\frac{1}{n}\left\| W^{\frac{K}{2}} (I - S_\lambda) \right\|^2}{\left[\frac{1}{n} trace\{W^{k-1}(I - S_\lambda)\}\right]^2}} \tag{21}$$

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Now the behavior of the minimize λ in GCV and UBR methods under the substituted value of k = 1 yield.

$$PSM\ (\lambda) = \frac{\dfrac{\left(y - \hat{f}\right)^{T} V^{-1}(y - f)}{[trace(I - S_{\lambda})]^{2}}}{\dfrac{\frac{1}{n}\left\| W^{\frac{1}{2}}(I - S_{\lambda})\right\|^{2}}{\left[\frac{1}{n} trace\{W^{0}(I - S_{\lambda})\}\right]^{2}}} \tag{22}$$

Factorizing equation (22)

$$PSM\ (\lambda) = \frac{\dfrac{\left(y - \hat{f}\right)^{T} V^{-1}(y - f)}{[trace(I - S_{\lambda})]^{2}}}{\dfrac{\frac{1}{n}\left\| W^{\frac{1}{2}}(I - S_{\lambda})\right\|^{2}}{\frac{1}{n}^{2}\,[trace(I - S_{\lambda})]^{2}}} \tag{23}$$

The Proposed Smoothing Method (PSM) we derived is the minimizer of V (λ) given by

$$PSM\ (\lambda) = \frac{\left(y - \hat{f}\right)^{T} V^{-1}(y - f)}{[trace(I - S_{\lambda})]^{2}} \times \frac{\frac{1}{n}^{2}\,[trace(I - S_{\lambda})]^{2}}{\frac{1}{n}\left\| W^{\frac{1}{2}}(I - S_{\lambda})\right\|^{2}} \tag{24}$$

$$PSM\ (\lambda) = \frac{\left(\frac{1}{n}\right)^{2}\left(y - \hat{f}\right)^{T} V^{-1}(y - f)}{\frac{1}{n}\left\| W^{\frac{1}{2}}(I - S_{\lambda})\right\|^{2}} \tag{25}$$

$$PSM\ (\lambda) = \frac{\frac{1}{n}\left(y - \hat{f}\right)^{T} V^{-1}(y - f)}{\left\| W^{\frac{1}{2}}(I - S_{\lambda})\right\|^{2}} \tag{26}$$

Where; n is Pairs of observations, λ is the Smoothing parameter, W and $\left(y - \hat{f}\right)^{T} V^{-1}(y - f)$ are the Autocorrelation structures and S_{λ} is the diagonal element of smoother matrix.

3.0 Material and method

3.1 Equation used for generating values in simulation

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A simulation study is conducted to evaluate and compare the performance of the four estimation methods presented in previous sections. The model considered is

$$y_t = \frac{Sin\pi_i}{t} + \varepsilon_t \, i = 1, 2, \ldots n, t = \in[0, 100] \qquad (27)$$

Where; ε's are generated by a first-order autoregressive process AR (1) with mean 0, standard deviations 0.8 and 1.0 and first-order correlations (i.e. $\rho = 0.2$, 0.5 and 0.8) and its 95% Bayesian confidence interval. Wahba, (1983) and Diggle, (1989).

3.2 Experimental design and data generation

The experimental plan applied in this research work was designed to have three time series sample Sizes (T) of 20, 60 and 100, three autocorrelation levels, i.e. α = 0.2, 0.5 and 0.8, four smoothing functions were considered i.e. λ = 1, 2, 3 and 4, two standard deviation were considered, i.e. σ = 0.8 and 1.0. The data were generated for 1000 replications for each of the $3 \times 3 \times 4 \times 2 = 72$ combinations of cases n, α , λ, and σ. The criterion used is the PMSE values to evaluate $\hat{f}_\lambda$ computed according to each of the estimation given as;

$$PMSE(\lambda) = E\left[\sum_{i=1}^{n}\left(f(x_i) - \hat{f}(x_i)\right)^2\right] \qquad (28)$$

The Predictive Mean Square Error can be divided into two terms, the first term is the sum of square biases of the fitted values while the second is the sum of variances of the fitted values. Where; $f(x_i)$ is the observed value and $\hat{f}(x_i)$ = fitted/predicted/estimated value. Aydin, Memmedli and Omay (2013). Simulation study was performed by using a program written in R, it was used to estimate all the model parameters, the criterion, the effect of autocorrelation on the estimated parameters and the performances of the four estimation methods i.e. Generalized

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Maximum Likelihood (GML), Generalized Crossed Validation (GCV), Unbiased Risk (UBR) and the Proposed Smoothing Method (PSM).

4.0 Result

In this study, the results of the proposed Spline smoothing estimation method was compared with three existing estimation methods namely; the Generalized Cross-Validation, Generalized Maximum Likelihood and Unbiased Risks, the Predictive mean square errors criterion was used to measure their efficiency.

4.1. Performance of the four smoothing methods based on predictive mean square error criterion when σ = 0.8.

Table one presents the predictive mean square error for the four estimators, three sample sizes, four spline smoothing levels and three correlation error levels at 0.8 sigma level. It was discovered that for GCV and for sample size 20 the predictive mean square error of 4.938284 at $\lambda = 1$, decreases to 2.789043 at $\lambda = 2$ and further decreased to 2.018062 when $\lambda = 4$. The predictive mean square error increases as the level of autocorrelation increases from 4.938284 when $\alpha = 0.2$ to 5.735483 when $\alpha = 0.5$ and to 5.70041 when $\alpha = 0.8$ for smoothing function (λ) = 1 and sample size = 20. It was also discovered that the predictive mean square error decreases as the sample size increases; at n = 20 the PMSE decreased from 4.938284 to 1.353605 at n = 60 and further deceases from 1.353605 to 0.394855 at n = 100 and for smoothing function (λ) = 1.

The predictive mean square error (PMSE) of GML decreases from 3.788134 at $\lambda = 1$, to 3.624478 at $\lambda = 3$ and then decreased to 3.615046 at $\lambda = 4$. At sample size 20 the predictive mean square error is 3.902353, it decreased to 2.328352 as the sample size increased to 60 and further decreased to 2.314015 as the sample size increased to 100. It is noticed that the PMSE of GML increases from 2.638143 to 2.804273 as the autocorrelation error level increases of 0.2 to 0.5, but

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decreases from 2.804273 to 2.625861 as the autocorrelation level increases from 0.5 to 0.8. For all the other increase in autocorrelation error levels, the PMSE increased correspondingly, thus there is efficiency in GML. For the Proposed Smoothing Method (PSM), it was discovered that the predictive mean square error increases as the autocorrelation level increases and decreases as the sample size increases. At sample size 20 the predictive mean square error of 4.208490 at λ = 2 decreases to 4.202272 at λ = 3 and further decreases to 3.615946 when λ = 4. The predictive mean square error of PSM decreases as the sample size increases, for λ = 1 and autocorrelation level of 0.2. PSM decreased from 4.188747 at sample size = 20 to 2.853925 at sample size 60 and further decreased to 2.287803 at sample size 100. The predictive mean square error of PSM increases from 2.853925 to 1.822216 as the autocorrelation error level increases of 0.2 to 0.5 for sample size is 60 and increases from 1.822216 and 1.812007 as the autocorrelation error level increases of 0.5 to 0.8 for sample size is 60. The predictive mean square error for UBR increases as the autocorrelation level increases and decreases as the smoothing levels and sample sizes increase. At sample size 20 the predictive mean square error of 3.777261 at λ = 1, decreases to 3.469432 at λ = 2, decreases to 3.416732 at λ = 3 but increased slightly to 3.98581 when λ = 4. The predictive mean square error of UBR decreases as the sample size increases, for λ = 2 and autocorrelation level of 0.5, UBR decreases from 3.469432 at sample size = 20 to 1.88788 at sample size 60 and further decreased to 1.431244 at sample size 100. The predictive mean square error of UBR increases from 3.416732 to 3.526772 as the autocorrelation error level increases of 0.2 to 0.5 for sample size is 20 and increases from 3.526772 and 3.611808 as the autocorrelation error level increases of 0.5 to 0.8 for sample size the same sample size.

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Table 1: The MSE result of the simulated study for GML, GCV, PSM and UBR in the presence of autocorrelation (α) = 0.3, 0.5 and 0.8 for n = 20, 60 and 100 when standard deviation (σ) = 0.8

		PMSE								
		N = 20			N = 60			N = 100		
Λ	Smoothing Methods	ρ= 0.2	ρ= 0.5	ρ= 0.8	ρ= 0.2	ρ= 0.5	ρ= 0.8	ρ= 0.2	ρ= 0.5	ρ= 0.8

λ = 1	GCV	4.938284	5.735483	5.700411	1.353605	3.179886	5.817303	0.394855	4.190077	4.753061
	GML	3.788134	3.902353	4.557857	2.328352	2.429546	2.625861	2.314015	2.836043	2.438085
	PSM(k=1)	4.188747	1.977449	2.05909	2.853925	1.822216	1.812007	2.287803	1.573442	1.605743
	UBR	3.777261	2.810875	1.449087	2.101405	2.317046	1.118518	1.913073	2.079789	0.841755
λ = 2	GCV	2.789043	3.755684	5.368908	1.123143	1.374032	4.406313	0.341562	2.96876	3.188995
	GML	2.638143	2.804237	1.300494	2.19448	2.018002	1.027948	2.040446	1.334802	0.171129
	PSM(k=1)	4.208498	2.018938	2.105152	2.823294	1.879530	1.778426	2.287803	1.573403	1.200836
	UBR	3.469432	2.506771	1.017353	1.88788	1.616574	1.230349	1.431244	0.220508	1.532589
λ = 3	GCV	3.175146	3.507623	4.218419	2.472227	1.730359	1.456264	0.334902	0.815361	1.992452
	GML	3.624478	3.802802	4.263339	2.094332	2.958588	2.996486	1.990265	2.22264	0.8030926
	PSM(k=1)	4.202272	2.025768	2.112142	1.816911	0.175471	1.765224	1.531958	0.467133	0.124897
	UBR	3.416732	3.526772	3.611808	1.857928	2.525618	2.564013	1.361115	1.866935	3.321139
λ = 4	GCV	2.018062	3.42688	2.169436	1.094332	0.173144	2.74644	0.332736	2.765412	2.928445
	GML	3.615946	2.800514	1.250932	2.175146	1.938749	5.985579	1.973208	1.984518	5.983278
	PSM(k=1)	4.11762	2.028096	2.114477	1.814626	1.701375	1.760514	1.500005	1.430172	1.098286
	UBR	3.398581	3.512612	4.927715	1.857928	1.94582	3.615934	1.337717	1.815722	3.257353

Table two presents the predictive mean square error for the four estimators, three sample sizes, four spline smoothing levels, three correlation error levels and at 1.0 sigma level. It was discovered that for GCV, at α = 0.5 and sample size 20 the predictive mean square error of 2.217985 at λ = 1, decreases to 2.038837 at λ = 2, decreases to 1.975886 at λ = 3 and further decreased to 0.873763 when λ = 4. The predictive mean square error increases as the level of autocorrelation increases from 2.217985 when α = 0.2 to 4.652218 when α = 0.5 and to 5.219997 when α = 0.8 for smoothing function (λ) = 1 and sample size = 20. It was also discovered that for smoothing function (λ) = 2, the predictive mean square error decreases as the sample size increases; at n = 20 the PMSE decreased from 2.038837 to 1.036064 at n = 60 and further deceased to 0.106917 at n = 100.

The predictive mean square error (PMSE) of GML decreases as the smoothing parameter increases. For small sample size and at α = 0.8, the predictive mean square error decreased from 1.460676 at λ = 1 to 1.191663 at λ = 2 then decreases to 1.152826 at λ = 3 and further decreased

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to 1.139958 at λ = 4. The predictive mean square error of GML decreases as the as the sample size increases. At sample size 20 the predictive mean square error is 1.402249, it decreased to 1.285324 as the sample size increased to 60 and further decreased to 0.917754 as the sample size increased to 100. It is noticed that the predictive mean square error of GML increases from 1.344602 to 2.150393 as the autocorrelation error level increases of 0.2 to 0.5, and increases from 2.150393 to 2.723054 as the autocorrelation level increases from 0.5 to 0.8. Thus there is efficiency in GML, but it was observed that predictive mean square error decreased as the autocorrelation error level increases.

For the Proposed Smoothing Method (PSM), it was discovered that the predictive mean square error decreases as the autocorrelation level, smoothing parameter and sample size increases. At sample size 20 the predictive mean square error of 4.188747 at λ = 1 increased to 4.208498 at λ = 2 but decreases to 4.02272 when λ = 3 and further decreases to 4.117621 when λ = 4. The predictive mean square error of PSM decreases as the sample size increases, for λ = 2 and autocorrelation level of 0.2. PSM decreased from 1.706005 at sample size = 20 to 1.337262 at sample size 60 and further decreased to 1.111343 at sample size 100. The predictive mean square error of PSM decreases from 1.9762941 to 1.878994 as the autocorrelation error level increases of 0.2 to 0.5 for sample size is 20 and further decreases from 1.878994 to 1.62727 as the autocorrelation error level increases of 0.5 to 0.8 for sample size is 20.

The predictive mean square error for UBR increases as the autocorrelation level decreases as the smoothing level and sample size increases.

At sample size 20 the predictive mean square error of 3.946115 at λ = 1, decreases to 2.285086 at λ = 2 to 2.166318 at λ = 3 and further decreases to 1.259853 when λ = 4. The predictive mean square error of UBR decreases as the sample size increases, for λ = 4 and autocorrelation level of

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0.8, UBR decreases from 2.549091 at sample size = 20 to 2.412688 at sample size 60 and further decreased to 1.540203 at sample size 100. The predictive mean square error of UBR increases from 2.166318 to 2.202126 as the autocorrelation error level increases of 0.2 to 0.5 for sample size is 20 and increases from 2.202126 to 2.563679 as the autocorrelation error level increases of 0.5 to 0.8 for sample size the same sample size, but it was observed that predictive mean square error decreased as the autocorrelation error level increases.

Table 2: The MSE result of the simulated study for GML, GCV, PSM and UBR in the presence of autocorrelation (α) = 0.3, 0.5 and 0.8 for n = 20, 60 and 100 when standard deviation (σ) = 1.0

		PMSE								
		N = 20			N = 60			N = 100		
Λ	Smoothing Methods	α = 0.2	α = 0.5	α = 0.8	α = 0.2	α = 0.5	α = 0.8	α = 0.2	α = 0.5	α = 0.8
λ = 1	GCV	2.217985	4.652218	5.219991	1.5079261	3.032906	3.355379	0.109678	0.205153	4.068174
	GML	1.402249	2.213838	2.854191	1.285324	2.424851	2.860878	0.917754	1.498209	1.460676
	PSM(k=1)	1.9762941	1.878994	1.62727	1.681525	1.655205	2.622758	1.625184	1.060796	1.814121
	UBR	3.946115	2.170123	2.854018	3.477279	1.895938	1.904192	0.715411	1.410622	1.391461
λ = 2	GCV	2.038837	1.550266	2.357644	1.036064	3.064901	3.686213	0.106917	0.204841	2.641265
	GML	2.353263	2.159928	2.742754	1.61744	1.745815	1.801702	0.916592	1.484834	1.191663
	PSM(k=1)	1.706005	1.883573	1.512748	1.337262	1.815278	1.258637	1.111343	1.555058	0.824054
	UBR	2.285086	2.043898	2.606053	1.686028	1.615925	1.94976	0.715436	0.391479	1.213843
λ = 3	GCV	1.975886	2.465147	2.230474	1.106586	1.865407	1.493562	0.914299	1.204822	1.462472
	GML	1.344602	2.150393	2.723054	2.376657	1.703152	1.747526	0.916174	0.482901	1.152826
	PSM(k=1)	1.691873	1.799777	1.490825	1.289702	1.65212	1.185653	1.188291	1.786081	1.525496
	UBR	2.166318	2.202126	2.563679	1.335866	2.149228	2.283664	0.715459	0.388746	1.832608
λ = 4	GCV	0.873763	1.437364	2.188967	0.106479	2.800442	1.430831	0.956241	0.204817	1.404276
	GML	1.341634	2.147087	2.716225	1.296255	2.050446	1.895078	0.916018	0.482256	1.139858
	PSM(k=1)	1.686857	1.794844	1.483121	1.2739570	1.659382	1.159813	1.104291	1.454671	1.259721
	UBR	1.259853	2.014616	2.549091	1.221922	1.578077	2.412688	0.715468	0.387835	1.540203

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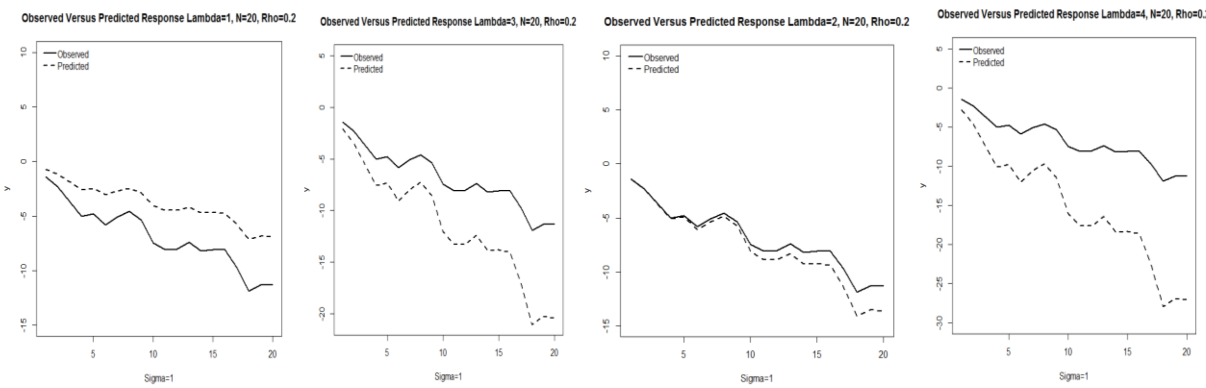
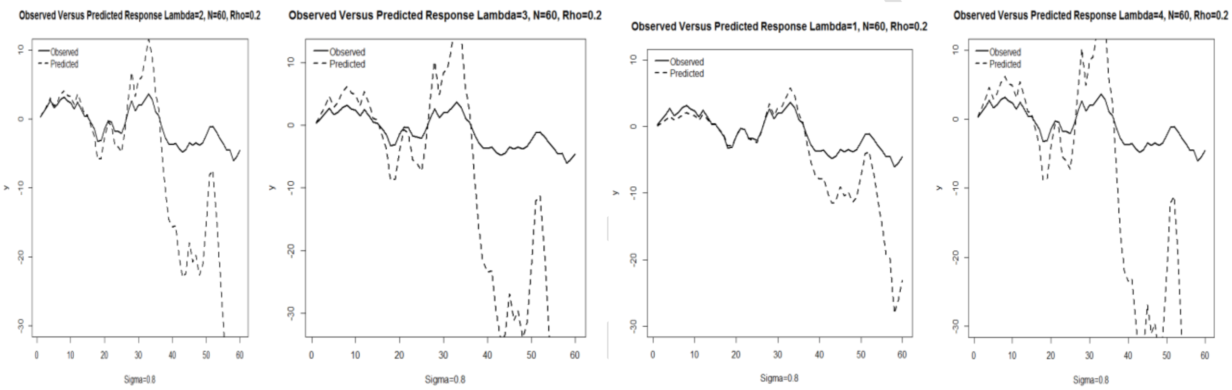


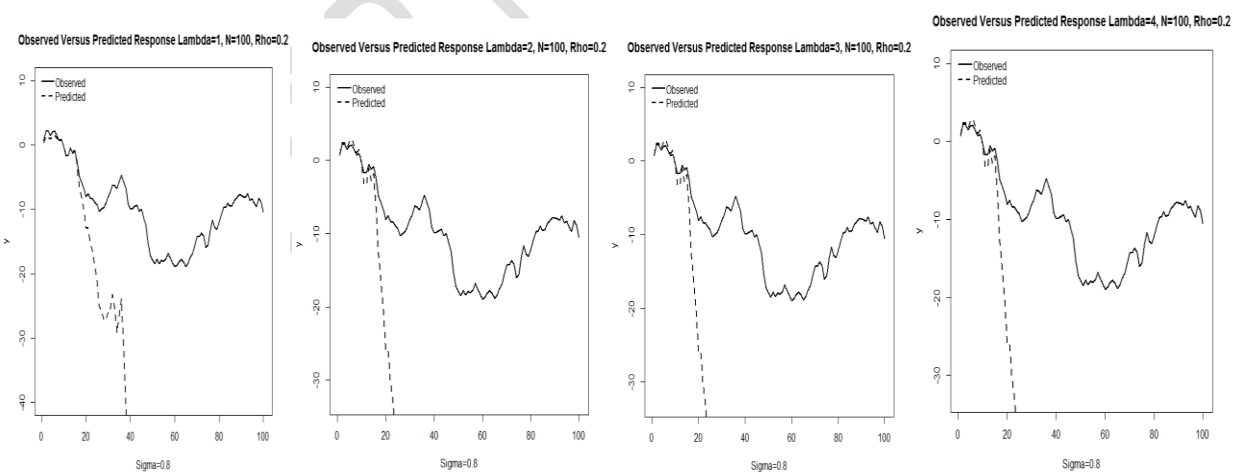
(a) (b) (c) (d)

Figure 1: Plots of the observations (. . .) and Estimates (---) With Smoothing Parameters Chosen by GCV (a), GML (b), PSM (c), and UBR (d) for n = 20



(a) (b) (c) (d)

Figure 2: Plots of the Observations (. . .) and Estimates (---) With Smoothing Parameters Chosen by GCV (a), GML (b), PSM (c),and UBR (d) for n = 60



(a) (b) (c) (d)

Figure 3: Plots of the Observations (. . .) and Estimates (---) With Smoothing Parameters Chosen by GCV (a), GML (b), PSM (c),and UBR (d) for n = 100

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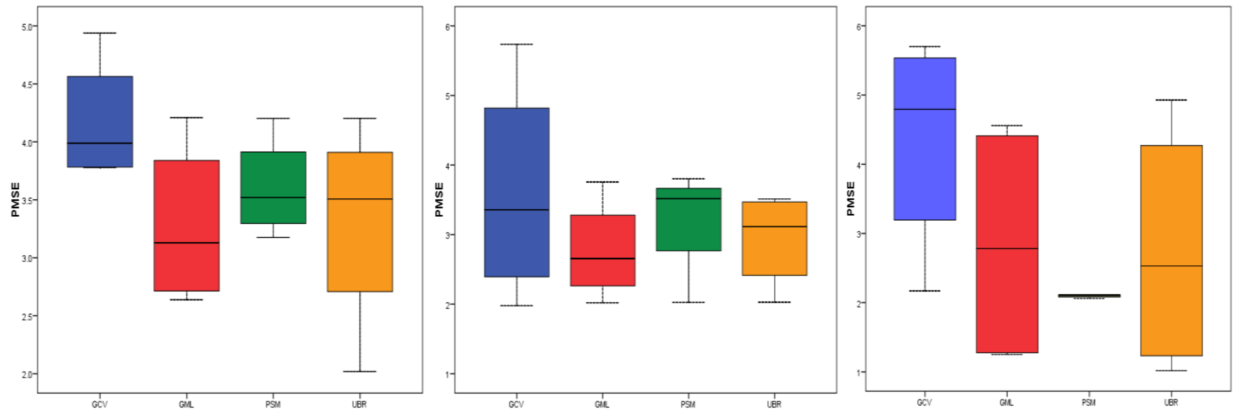

Figure 4: The plot of the GML, GCV, PSM and UBR of the MSE of the simulated study in the presence of autocorrelation when σ = 1, ρ = 0.2 and n = 20

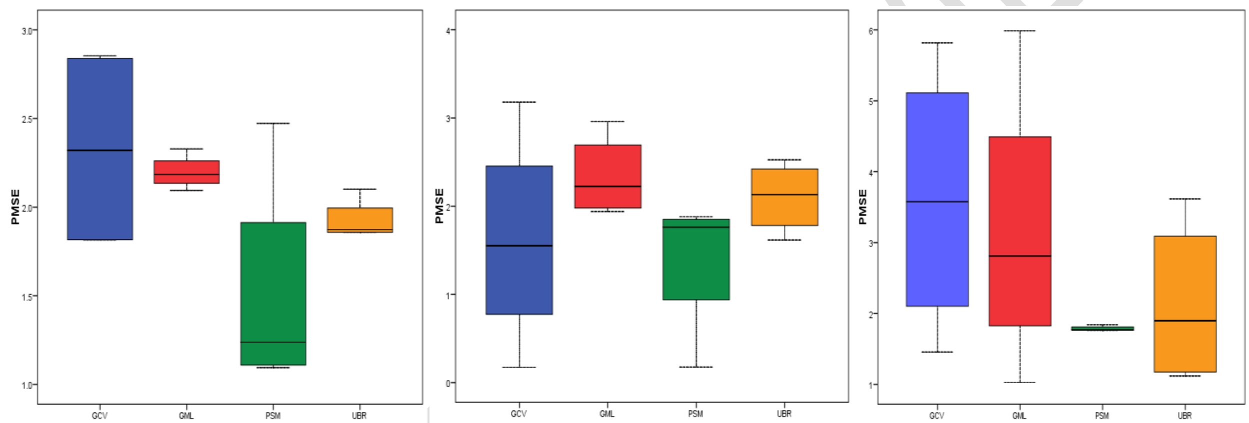

Figure 5: The plot of the GML, GCV, PSM and UBR of the MSE of the simulated study in the presence of autocorrelation when σ = 1, ρ = 0.2 and n = 60

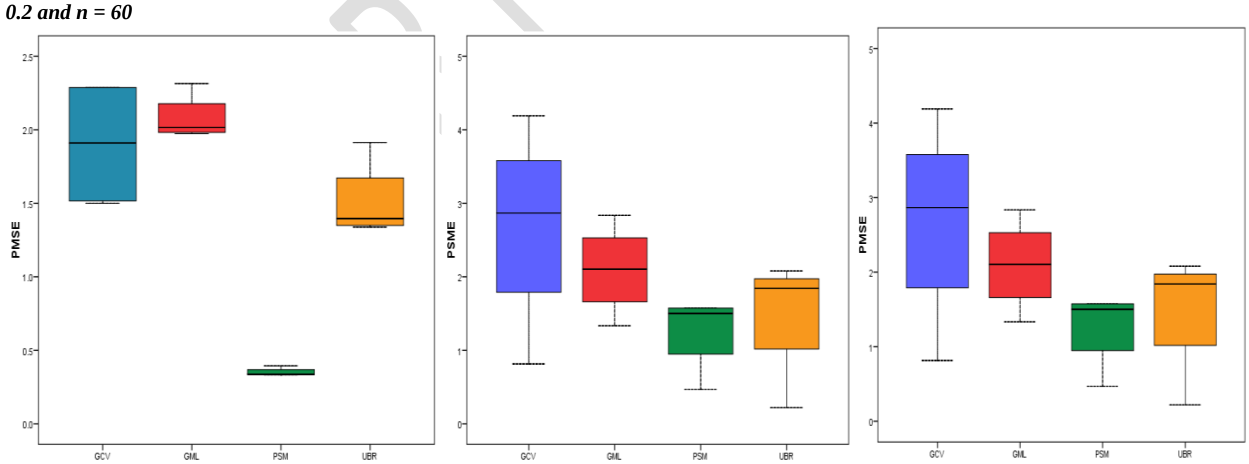

Figure 6: The plot of the GML, GCV, PSM and UBR of the MSE of the simulated study in the presence of autocorrelation when σ = 1, ρ = 0.2 and n = 100

Figure 1 and 5 presents the predictive mean square error estimates of GCV, GML, PSM and in 1000 replications. From these plots we can see that the PSM and UBR estimates have small PSMEs compare with GCV and GML. We conclude that all four methods estimate the smoothing parameters and the functions well but the PSM and UBR provide better estimates than GCV and GML in terms of mean-square error. The PSM method is more stable when the sample size is small, such as when N = 20 while UBR method performs slightly better when N = 60. In this case there were several replications where GCV and GML providing more estimates of smoothing parameters which lead to undersmoothing of the data. This behavior of the GCV method was investigated in Wahba and Wang (1993) and Wang (1998).

Table 3: Summary of the predictive mean square error and ranks of the smoothing methods in the presence of autocorrelation error

Autocorrelation levels	*Smoothing method* *GCV*	*GML*	*PSM (k=1)*	*UBR*
α = 0.2	1.08	1.39	1.47	1.63
α = 0.5	1.89	1.71	1.66	1.48
α = 0.8	2.63	1.99	1.27	2.09
Grand mean	1.87	1.70	1.47	1.73
Rank	4	2	1	3

Table 4: Summary of the predictive mean square error and ranks of the smoothing methods based on sample size

Sample size	*Smoothing method* *GCV*	*GML*	*PSM (k=1)*	*UBR*
n = 20	2.434	2.179	1.711	2.326
n = 60	2.041	1.900	1.549	1.921
n = 100	1.124	1.047	1.145	0.951
Grand mean	1.867	1.709	1.468	1.732
Ranks	4	2	1	3

5.0 Conclusion

In this study, Spline smoothing estimation method for time series observations in the presence of Autocorrelated errors were compared based on three sample sizes. The simulation result under

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the finite sampling properties of PMSE criterion shows that all smoothing methods were consistent but adversely affected by the presence of Autocorrelation in the error term, the smoothing methods ranks as follows, PSM, GML, UBR and GCV. The result suggested that PSM should be preferred when Autocorrelation level is mild and high (α = 0.5 – 0.8) and for low Autocorrelation levels in the observations, (i.e. α = 0.2 – 0.5) the Unbiased Risk (UBR) should be considered. It was also observed that GCV and GML were mostly affected by the presence of Autocorrelation and therefore had an asymptotically similar behavioural pattern. The study also discovered that the Proposed Smoothing method is preferred mostly at the large sample size and the proposed Smoothing method do not over fit, as shown in the figures above.

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