Theoretical verification of formula for charge function in time q = c * v in RC circuit for charging/discharging of fractional & ideal capacitor

Abstract

In this paper we apply the newly developed charge storage expression as a function of time i.e. via convolution operation of time varying capacity function and applied voltage function to a capacitor i.e. q = c* v. This new formula is different to usual and conventional way of writing capacitance multiplied by voltage to get charge stored in a capacitor i.e. q = cv. We apply this new formula to RC circuit as charging/discharging the capacitors (ideal and fractional ones) via constant dc voltage or current sources. This paper gives validity of usage of this new formula in RC circuits.

Keywords

Mittag-Leffler function, Time varying Capacity Function, Fractional Capacitor, Ideal loss-less Capacitor, Convolution Operation, Laplace Transform, Fractional derivative, Supercapacitor

1. Introduction

This is continuation of our earlier deliberations regarding verification of the new formula $q(t)=c(t)*v(t)$; [1], [40]. This paper is from deliberations regarding usage of this formula in Project: Design & Development of Power-packs with Aerogel Supercapacitors & Fractional Order Modeling BRNS Santion No. 36(3)/14/50B/2014-BRNS/2620 dated 11.03.2015; where we wish to use this new developed formula.

The voltage change when appears at a capacitor, it reacts or relaxes via relaxation current. The time varying capacity function $c(t)$ is the one that defines the response function; and by principle of causality [1] we write $q(t)=c(t)*v(t)$ where $v(t)$ is the input impressed voltage. This is different to usual formula $q(t)=c(t)v(t)$. This new formulation is deliberated in detail with $c(t)$ as for ideal loss less capacitor case, as well as time varying capacity function (fractional capacitor case) in [1]. The capacity function $c(t)$ is the function which decays with time, and has the form $c(t)\propto t^{-\alpha}$; $0<\alpha<1$ and acts only at the time of application of voltage change. For ideal case of loss-less capacitor the capacity function is $c(t)\propto\delta(t)$; [1]. In this paper we will always take the power-exponent of power-law of decaying capacity function i.e. α as between zero and one, i.e. $0<\alpha<1$. This power-law decay function is in singular at origin and is in tune with singular power law decay relaxation current given by Curie-von Schweidler (universal law) of dielectric relaxation [2]-[5]. In this universal dielectric relaxation law, the relaxing current is a decaying power-law as $i(t)\propto t^{-\alpha}$, when uncharged system of dielectric is stressed by a constant voltage. The use of this universal dielectric relaxation law gives current voltage relation of a capacitor as given by fractional derivative [6]-[10]. The non-singular decaying function gives all together different form of current voltage relations in capacitor is discussed in [11], [38]. The use of non-singular kernel in integration for the formula for fractional derivative and application is developing topic. This concept is used and studied in pioneering works [23]-[36], for several dynamic systems.

Here we are taking singular function $c(t)$ as 'time varying capacity function', as because the same gets derived from basic universal dielectric relaxation law $i(t) \propto t^{-\alpha}$ of Curie-von Schweidler [1]. In this paper we will take capacitor with time varying capacity function $c(t) = C_\alpha t^{-\alpha}$ (i.e. a fractional capacitor), and will use the formula [1], where the voltage excitation $v(t)$ is applied at time $t = 0$ to an uncharged capacitor

$$q(t) = c(t) * v(t) = \int_0^t c(t-\tau)v(\tau)d\tau = \int_0^t c(\tau)v(t-\tau)d\tau$$

With this new formula $q(t) = c(t) * v(t)$ applied we discuss various cases of $q(t)$ i.e. charge stored in capacitor and $i(t)$, the circuital current etc. for RC charging/discharging circuit with ideal capacitor and fractional capacitor.

We note a priori that the constant C_α is proportionality constant of the relation of time varying capacity function i.e. $c(t) \propto t^{-\alpha}$, and not Fractional Capacity. The fractional capacity of a fractional capacitor we will represent as $C_{F-\alpha}$ which has units of $\text{Farad}/\sec^{1-\alpha}$, and we will use $C_{F-\alpha} = C_\alpha \Gamma(1-\alpha)$ to relate the two [1], [40]. The equation of current and voltage, and impedance for fractional capacitor is following, given by fractional derivative $D_t^\alpha \equiv d^\alpha / dt^\alpha$ [6], [7] [8], [12], [13]; is following

$$i(t) = C_{F-\alpha} \frac{d^\alpha v(t)}{dt^\alpha}; \quad Z(s) = \frac{1}{s^\alpha C_{F-\alpha}}; \quad 0 < \alpha < 1$$

With limit $\alpha \to 1$ we get classical ideal loss less capacitor that is following

$$i(t) = C \frac{d\ v(t)}{dt}; \quad Z(s) = \frac{1}{s\ C}$$

The fractional capacitor appears in studies with super-capacitors and other memory based relaxation phenomena [14]-[22]. We assume that the fractional capacitor has no resistance, (like ideal capacitor has no resistance) and is excited by ideal voltage sources (that has zero output impedance), in the RC charging circuits. We will use Laplace Transform technique in all our analysis. In all the cases in subsequent sections, we will apply this new formula $q(t) = c(t) * v(t)$ and give the validity justification. Recently this formula $q(t) = c(t) * v(t)$ is getting experimentally validated [39], for super-capacitors.

Therefore charge in a capacitor is $q(t) = c(t) * v(t)$, is given via convolution operation and not with the usual way that we write as $q(t) = c(t)v(t)$. Let us have a capacitor with capacity function in time as power-law $c(t) = C_\alpha t^{-\alpha}$ ($0 < \alpha < 1$), that is fractional capacitor, is charged via resistance R. Let a voltage $v_{in}(t)$ or current $i_{in}(t)$ be applied to an uncharged capacitor in the RC circuit at time $t = 0$. Then charge function in time is given as convolution ($*$) operation as $q(t) = c(t) * v_0(t)$, with $v_0(t)$ is the voltage profile on the capacitor, in the RC circuit of Figure-1. This charge $q(t)$ is also $q(t) = \int_0^t i(\tau)d\tau$, where $i(t)$ is current flowing through the capacitor in the RC circuit. This comes from normal circuit theory application, and we will show that this $q(t)=c(t)*v_0(t)$ is same that we get from normal circuit theory. For each case we also study the ideal loss less capacitor given by capacity function as $c(t)=C\delta(t)$, and apply $q(t)=c(t)*v_0(t)$.

We will validate and verify this new formula $q(t)=c(t)*v(t)$ in circuit theory with RC circuit, in this paper. The aim of the paper is not to show profiles of circuit voltage current or charge, with variation of α; but rather validate the new formula $q(t)=c(t)*v_0(t)$, with that of solution obtained by circuit theory techniques. Thus we are not drawing MATLAB simulated figures for voltage current and charge functions.

2. Charge storage $q(t)$ by step input voltage $v_{in}(t) = V_m u(t)$ excitation to RC circuit with ideal loss less capacitor

In classical circuit theory, if we charge an ideal capacitor, C (initially uncharged) through a resistor R, via a step input voltage $v_{in}(t) = V_m u(t)$ (Figure-1) we get voltage across capacitor as exponential rise as $v_0(t)=V_m(1-e^{-t/RC})$. In Figure-1 consider $Z_1(s)=R$, and $Z_2(s)$ is ideal capacitor with capacity function as $c(t)=C\delta(t)$. Therefore we have following impedance function

$$Z_2(s)=\frac{1}{s\mathcal{L}\left\{c(t)\right\}}=\frac{1}{s\mathcal{L}\left\{C\delta(t)\right\}}=\frac{1}{sC} \tag{1}$$

The above Eq. (1) is new way of writing $Z(s)$ for capacitor ideal or fractional we got from application of formula $q(t)=c(t)*v(t)$ in our earlier discussion [40]. That we got by differentiating this convolution expression to get $i(t)$ and taking Laplace transform to arrive at Eq. (1), i.e. $Z(s)=V(s)/I(s)=\left(s\mathcal{L}\left\{c(t)\right\}\right)^{-1}$.

We have from circuit theory and Figure-1 the following expressions

$$\begin{aligned} V_0(s) &= \frac{Z_2(s)}{Z_1(s)+Z_2(s)}\mathcal{L}\left\{v_{in}(t)\right\}, \quad v_{in}(t)=V_m u(t), \qquad \mathcal{L}\left\{v_{in}(t)\right\}=\frac{V_m}{s} \\ &= \frac{V_m}{RCs\left(s+\frac{1}{RC}\right)}=V_m\left(\frac{1}{s}-\frac{1}{s+\frac{1}{RC}}\right) \end{aligned} \tag{2}$$

The inverse Laplace Transform of Eq. (2) gives following voltage charging equation for capacitor

$$v_0(t)=V_m(1-e^{-t/RC}); \quad t\ge 0 \tag{3}$$

The current flowing in the RC circuit at $t\ge 0$ is following

$$i(t)=\mathcal{L}^{-1}\left\{\frac{V_m/s}{R+\frac{1}{Cs}}\right\}=\mathcal{L}^{-1}\left\{\frac{V_m}{R}\left(\frac{1}{s+\frac{1}{RC}}\right)\right\}=\frac{V_m}{R}e^{-t/RC} \tag{4}$$

Therefore the charge function $q(t)$ is

$$\begin{aligned} q(t) &= \int_0^t i(\tau)d\tau=\int_0^t \frac{V_m}{R}e^{-\tau/RC} \\ &= V_m C(1-e^{-t/RC}); \quad t\ge 0 \end{aligned} \tag{5}$$

We apply the formula $q(t)=c(t)*v(t)$ to ideal capacitor given by $c(t)=C\delta(t)$ across which we are having a voltage profile as $v_0(t)=V_m(1-e^{-t/RC})$, to write following

$$Q(s) = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right)$$
$$= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\{V_m(1-e^{-t/RC})\}\right) = C\left(\frac{V_m}{s} - \frac{V_m}{\left(s+\frac{1}{RC}\right)}\right) \tag{6}$$

The inverse Laplace transform of Eq. (6) above gives

$$q(t) = V_m C(1-e^{-t/RC}) \tag{7}$$

Eq. (7) is same as Eq. (5) that we got via circuit theory applying $q(t) = \int_0^t i(\tau)d\tau$. This gives validation of formula $q(t) = c(t) * v(t)$ for classical ideal loss less capacitor case.

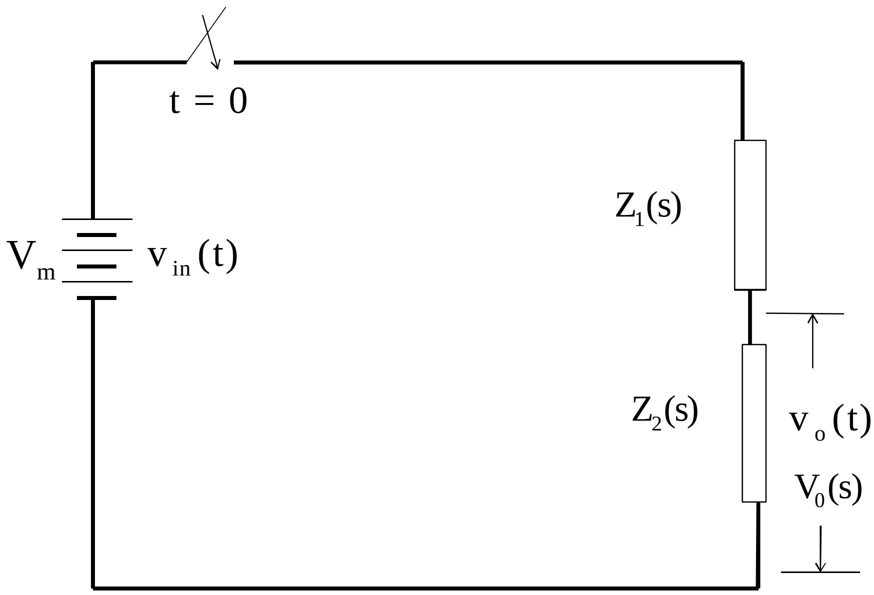

Figure- 1: The constant voltage charging RC circuit

3. Charge storage $q(t)$ by step input voltage $v_{in}(t) = V_m u(t)$ excitation to RC circuit with fractional capacitor

In Figure-1 consider $Z_1(s) = R$, and $Z_2(s)$ is fractional capacitor with capacity function as $c(t) = C_\alpha t^{-\alpha}$; with $0 < \alpha < 1$. Therefore we have following impedance function [40]

$$Z_2(s) = \frac{1}{s\mathcal{L}\{c(t)\}} = \frac{1}{s\mathcal{L}\{C_\alpha t^{-\alpha}\}} = \frac{1}{s\left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)}$$
$$= \frac{1}{s^\alpha C_\alpha \Gamma(1-\alpha)} = \frac{1}{s^\alpha C_{F-\alpha}}; \quad C_{F-\alpha} = C_\alpha \Gamma(1-\alpha) \tag{8}$$

Here we will use a constant voltage excitation of V_m from time $t = 0$, to time $t = T_c$ (as charging phase, through a known resistor R) and thereafter we will switch to discharging phase i.e. voltage source will be made zero (Figure-2). By this we record the charging and discharging profile $v_0(t)$, and then apply $q(t)=c(t)*v_0(t)$ to get charge, and then current. From the circuit diagram of Figure-1, we write the following [37]

$$\begin{aligned} V_0(s) &= \frac{Z_2(s)}{Z_1(s)+Z_2(s)}\mathcal{L}\{v_{in}(t)\}, \quad v_{in}(t)=V_m u(t), \quad \mathcal{L}\{v_{in}(t)\}=\frac{V_m}{s} \\ &= \frac{V_m}{RC_{F-\alpha}s\left(s^\alpha+\frac{1}{RC_{F-\alpha}}\right)}=\frac{V_m k s^{-1}}{(s^\alpha+k)}; \quad k=\frac{1}{RC_{F-\alpha}} \end{aligned} \tag{9}$$

Now use $\mathcal{L}\left\{t^{\alpha p+\beta-1}E^{(p)}_{\alpha,\beta}(at^\alpha)\right\}=\frac{p!s^{\alpha-\beta}}{s^\alpha\text{-}a}$ [10], [12], [13] to get $\mathcal{L}^{-1}\left\{\frac{s^{-1}}{s^\alpha\text{-}a}\right\}=t^\alpha E_{\alpha,\alpha+1}(at^\alpha)$, by putting $p=0$, $\alpha=\alpha$, $\beta=\alpha+1$, where the $E_{\alpha,\beta}(at^\alpha)$ is two parameter Mittag-Leffler function; as defined in infinite series in following expression

$$E_{\alpha,\beta}(x)=\sum_{m=0}^{¥}\frac{(x)^m}{\Gamma(\alpha m+\beta)}, \qquad E_{\alpha,(\alpha+1)}(\text{-}kt^\alpha)=\sum_{m=0}^{¥}\frac{(\text{-}kt^\alpha)^m}{\Gamma(m\alpha+\alpha+1)} \tag{10}$$

With this we obtain the following from Laplace inverse of Eq. (9)

$$\begin{aligned} v_0(t) &= \mathcal{L}^{-1}\left\{\frac{V_m k}{s(s^\alpha+k)}\right\}=V_m k t^\alpha E_{\alpha,\alpha+1}(\text{-}kt^\alpha) \\ &= \frac{V_m}{RC_{F-\alpha}}t^\alpha E_{\alpha,\alpha+1}\left(\text{-}\frac{t^\alpha}{RC_{F-\alpha}}\right) \end{aligned} \tag{11}$$

We have alternate derivation via series expansion [13], [37] as follows

$$\begin{aligned} V_0(s) &= \frac{V_m k}{s(s^\alpha+k)}=\frac{V_m k}{s^{\alpha+1}}\left(1+\frac{k}{s^\alpha}\right)^{-1}; \quad (1+x)^{-1}=1-x+x^2-x^3+\dots \\ &= \frac{V_m k}{s^{\alpha+1}}\left(1-\frac{k}{s^\alpha}+\frac{k^2}{s^{2\alpha}}-\frac{k^3}{s^{3\alpha}}+\dots\right) \\ &= V_m\left(\frac{k}{s^{\alpha+1}}-\frac{k^2}{s^{2\alpha+1}}+\frac{k^3}{s^{3\alpha+1}}-..\right) \end{aligned} \tag{12}$$

Use Laplace pair $\frac{\Gamma(n+1)}{s^{n+1}}=\mathcal{L}\left\{t^n\right\}$ to invert term by term the above Eq. (12) to get following

$$\begin{aligned} v_0(t) &= V_m\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}-\frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)}-\dots\right) \\ &= V_m\left(1-\left[1-\frac{kt^\alpha}{\Gamma(\alpha+1)}+\frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)}+\dots\right]\right) \\ &= V_m\left(1-\sum_{n=0}^{\infty}\frac{(-kt^\alpha)^n}{\Gamma(n\alpha+1)}\right)=V_m\left[1-E_\alpha(\text{-}kt^\alpha)\right]=V_m\left[1-E_\alpha\left(\text{-}\frac{t^\alpha}{RC_{F-\alpha}}\right)\right] \end{aligned} \tag{13}$$

Where, $E_\alpha(x)$ is one parameter Mittag-Leffler function used in Eq. (13), with $E_1(x)=e^x$. Therefore for classical ideal capacitor with limit $\alpha\to 1$, we have normal exponential charging $v_0(t)=V_m(1-e^{-t/RC})$; writing $C_{F-\alpha}\big|_{\alpha\to 1}\equiv C$.

For voltage charging expression for fractional order impedance $Z_2(s)=s^{-\alpha}\,C^{-1}_{F-\alpha}$, Eq. (8) we have from Eq. (11) and Eq. (13) the following

$$v_0(t) = V_m\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right) = \frac{V_m}{RC_{F\text{-}\alpha}} t^\alpha E_{\alpha,\alpha+1}\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right) \tag{14}$$

For charging current of circuit of Figure-1 with $Z_1 = R$ and $Z_2(s) = \frac{1}{s^\alpha C_{F\text{-}\alpha}}$, we have $Z(s) = Z_1(s) + Z_2(s)$ and write the following

$$I(s) = \frac{1}{Z(s)}\left(\frac{V_m}{s}\right) = \frac{V_m}{s\left(R + \frac{1}{s^\alpha C_{F\text{-}\alpha}}\right)} = \frac{V_m}{R}\left(\frac{s^{\alpha-1}}{s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}}\right) \tag{15}$$

Using $\mathcal{L}\left\{E_n(at^n)\right\} = \frac{s^{n-1}}{s^n - a}$, [10], [12], [13] we get inverse Laplace transform of above Eq. (15) as

$$i(t) = \frac{V_m}{R} E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right) \tag{16}$$

Clearly for ideal i.e. in limit $\alpha \to 1$ case we get $i(t) = \frac{V_m}{R} e^{-t/RC}$. Therefore the charge $q(t)$ is from Eq. (16) the following

$$q(t) = \int_0^t i(\tau)d\tau = \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau \tag{17}$$

We apply the formula $q(t) = c(t) * v(t)$ to fractional capacitor given by $c(t) = C_\alpha t^{-\alpha}$ across which we are having a voltage profile as $v_0(t) = V_m\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right)$, to write following steps

$$\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right) \\
&= \left(\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{V_m\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right)\right\}\right) \\
&= \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\left(\tfrac{V_m k}{s(s^\alpha + k)}\right) = \frac{V_m C_{F\text{-}\alpha}\left(\frac{1}{RC_{F\text{-}\alpha}}\right)}{s^{2-\alpha}\left(s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}\right)}; \quad k = \tfrac{1}{RC_{F\text{-}\alpha}}, \quad \tfrac{C_{F\text{-}\alpha}}{\Gamma(1-\alpha)} = C_\alpha \\
&= \left(\frac{V_m}{R}\right)\frac{s^{\alpha-2}}{\left(s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}\right)} \qquad \mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha + k} \\
&= \left(\frac{V_m}{R}\right)\left(s^{-1}\left(\frac{s^{\alpha-1}}{\left(s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}\right)}\right)\right) = \left(\frac{V_m}{R}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right\}\right)
\end{aligned} \tag{18}$$

Taking inverse Laplace transform of Eq. (18) by recognizing $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = s^{-1}F(s)$ we write

$$q(t) = \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau \tag{19}$$

The same result as in Eq. (17) we got by using $q(t) = \int_0^t i(\tau)d\tau$ validates the verification of formula $q(t) = c(t) * v(t)$. Put $\alpha = 1$ in Eq. (19) and we get ideal loss-less capacitor with $C_{F\text{-}\alpha} = C$, and $E_1(x) = e^x$ to write the following case

$$\begin{aligned}
q(t) &= \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau\Big|_{\alpha=1} = \int_0^t \frac{V_m}{R} e^{-\tau/RC} d\tau \\
&= V_m C\left(1 - e^{-t/RC}\right)
\end{aligned} \tag{20}$$

The above Eq. (20) is charge build up relation for ideal-loss less capacitor, same as Eq. (5) and Eq. (7).

We take the integration of Mittag-Leffler function as $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ with $E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)}$. This we will prove this in next section. So we have charge build up function on a fractional capacitor in RC charging circuit as follows

$$\begin{aligned} q(t) &= \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau \\ &= \frac{V_m t}{R}\left(E_{\alpha,2}(-t^\alpha / RC_{F\text{-}\alpha})\right); \quad t \geq 0 \end{aligned} \tag{21}$$

4. Proof of formula $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ used

We verify the formula used $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ as in following steps

$$\begin{aligned} \int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau &= \int_0^t\left(1-\frac{k\tau^\alpha}{\Gamma(\alpha+1)}+\frac{k^2\tau^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{k^3\tau^{3\alpha}}{\Gamma(3\alpha+1)}+....\right)d\tau \\ &= t-\frac{kt^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}+\frac{k^2t^{2\alpha+1}}{(2\alpha+1)\Gamma(2\alpha+1)}-\frac{k^3t^{3\alpha+1}}{(3\alpha+1)\Gamma(3\alpha+1)}+... \\ &= t\left(1-\frac{kt^\alpha}{\Gamma(\alpha+2)}+\frac{k^2t^{2\alpha}}{\Gamma(2\alpha+2)}-\frac{k^3t^{3\alpha}}{\Gamma(3\alpha+2)}+...\right), \quad \Gamma(m+1) = m\Gamma(m) \\ &= t\left(E_{\alpha,2}(-kt^\alpha)\right) \quad ; \quad E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)} \end{aligned} \tag{22}$$

Let us verify this for $\alpha = 1$ from Eq. (22)

$$\begin{aligned} q(t) &= \frac{V_m t}{R}\left(E_{\alpha,2}(-t^\alpha / RC_{F\text{-}\alpha})\right)\Big|_{\alpha=1;C_{F\text{-}\alpha}=C} \quad ; \quad E_{\alpha,2}(-a\,x^\alpha) = \sum_{m=0}^{\infty}\frac{(-1)^m a^m x^{\alpha m}}{\Gamma(\alpha m+2)} \\ &= \frac{V_m}{R}t\left(1-\frac{t}{(RC)\Gamma(3)}+\frac{t^2}{(RC)^2\Gamma(4)}-\frac{t^3}{(RC)^3\Gamma(5)}+...\right) \\ &= \frac{V_m C}{RC}\left(t-\frac{t^2}{(RC)(2)!}+\frac{t^3}{(RC)^2(3!)}-\frac{t^4}{(RC)^3(4)!}+...\right) \\ &= V_m C\left(1-1+\frac{\left(\frac{t}{RC}\right)}{1!}-\frac{\left(\frac{t}{RC}\right)^2}{2!}+\frac{\left(\frac{t}{RC}\right)^3}{3!}-\frac{\left(\frac{t}{RC}\right)^4}{4!}+...\right) \\ &= V_m C\left(1-\left(1-\frac{\left(\frac{t}{RC}\right)}{1!}+\frac{\left(\frac{t}{RC}\right)^2}{2!}-\frac{\left(\frac{t}{RC}\right)^3}{3!}+\frac{\left(\frac{t}{RC}\right)^4}{4!}-...\right)\right) \\ &= V_m C\left(1-e^{-t/RC}\right) \end{aligned} \tag{23}$$

Thus we have verified the validity of formula $q(t) = c(t) * v(t)$ in RC charging circuit with fractional capacitor.

5. Charging/discharging a super-capacitor in RC circuit

5-a) Charging Phase

The differential equation corresponding to Figure-1 for $\alpha = 1$, is ordinary differential equation (ODE), with $Z_1(s) = R$ and $Z_2(s) = \frac{1}{sC}$ is following

$$RC\frac{dv_0(t)}{dt} + v_0(t) = v_{in}(t) \tag{24}$$

For $\alpha \neq 1$ we get fractional differential equation (FDE), with $Z_1(s) = R$ and $Z_2(s) = \frac{1}{s^\alpha C_{F\text{-}\alpha}}$ is following

$$RC_{F\text{-}\alpha}\frac{d^\alpha v_0(t)}{dt^\alpha} + v_0(t) = v_{in}(t) \tag{25}$$

A super-capacitor is modeled as Equivalent Series Resistance (ESR) series with impedance of a Fractional Capacitor of order α [15]-[22]. We now consider a lumped ESR (R_s) for super-capacitor, thus for Figure-1 we have $Z_2(s) = R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}} = \frac{s^\alpha R_s C_{F\text{-}\alpha}+1}{s^\alpha C_{F\text{-}\alpha}}$ while charging impedance remains at $Z_1(s) = R$. Therefore for any input voltage $V_{in}(s) = \mathcal{L}\{v_{in}(t)\}$, we write the charging current (in Laplace domain) as

$$I_{CH}(s) = \frac{V_{in}(s)}{R + R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}}} = \frac{s^\alpha C_{F\text{-}\alpha} V_{in}(s)}{s^\alpha C_{F\text{-}\alpha}(R + R_s) + 1} \tag{26}$$

Output voltage across $Z_2(s)$ in Laplace domain is therefore is

$$\begin{aligned}
V_0(s) &= \left(I_{CH}(s)\right)\left(Z_2(s)\right) = \left(\frac{V_{in}(s)s^\alpha C_{F\text{-}\alpha}}{s^\alpha C_{F\text{-}\alpha}(R+R_s)+1}\right)\left(\frac{s^\alpha R_s C_{F\text{-}\alpha}+1}{s^\alpha C_{F\text{-}\alpha}}\right) \\
&= \frac{V_{in}(s) + V_{in} s^\alpha R_s C_{F\text{-}\alpha}}{s^\alpha C_{F\text{-}\alpha}(R+R_s)+1} = \frac{\frac{V_{in}(s)}{C_{F\text{-}\alpha}(R+R_s)} + \frac{V_{in}(s)s^\alpha R_s}{(R+R_s)}}{s^\alpha + \frac{1}{C_{F\text{-}\alpha}(R+R_s)}} \qquad \text{put} \qquad V_{in}(s) = \frac{V_m}{s} \\
&= \left(\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}\right)\left(\frac{1}{s\left(s^\alpha + \frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right) + \left(\frac{V_m R_s}{R+R_s}\right)\left(\frac{s^{\alpha-1}}{s^\alpha + \frac{1}{C_{F\text{-}\alpha}(R+R_s)}}\right)
\end{aligned} \tag{27}$$

To get $v_0(t)$ we do inverse Laplace transform of Eq. (27) as following

$$v_0(t) = \mathcal{L}^{-1}\{V_0(s)\} = \mathcal{L}^{-1}\left\{\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)s\left(s^\alpha + \frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right\} + \mathcal{L}^{-1}\left\{\frac{V_m R_s s^{\alpha-1}}{(R+R_s)\left(s^\alpha + \frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right\} \tag{28}$$

Use formula $\mathcal{L}\left\{t^{\alpha p+\beta-1}E^{(p)}_{\alpha,\beta}(at^\alpha)\right\} = p!\frac{s^{\alpha-\beta}}{s^\alpha - a}$. [10], [12], [13] with $p = 1, \alpha = \alpha$, $\beta = \alpha + 1$ and $p = 0$, $\alpha = \alpha, \beta = 1$, to write from Eq. (28) the inverse Laplace as

$$v_0(t) = \frac{V_m}{C_{F\text{-}\alpha}(R+R_s)} t^\alpha E_{\alpha,\alpha+1}\left(-\frac{t^\alpha}{C_{F\text{-}\alpha}(R+R_s)}\right) + \frac{V_R R_s}{R+R_s} E_{\alpha,1}\left(-\frac{t^\alpha}{C_{F\text{-}\alpha}(R+R_s)}\right) \tag{29}$$

Let us keep the step input from time $t = 0$ to $t = T_c$, and then at time $t = T_c$, the output voltage is

$$v_0(T_c) = \frac{V_R T_c^\alpha}{C_{F\text{-}\alpha}(R+R_s)} E_{\alpha,\alpha+1}\left(-\frac{T_c^\alpha}{C_{F\text{-}\alpha}(R+R_s)}\right) + \frac{V_R R_s}{R+R_s} E_{\alpha,1}\left(-\frac{T_c^\alpha}{C_{F\text{-}\alpha}(R+R_s)}\right) \tag{30}$$

The charge $q(t)$ will be held only in the element $C_{F\text{-}\alpha}$. We calculate now the voltage profile $v_c(t)$ and then voltage at $t=T_c$, i.e. $v_c(T_c)$ for only fractional impedance part i.e. $\frac{1}{s^\alpha C_{F\text{-}\alpha}}$ of the impedance $Z_2(s)$ comprising of R_s plus this fractional impedance $\frac{1}{s^\alpha C_{F\text{-}\alpha}}$, the voltage is thus

$$\begin{aligned} V_c(s) &= I_{CH}\left(\frac{1}{s^\alpha C_{F\text{-}\alpha}}\right)=\left(\frac{s^\alpha C_{F\text{-}\alpha}V_{in}(s)}{s^\alpha C_{F\text{-}\alpha}(R+R_s)+1}\right)\left(\frac{1}{s^\alpha C_{F\text{-}\alpha}}\right) \quad \text{put} \quad V_{in}(s)=\frac{V_m}{s} \\ &= \left(\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}\right)\left(\frac{1}{s\left(s^\alpha+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right) \end{aligned} \tag{31}$$

Using the Laplace identity of Mittag-Leffler function $\mathcal{L}\left\{E_n(at^n)\right\}=\frac{s^{n-1}}{s^n-a}$, [10], [12], [13] we write

$$\begin{aligned} v_c(t) &= \frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}t^\alpha E_{\alpha,\alpha+1}\left(-\frac{t^\alpha}{C_{F\text{-}\alpha}(R+R_s)}\right) \\ v_c(t) &= V_m\left(1-E_\alpha\left(-\frac{t^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right), \quad 0\le t\le T_c \end{aligned} \tag{32}$$

At $t=T_c$ we thus have the voltage at the fractional impedance

$$v_c(T_c)=\frac{V_m T_c^\alpha}{C_{F\text{-}\alpha}(R+R_s)}E_{\alpha,\alpha+1}\left(-\frac{T_c^\alpha}{C_{F\text{-}\alpha}(R+R_s)}\right)=V_m\left(1-E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \tag{33}$$

The charge $q(t)$ is $q(t)=c(t)*v_c(t)$ with fractional capacitor with capacity function as $c(t)=C_\alpha t^{-\alpha}$ having voltage profile and that is $v_c(t)=V_m\left(1-E_\alpha\left(-\frac{t^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$ as following

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\left\{c(t)\right\}\right)\left(\mathcal{L}\left\{v_c(t)\right\}\right) \\ &= \left(\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{V_m\left(1-E_\alpha\left(-\frac{t^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)\right\}\right) \\ &= \left(C_\alpha\Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{V_m\left(\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}{s\left(s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}\right)=\frac{V_m C_{F\text{-}\alpha}\left(\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}{s^{2-\alpha}\left(s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}; \quad k=\frac{1}{(R+R_s)C_{F\text{-}\alpha}}, \quad \frac{C_{F\text{-}\alpha}}{\Gamma(1-\alpha)}=C_\alpha \\ &= \left(\frac{V_m}{R+R_s}\right)\frac{s^{\alpha-2}}{\left(s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)} \qquad \mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\}=\frac{s^{\alpha-1}}{s^\alpha+k} \\ &= \left(\frac{V_m}{R+R_s}\right)\left(s^{-1}\left(\frac{s^{\alpha-1}}{\left(s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}\right)\right) \\ &= \left(\frac{V_m}{R+R_s}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\frac{t^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right\}\right) \end{aligned} \tag{34}$$

Taking inverse Laplace transform of Eq. (34) by recognizing $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=s^{-1}F(s)$ we write

$$q(t)=\int_0^t\frac{V_m}{R+R_s}E_\alpha\left(-\frac{\tau^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau=\frac{V_m t}{R+R_s}\left(E_{\alpha,2}(-t^\alpha/(R+R_s)C_{F\text{-}\alpha})\right) \tag{35}$$

At $t=T_c$ we have charge as

$$q(T_c)=\frac{V_mT_c}{R+R_s}E_{\alpha,2}\left(-\frac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right) \tag{36}$$

For $Z_2(s)=R_s+\frac{1}{sC}$ i.e. with an ideal capacitor with ESR, we have the following

$$\begin{aligned}
Q(s)&=\left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right)\\
&=\left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\left\{V_m\left(1-e^{-\frac{t}{(R+R_s)C}}\right)\right\}\right)\\
&=C\left(\frac{V_m\left(\frac{1}{(R+R_s)C}\right)}{s\left(s+\frac{1}{(R+R_s)C}\right)}\right)=\frac{V_mC\left(\frac{1}{(R+R_s)C}\right)}{s\left(s+\frac{1}{(R+R_s)C}\right)}=V_mC\left(\frac{1}{s}-\frac{1}{s+\frac{1}{(R+R_s)C}}\right)
\end{aligned} \tag{37}$$

$$q(t)=V_mC\left(1-e^{-\frac{t}{(R+R_s)C}}\right)$$

Charge at the end of $t=T_c$ is

$$q(T_c)=V_mC\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right) \tag{38}$$

The charging current is following from Eq. (37)

$$i_{CH}(t)=\frac{dq(t)}{dt}=\frac{V_me^{-\frac{t}{(R+R_s)C}}}{(R+R_s)},\quad 0\le t\le T_c \tag{39}$$

The voltage at the end of $t=T_c$ is $v_c(T_c)=V_m(1-e^{-\frac{T_c}{(R+R_s)C}})$.

5-b) Discharging Phase

After $t=T_c$ we make the voltage $v_{in}(t)=0$ i.e. we are draining out the stored charge i.e. $q(T_c)=V_mC(1-e^{-T_c/((R+R_s)C)})$ during the discharge phase ($t\ge T_c$); Figure-2. In the discharge phase the voltage $v_c(T_c)$ will decay as $v_c(t')=\left(v_c(T_c)\right)e^{-t'/(R+R_s)C}$, for $t\ge T_c$, writing $t'=t-T_c$. At this point the capacity function $c(t')=C\delta(t')$ will again appear, as there is sudden change (differentiability is lost) in voltage from V_m to 0 at $t'=0$ (i.e. $t=T_c$). Therefore the discharging charge profile $q(t')$ we write as $q(t')=c(t')*v_c(t')$ as follows

$$\begin{aligned}
Q(s)&=\left(\mathcal{L}\{c(t')\}\right)\left(\mathcal{L}\{v_c(t')\}\right),\quad t\ge T_c\\
&=\left(\mathcal{L}\{C\delta(t')\}\right)\left(\mathcal{L}\left\{\left(v_c(T_c)\right)e^{-t'/(R+R_s)C}\right\}\right)\\
&=(C)\left(\frac{\left(v_c(T_c)\right)}{s+\frac{1}{(R+R_s)C}}\right)
\end{aligned} \tag{40}$$

$$\begin{aligned}
q(t')&=Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}};\quad v_c(T_c)=V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right)\\
&=V_mC\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right)e^{-\frac{t'}{(R+R_s)C}};\quad t'\ge 0;\quad t\ge T_c
\end{aligned}$$

The discharging current $t\ge T_c$ is as follows

$$
\begin{aligned}
i_{DIS}(t') &= \frac{dq(t')}{dt'} = Cv_c(T_c)\frac{de^{-\frac{t'}{(R+R_s)C}}}{dt'}, \quad t \geq T_c \\
&= -\frac{v_c(T_c)}{(R+R_s)}e^{-\frac{t'}{(R+R_s)C}} = -\frac{V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right)}{(R+R_s)}e^{-\frac{t'}{(R+R_s)C}}
\end{aligned} \tag{41}
$$

The negative sign in Eq. (41) indicates that discharge current is opposite to that of charging current. Now we carry on with the above logic for a fractional capacitor with $Z_2(s) = R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}}$.

This value $v_c(T_c) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$; Eq. (33) of voltage becomes the initial voltage while we discharge the super-capacitor with time defined as $t' = t - T_c$, for discharge phase where $v_{in}(t') = 0$.

Now we see the discharge profile, as the charged fractional capacitor $C_{F\text{-}\alpha}$ with above value $v_c(T_c)$ Eq. (33) discharges through R . The discharge current is now for $t' \geq 0$, negative to the charging current is following

$$
I_{DIS}(s) = -\frac{v_c(T_c)/s}{R + R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}}} = -\frac{v_c(T_c)s^{\alpha-1}}{(R+R_s)\left(s^\alpha + \frac{1}{s^\alpha C_{F\text{-}\alpha}(R+R_s)}\right)} \tag{42}
$$

The inverse Laplace transform of Eq. (42) gives discharge current for $t \geq T_c$ as following

$$
\begin{aligned}
i_{DIS}(t') &= \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)/s}{R+R_s+\frac{1}{s^\alpha C_{F\text{-}\alpha}}}\right\} \\
&= -\frac{v_c(T_c)}{R+R_s}E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right); \quad t \geq T_c, \quad v_c(0) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)
\end{aligned} \tag{43}
$$

For $\alpha = 1$ we have for ideal loss less capacitor $C_{F\text{-}\alpha} = C$ from Eq. (43)

$$
i_{DIS}(t') = \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)/s}{R+R_s+\frac{1}{sC}}\right\} = -\frac{v_c(T_c)}{R+R_s}e^{-\frac{t'}{(R+R_s)C}}; \quad t \geq T_c, \quad v_c(T_c) = V_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right) \tag{44}
$$

The discharging profile of $q(t')$ with initial charge $q(0) = q(T_c)$ is

$$
\begin{aligned}
q(t') &= q(0) + \int_0^{t'} -\frac{v_c(T_c)}{R+R_s}e^{-\frac{\tau}{(R+R_s)C}}d\tau = \left[Cv_c(T_c)e^{-\frac{\tau}{(R+R_s)C}}\right]_{\tau=0}^{\tau=t'}; \quad t > T_c \\
&= q(0) + Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}} - Cv_c(T_c) \\
q(T_c) &= q(0) = V_m C\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right) = Cv_c(T_c)
\end{aligned} \tag{45}
$$

Thus we get $q(t')$ for $t \geq T_c$ with $t' = t - T_c$ as following

$$
q(t') = Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}}; \quad v_c(T_c) = V_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right); \quad t \geq T_c \tag{46}
$$

The voltage profile across the fractional capacitor while discharging process is

$$
v_c(t') = v_c(T_c)E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right), \quad t \geq T_c, \quad v_c(T_c) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \tag{47}
$$

The charge $q(t')$ profile during the discharge phase is $q(t') = c(t') * v_c(t')$ for $t \geq T_c$ is following

$$\begin{aligned}Q(s) &= \left(\mathcal{L}\left\{c(t')\right\}\right)\left(\mathcal{L}\left\{v_c(t')\right\}\right)\\ &= \left(\mathcal{L}\left\{C_\alpha (t')^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{v_c(T_c)E_\alpha\left(-\tfrac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right\}\right);\quad C_{F\text{-}\alpha}=C_\alpha\Gamma(1-\alpha)\\ &= \left(C_\alpha\Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{v_c(T_c)s^{\alpha-1}}{s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}}\right)=C_{F\text{-}\alpha}v_c(T_c)s^{\alpha-1}\frac{s^{\alpha-1}}{s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}}\\ &= C_{F\text{-}\alpha}v_c(T_c)\left(s^{-1}\mathcal{L}\left\{D_{t'}^\alpha E_\alpha(-kt'^\alpha)\right\}\right);\quad k=\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\end{aligned}\tag{48}$$

In above steps Eq. (48), we have used $s^\alpha F(s)\equiv D_t^\alpha f(t)$, for $F(s)=\frac{s^{\alpha-1}}{s^\alpha+k}$, $f(t)=E_\alpha(-kt^\alpha)$. Consider the fractional derivative operator D_t^α as Caputo fractional derivative. We have the Caputo fractional derivative of Mittag-Leffler function $E_\alpha(\lambda x^\alpha)$ as $D_x^\alpha E_\alpha(\lambda x^\alpha)=\lambda E_\alpha(\lambda x^\alpha)$; [13]. Using this we write the following

$$Q(s)=C_{F\text{-}\alpha}v_c(T_c)\left(s^{-1}\mathcal{L}\left\{-kE_\alpha(-kt'^\alpha)\right\}\right);\quad k=\tfrac{1}{(R+R_s)C_{F\text{-}\alpha}}\tag{49}$$

Using inverse Laplace Transform we have

$$\begin{aligned}q(t') &= q(0)+\left(-C_{F\text{-}\alpha}v_c(T_c)\int_0^{t'}kE_\alpha(-k\tau^\alpha)d\tau\right);\quad k=\tfrac{1}{(R+R_s)C_{F\text{-}\alpha}}\\ &= q(0)+\left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'}E_\alpha(-k\tau^\alpha)d\tau\right);\quad t\geq T_c\end{aligned}\tag{50}$$

Where we have $q(0)=q(T_c)=\frac{V_m T_c}{R+R_s}E_{\alpha,2}\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)$ and $v_c(T_c)=V_m\left(1-E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$

We use $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau=t\left(E_{\alpha,2}(-kt^\alpha)\right)$ (that we derived in Section-4) and write the following

$$\begin{aligned}q(t') &= q(0)+\left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'}E_\alpha\left(-\tfrac{\tau^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau\right);\quad t\geq T_c\\ &= \frac{V_m T_c}{R+R_s}E_{\alpha,2}\left(-\tfrac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)-\frac{V_m\left(1-E_\alpha\left(-\tfrac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)}{(R+R_s)}\left[t'\left(E_{\alpha,2}\left(-\tfrac{t'^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)\right]\end{aligned}\tag{51}$$

We put $\alpha=1$ in $q(t')=q(0)+\left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'}E_\alpha\left(-\frac{\tau^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau\right)$; Eq. (51) and we get what we got for classical ideal capacitor $C_{F\text{-}\alpha}=C$, i.e. $q(t')=q(0)+\left(-\frac{v_c(T_c)}{R+R_s}\int_0^{t'}e^{-\frac{\tau}{(R+R_s)C}}d\tau\right)$, Eq. (45).

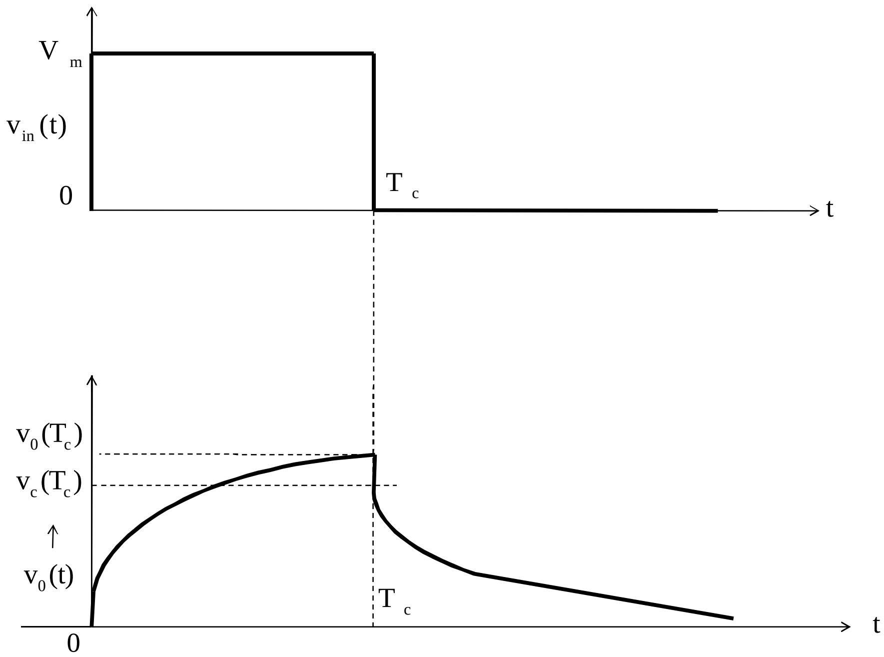

Figure-2: Constant voltage charging and discharging voltage profile at super-capacitor

The Figure-2 displays the curve of voltage profile for a constant voltage charge and discharge case. Here we point out that the charging curve though similar to exponential charging of a text book capacitor $v_0(t) \propto (1-e^{-t/RC})$, but it is not so, for fractional capacitor that is described via Mittag-Leffler function. Similarly the discharge profile though similar to exponential decay $v_0(t) \propto e^{-t/RC}$, but is not so for fractional capacitor; here too described by Mittag-Leffler function. All the relations we obtained and also verified our formula $q(t) = c(t) * v(t)$.

6. Charge storage $q(t)$ by step input constant current $i_{in}(t) = I_m u(t)$ excitation to RC circuit with fractional capacitor and ideal capacitor

In the Figure-1 we take $Z_1(s) = R$, $Z_2(s) = \frac{1}{s^\alpha C_{F\text{-}\alpha}}$ and instead of $v_{in}(t) = V_m u(t)$, that is voltage source, we take, that as an ideal constant current source i.e. $i_{in}(t) = I_m u(t)$. This constant current charging we apply to initially uncharged fractional capacitor, with capacity function $c(t) = C_\alpha t^{-\alpha}$. The fractional capacitor will develop a voltage across it by law governed by fractional derivative and fractional integral as follows

$$i(t) = C_{F\text{-}\alpha} \frac{d^\alpha v(t)}{dt^\alpha}; \quad v(t) = \frac{1}{C_{F\text{-}\alpha}} \int_0^t i(\tau)\left(d\tau\right)^\alpha = \frac{1}{C_{F\text{-}\alpha}} D_t^{-\alpha} i(t); \quad 0 < \alpha < 1 \tag{52}$$

Therefore, for constant current $i(t) = I_m$ the voltage is fractional integral of a constant I_m

$$v(t) = \frac{1}{C_{F\text{-}\alpha}} D_t^{-\alpha} i(t) = \frac{1}{C_{F\text{-}\alpha}} D_t^{-\alpha} I_m = \frac{I_m}{C_{F\text{-}\alpha}\Gamma(1+\alpha)} t^\alpha; \quad t \ge 0 \tag{53}$$

for $t \ge 0$ [12], [13], [37]; we used formula $D_t^{-n} t^m = \frac{\Gamma(m+1)}{\Gamma(m+1+n)} t^{m+n}$ in Eq. (53). Therefore the charge function $q(t)$ is $q(t) = c(t) * v(t)$ as follows

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right) \\
&= \left(\mathcal{L}\left\{C_\alpha\, t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{I_m \tfrac{1}{C_{F\text{-}\alpha}\Gamma(1+\alpha)}\, t^{\alpha}\right\}\right); \quad \tfrac{\Gamma(n+1)}{s^{n+1}} = \mathcal{L}\left\{t^n\right\} \\
&= \left(C_\alpha \Gamma(1-\alpha) s^{\alpha-1}\right)\left(I_m \tfrac{1}{C_{F\text{-}\alpha}\Gamma(1+\alpha)} \tfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right); \quad C_\alpha\Gamma(1-\alpha) = C_{F\text{-}\alpha} \\
&= \frac{I_m}{s^2}
\end{aligned}
\tag{54}
$$

Thus we have charge function by taking Laplace inverse of above Eq. (54) as

$$q(t) = I_m t; \quad t \geq 0 \tag{55}$$

The Eq. (55) can be expressed as $q(t) = I_m r(t)$, where $r(t)$ is unit ramp function at $t = 0$. That is $r(t) = t$ for $t \geq 0$ and $r(t) = 0$ for $t < 0$. This Eq. (55) is matter of fact is the current flowing through R and $C_{F\text{-}\alpha}$ is $i(t) = I_m$ for $t \geq 0$, and thus the charge will be

$$q(t) = \int_0^t i(\tau)d\tau = \int_0^t I_m d\tau = I_m t = I_m r(t); \quad t \geq 0 \tag{56}$$

For an ideal capacitor with $c(t) = C\delta(t)$ the voltage is $v(t) = \frac{1}{C}\int_0^t I_m d\tau = \frac{I_m}{C} t$ so the charge is $q(t) = c(t) * v(t)$ as follows

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right) \\
&= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\left\{I_m \tfrac{1}{C} t\right\}\right); \quad \tfrac{1}{s^2} = \mathcal{L}\{t\} = \mathcal{L}\{r(t)\} \\
&= (C)\left(I_m \tfrac{1}{Cs^2}\right) = \frac{I_m}{s^2}
\end{aligned}
\tag{57}
$$

$$q(t) = I_m t = I_m r(t); \quad t \geq 0$$

Thus in the case of constant current charging, we verified the validity of $q(t) = c(t) * v(t)$ as for any capacitor fractional or ideal loss less capacitor, the $q(t) = I_m t$; that is always integration of current function, i.e. $q(t) = \int_0^t i(\tau)d\tau$, for $t \geq 0$.

7. Charge storage $q(t)$ by step input current of a square pulse $i_{in}(t)$ to RC circuit with fractional capacitor and ideal capacitor

Let the square pulse of current be described as follows

$$i(t) = I_m u(t) - 2I_m u(t - T_c) + I_m u(t - T_d) \tag{58}$$

Where $u(t - T) = 1$ for $t \geq T$ and $u(t - T) = 0$ for $t < T$, i.e. unit step function at time $t = T$ .Then with identity $\mathcal{L}\{f(t - T)\} = e^{-sT}F(s)$ with $f(t - T) = 0$ for $t < T$; we write

$$I(s) = \mathcal{L}\{i(t)\} = \frac{I_m}{s} - \frac{2I_m}{s} e^{-sT_c} + \frac{I_m}{s} e^{-sT_d} \tag{59}$$

We have voltage across $Z_2(s) = \frac{1}{s^\alpha C_{F\text{-}\alpha}}$ as follows

$$
\begin{aligned}
V(s) &= Z_2(s)I(s) \\
&= \left(\frac{1}{C_{F\text{-}\alpha}s^\alpha}\right)\left(\frac{I_m}{s} - \frac{2I_m}{s} e^{-sT_c} + \frac{I_m}{s} e^{-sT_d}\right) = \frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}} - \frac{2I_m}{C_{F\text{-}\alpha}s^{\alpha+1}} e^{-sT_c} + \frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}} e^{-sT_d}
\end{aligned}
\tag{60}
$$

Then taking inverse Laplace of Eq. (60) we get voltage profile across $C_{F\text{-}\alpha}$ as

$$\begin{aligned} v(t) &= \frac{I_m t^\alpha}{C_{F\text{-}\alpha}\Gamma(\alpha+1)}u(t) - \frac{2I_m(t-T_c)^\alpha}{C_{F\text{-}\alpha}\Gamma(\alpha+1)}u(t-T_c) + \frac{I_m(t-T_d)^\alpha}{C_{F\text{-}\alpha}\Gamma(\alpha+1)}u(t-T_d) \\ &= \frac{I_m r_\alpha(t)}{C_{F\text{-}\alpha}\Gamma(\alpha+1)} - \frac{2I_m r_\alpha(t-T_c)}{C_{F\text{-}\alpha}\Gamma(\alpha+1)} + \frac{I_m r_\alpha(t-T_d)}{C_{F\text{-}\alpha}\Gamma(\alpha+1)} \end{aligned} \tag{61}$$

We note that $\mathcal{L}^{-1}\left\{e^{-sT}F(s)\right\} = f(t-T)$, where $f(t-T)=0$ for $t<T$. We can write explicitly $\mathcal{L}^{-1}\left\{e^{-sT}F(s)\right\} = f(t-T)u(t-T)$, where $u(t-T)$ is unit step function at $t=T$. This we used in Eq. (61). Also in Eq. (61) we define function r_α as $r_\alpha(t-\tau)=(t-\tau)^\alpha$ for $t\ge\tau$ and $r_\alpha(t-\tau)=0$ for $t<\tau$. The Laplace transform of r_α is, $\mathcal{L}\left\{r_\alpha(t)\right\} = \Gamma(\alpha+1)s^{-(\alpha+1)}$ therefore we have the identity $\mathcal{L}\left\{r_\alpha(t-\tau)\right\} = e^{-s\tau}\Gamma(\alpha+1)s^{-(\alpha+1)}$, which is used in Eq. (60) to get Eq. (61).

The charge function is $q(t)=c(t)*v(t)$ as follows, when the voltage profile $v(t)$; Eq. (60) is across a fractional capacitor $c(t)=C_\alpha t^{-\alpha}$. This $c(t)=C_\alpha t^{-\alpha}$ gets applied at $t=0$, $t=T_c$ and $t=T_d$; that is where there is sudden change of state of $v(t)$; (that is at points where the differentiability of $v(t)$ is lost). We write

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\left\{C_\alpha\, t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{v(t)\right\}\right); \quad \tfrac{\Gamma(n+1)}{s^{n+1}} = \mathcal{L}\left\{t^n\right\}; \quad C_{F\text{-}\alpha} = C_\alpha\Gamma(1-\alpha) \\ &= \left(C_\alpha\Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}} - \frac{2I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_c} + \frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_d}\right) \\ &= C_{F\text{-}\alpha}s^{\alpha-1}\frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}} - C_{F\text{-}\alpha}s^{\alpha-1}\frac{2I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_c} + \frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_d} \\ &= \frac{I_m}{s^2} - \frac{2I_m}{s^2}e^{-sT_c} + \frac{I_m}{s^2}e^{-sT_d} \\ q(t) &= I_m t - 2I_m(t-T_c)u(t-T_c) + I_m(t-T_d)u(t-T_d) \\ &= I_m r(t) - 2I_m r(t-T_c) + I_m r(t-T_d) \end{aligned} \tag{62}$$

In Eq. (62) we define unit ramp function r as $r(t-\tau)=(t-\tau)$ for $t\ge\tau$ and $r(t-\tau)=0$ for $t<\tau$. The Laplace transform of r is, $\mathcal{L}\left\{r(t)\right\} = s^{-2}$ therefore we have the identity $\mathcal{L}\left\{r(t-\tau)\right\} = e^{-s\tau}s^{-2}$, which is used in Eq. (62).

This shows verification of our formula $q(t)=c(t)*v(t)$. In similar way we can analyze the ideal loss less capacitor $c(t)=C\delta(t)$, for this wave form of current pulse.

8. Charging/discharging when R is zero ohms in RC circuit with voltage pulses

In this case Figure-1 has $Z_1(s)=0$. Therefore the voltage source directly gets connected to the fractional or ideal capacitor represented by impedance $Z_2(s)$. This case we have studied for step, ramp and sinusoidal voltage excitation in [40]. Here we take square wave case and triangular wave case, as extension of [40].

8-a) Charge storage $q(t)$ in a square wave voltage-on for time T_c and thereafter zero

The following excitation of a square wave pulse is applied to uncharged capacitor

$$v(t)=\begin{cases}0 & , \quad t<0\\ V_m & , \quad 0\le t\le T_c\\ 0 & , \quad t>T_c\end{cases} \tag{63}$$

We construct the above Eq. (63) excitation with $u(t-\tau)=1$ for $t\ge\tau$ and $u(t-\tau)=0$ for $t<\tau$; that is unit step function at $t=\tau$ as $v(t)=V_m u(t)-V_m u(t-T_c)$. The Laplace transform is

$$V(s)=\mathcal{L}\{V_m u(t)\}-\mathcal{L}\{V_m u(t-T_c)\}=\frac{V_m}{s}-\frac{V_m}{s}e^{-sT_c} \tag{64}$$

We used $\mathcal{L}\{f(t-t_d)\}=e^{-st_d}\mathcal{L}\{f(t)\}=e^{-st_d}F(s)$ with $f(t-t_d)=0$ for $t<t_d$ in above Eq. (64). When this voltage is applied to a time varying capacity function $c(t)=C_1\delta(t)$ i.e. ideal loss less capacitor we write from $q(t)=c(t)*v(t)$ the following

$$\begin{aligned}Q(s)=\mathcal{L}\{q(t)\}&=\left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right)=\left(C_1\right)\left(\frac{V_m}{s}-\frac{V_m}{s}e^{-sT_c}\right)\\ &=\frac{V_mC_1}{s}-e^{-sT_c}\frac{V_mC_1}{s}\end{aligned} \tag{65}$$

Taking inverse Laplace transform of Eq. (65) we get

$$q(t)=V_mC_1u(t)-V_mC_1u(t-T_c)=\begin{cases}0 & , \quad t<0\\ V_mC_1 & , \quad 0\le t\le T_c\\ 0 & , \quad t>T_c\end{cases} \tag{66}$$

Now when this square-wave is applied for a time varying capacity function as $c(t)=C_\alpha t^{-\alpha}$ i.e. for fractional capacitor we write from $q(t)=c(t)*v(t)$ the following

$$\begin{aligned}Q(s)=\mathcal{L}\{q(t)\}&=\left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right)=\left(\frac{C_\alpha\Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(\frac{V_m}{s}-\frac{V_m}{s}e^{-sT_c}\right)\\ &=\frac{V_mC_\alpha\Gamma(1-\alpha)}{s^{2-\alpha}}-e^{-sT_c}\frac{V_mC_\alpha\Gamma(1-\alpha)}{s^{2-\alpha}}\end{aligned} \tag{67}$$

Taking inverse Laplace Transform of above Eq. (67) we obtain

$$q(t)=\frac{V_mC_\alpha t^{1-\alpha}u(t)}{1-\alpha}-\frac{V_mC_\alpha(t-T_c)^{1-\alpha}u(t-T_c)}{1-\alpha}=\begin{cases}0 & , \quad t<0\\ \dfrac{V_mC_\alpha}{1-\alpha}t^{1-\alpha} & , \quad 0\le t\le T_c\\ \dfrac{V_mC_\alpha}{1-\alpha}t^{1-\alpha}-\dfrac{V_mC_\alpha}{1-\alpha}(t-T)^{1-\alpha} & , \quad t>T_c\end{cases} \tag{68}$$

The charge at $t=T_c$ is $q(T_c)=\frac{V_mC_\alpha T_c^{1-\alpha}}{1-\alpha}$, charge at $t=2T_c>T_c$ $q(2T_c)=\frac{V_mC_\alpha T_c^{1-\alpha}}{1-\alpha}(2^{1-\alpha}-1)$, charge at $t=3T_c$ is $q(3T_c)=\frac{V_mC_\alpha T_c^{1-\alpha}}{(1-\alpha)}(3^{1-\alpha}-2^{1-\alpha})$. We observe that for a fractional capacitor while the voltage is zero, after $t=T_c$, there still is charge holding, as compared with ideal capacitor Eq. (66). The current wave form is

$$i(t)=\frac{dq(t)}{dt}=V_mC_\alpha(t^{-\alpha}-(t-T_c)^{-\alpha})=\begin{cases}0 & , \quad t<0\\ V_mC_\alpha t^{-\alpha} & , \quad 0\le t\le T_c\\ V_mC_\alpha(t^{-\alpha}-(t-T_c)^{-\alpha}) & , \quad t>T_c\end{cases} \tag{69}$$

8-b) Charge storage by voltage as triangular input of voltage

The following excitation of a square wave pulse is applied to uncharged capacitor

$$v(t)=\begin{cases}0 & , \quad t<0\\ \frac{V_m}{T}t & , \quad 0\le t\le T\\ \frac{V_m}{T}t-\frac{2V_m}{T}(t-T) & , \quad T\le t\le 2T\\ 0 & , \quad t\ge 2T\end{cases} \tag{70}$$

We can write the above excitation as $v(t)=\left(V_m/T\right)r(t)-\left(2V_m/T\right)r(t-T)$ for $0\le t\le 2T$. With $r(t)$ unit ramp at $t=0$ and is zero for $t<0$ and $r(t-T)$ as unit ramp at $t=T$ and zero at $t<T$. With this applied to a ideal capacitor, with $c(t)=C_1\delta(t)$, we get the following by application of $q(t)=c(t)*v(t)$

$$\begin{aligned}Q(s)=\mathcal{L}\{q(t)\}&=\left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right)=\left(C_1\right)\left(\frac{V_m}{Ts^2}-\frac{2V_m}{Ts^2}e^{-sT}\right)\\ &=\frac{V_mC_1}{Ts^2}-e^{-sT}\frac{2V_mC_1}{Ts^2}\end{aligned} \tag{71}$$

Doing inverse Laplace transform of Eq. (71) we get

$$q(t)=\frac{V_mC_1}{T}r(t)-\frac{2V_mC_1}{T}r(t-T)=\begin{cases}0 & , \quad t<0\\ \frac{V_mC_1}{T}t & , \quad 0\le t\le T\\ \frac{V_mC_1}{T}t-\frac{2V_mC_1}{T}(t-T) & , \quad T\le t\le 2T\\ 0 & , \quad t\ge 2T\end{cases} \tag{72}$$

Current is got by differentiation of above Eq. (72)

$$i(t)=\frac{dq(t)}{dt}=\frac{V_mC_1}{T}u(t)-\frac{2V_mC_1}{T}u(t-T)=\begin{cases}0 & , \quad t<0\\ \frac{V_mC_1}{T} & , \quad 0\le t\le T\\ -\frac{V_mC_1}{T} & , \quad T\le t\le 2T\\ 0 & , \quad t\ge 2T\end{cases} \tag{73}$$

We take a fractional capacitor and do the following as done above as in Eq. (71) by applying the formula $q(t)=c(t)*v(t)$

$$\begin{aligned}Q(s)=\mathcal{L}\{q(t)\}=\left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right)&=\left(\frac{C_\alpha\Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(\frac{V_m}{Ts^2}-\frac{2V_m}{Ts^2}e^{-sT}\right)\\ &=\frac{V_mC_\alpha\Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}}-e^{-sT}\frac{2V_mC_\alpha\Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}}\end{aligned} \tag{74}$$

We take inverse Laplace transform of above Eq. (74) with following definition of a function $r_m(t-\tau)$ defined as

$$r_m(t-\tau)=\begin{cases}(t-\tau)^m, & t\ge\tau\\ 0, & t<\tau\end{cases}; \qquad \mathcal{L}\{r_m(t-\tau)\}=\frac{e^{-s\tau}\Gamma(1+m)}{s^{1+m}} \tag{75}$$

Thus the charge function $q(t)$ is following from Eq. (74) and Eq. (75)

$$\begin{aligned}q(t)&=\mathcal{L}^{-1}\left\{\frac{V_mC_\alpha\Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}}\right\}-\mathcal{L}^{-1}\left\{e^{-sT}\frac{2V_mC_\alpha\Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}}\right\}\\ &=\frac{V_mC_\alpha\Gamma(1-\alpha)}{T\Gamma(3-\alpha)}r_{2-\alpha}(t)-\frac{2V_mC_\alpha\Gamma(1-\alpha)}{T\Gamma(3-\alpha)}r_{2-\alpha}(t-T)\\ &=\frac{V_mC_\alpha}{T(1-\alpha)(2-\alpha)}r_{2-\alpha}(t)-\frac{2V_mC_\alpha}{T(1-\alpha)(2-\alpha)}r_{2-\alpha}(t-T)\end{aligned} \tag{76}$$

We used $\Gamma(1+m)=m\Gamma(m)$ in above Eq. (76). We re-write above Eq. (76) as using Eq. (75)

$$q(t)=\frac{V_mC_\alpha r_{2-\alpha}(t)}{T(1-\alpha)(2-\alpha)}-\frac{2V_mC_\alpha r_{2-\alpha}(t-T)}{T(1-\alpha)(2-\alpha)}=\begin{cases}0 & , \quad t<0\\ \frac{V_mC_\alpha t^{2-\alpha}}{T(1-\alpha)(2-\alpha)} & , \quad 0\le t\le T\\ \frac{V_mC_\alpha t^{2-\alpha}}{T(1-\alpha)(2-\alpha)}-\frac{2V_mC_\alpha(t-T)^{2-\alpha}}{T(1-\alpha)(2-\alpha)} & , \quad T\le t\le 2T\end{cases} \tag{77}$$

We have at $t = T$, $q(T) = \frac{V_m C_\alpha T^{1-\alpha}}{(1-\alpha)(2-\alpha)}$ at $t = 2T$, $q(2T) = \frac{V_m C_\alpha T^{1-\alpha}(2^{2-\alpha}-2)}{(1-\alpha)(2-\alpha)}$. We observe that at $t = 2T$, the voltage is zero, but we have charge as non-zero. With $\alpha \approx 1$, we get $q(2T) \approx 0$, Eq. (72) that we have analyzed for an ideal loss less capacitor. Differentiating the above we write current as

$$i(t) = \frac{dq(t)}{dt} = \begin{cases} 0 & , \quad t < 0 \\ \frac{V_m C_\alpha t^{1-\alpha}}{T(1-\alpha)} & , \quad 0 \le t \le T \\ \frac{V_m C_\alpha t^{1-\alpha}}{T(1-\alpha)} - \frac{2V_m C_\alpha (t-T)^{1-\alpha}}{T(1-\alpha)} & , \quad T \le t \le 2T \end{cases} \tag{77}$$

Thus we verified $q(t) = c(t) * v(t)$ the formula in RC circuits with charging resistance as zero, for triangular and square pulse of voltage excitation.

9. Conclusions

We have applied the new formula of charge storage i.e. via convolution operation $q(t) = c(t) * v(t)$, of time varying capacity function and voltage stress for a fractional capacitor and ideal loss-less capacitor; for verification in RC charging/discharging circuit; with dc voltage and current sources. This new formulation is different to the earlier used formula of multiplication of capacity and voltage function. The circuit analysis that we described for each cases verifies this formula. Thus this new formulation of stored charge via convolution operation is applicable, and can be taken as general formula applicable to fractional capacitor as well as ideal capacitor.

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