Magic Polygons and Degenerated Magic Polygons: Characterization and Properties

Original Research Article

Abstract

In this work we define Magic Polygons $P(n, k)$ and Degenerated Magic Polygons $D(n, k)$ and we obtain their main properties, such as the magic sum and the value corresponding to the root vertex. The existence of magic polygons $P(n, k)$ and degenerated magic polygons $D(n, k)$ are discussed for certain values of n and k.

Keywords: Combinatorics; Magic Polygons; Degenerated Magic Polygons

2010 Mathematics Subject Classification: 65R05; 00A08; 97A20

1 Introduction

Magic Squares have been known for a long time in different people and cultures that, sometimes, has attributed mystic meanings [Andress (1960); Cammann (1960); Rosser (1939)]. In addiction to being used for recreational purposes, we can now find applications for magic squares in Physics, in Computer Science, in Image Processing and in Criptography [Chu, Drury, Styan and Trenkler (2011); Ganapathy and Mani (2009); Loly (2007)], among others. In this way, have been developed several methods to construct magic squares that satisfies some particular properties and some generalization have been created, as we can see in [Chan (2014); Kim and Yoo (2008); Mattingly (2000); Nordgren (2012); Ollerenshaw and Brée (1998); Planck (1919)].

We can see in [Jakicic and Bouchat (2018)] a generalization of the same idea of the representation magic squares of order 3 using vertices, midpoints and the geometric center of a square, where we can find some properties and the condiction of existence of magic polygons and a construction for magic square of order 4, for each n even. In [Pickover (2003)] others similar structures was proposed.

In this work we will cover a generalization of the work proposed in [Jakicic and Bouchat (2018)] and we show that some valid properties of magic polygons for a given order sometimes are not valid in general. In addition, we will introduce another class of polygonal structure known as the class of degenerate magic polygons.

2 Magic Polygons $P(n,k)$

Let Ω be a set of $\frac{k}{2}$ regular polygons on plane with n sides and corresponding parallel sides and centered in a central point C.

A magic polygon $P(n,k)$ of n sides and order $k+1$ is a set of $\frac{k^2n}{2}+1$ points satisfying the following conditions:

(i) Points of magic polygon are labeled by distict values from 1 to $\frac{k^2n}{2}+1$;

(ii) One point of a magic polygon is the central point C;

(iii) $\frac{kn}{2}$ points of magic polygon are vertices of the $\frac{k}{2}$ regular polygons of Ω;

(iv) The magic polygon has $k-1$ intermediate points on each edge of regular polygons in Ω, which gives a total of $\dfrac{(k-1)nk}{2}$ intermediate points.

(v) Segments with diametrically opposite ends of the larger polygon of Ω intersecting the central vertex contain $k+1$ points of the magic polygon;

(vi) Segments with ends at two adjacent vertices of a polygon of Ω contains $k+1$ points of the magic polygon;

(vii) The sum of values corresponding to the $k+1$ points on each segment defined in (iv) and (v) is a fixed value u, called of **magic sum.**

In Figures 1 e 2 we can see examples of Magic Polygons $P(4,4)$ and $P(8,2)$, respectively.

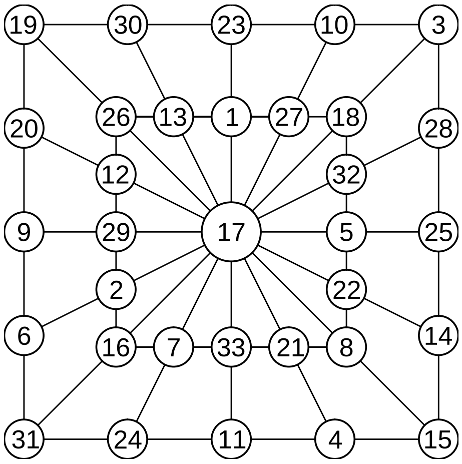

Figure 1: Example of Magic Polygon $P(4,4)$

Theorem 2.1. *In a magic polygon* $P(n,k)$, *we have the following properties:*

(i) the magic sum is $(k+1)\frac{k^2n+4}{4}$;

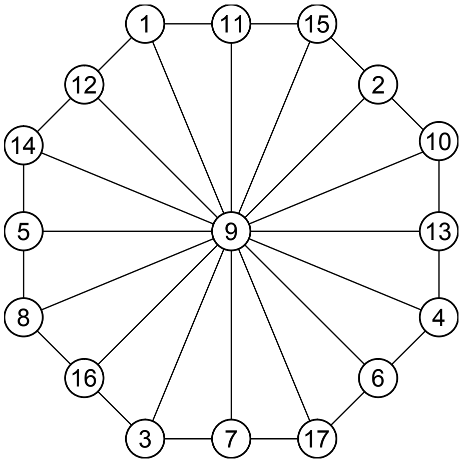

Figure 2: Example of Magic Polygon $P(8,2)$

(ii) the value corresponding to the root vertex is $c = \frac{k^2n+4}{4}$;

(iii) the sum S_j *of the values representing to* j*-th points partitioning each edge on magic polygon chosen clockwise is*

$$S_j = \frac{kn[k^2n+4]}{8}$$

Proof. Let $x_{(t-1)nk+(i-1)k+j}$ be the value correspondig to the point $P_{(t-1)nk+(i-1)k+j}$, j-th point of the i-th edge of the i-th regular polygon.

Each point $P_{(t-1)nk+(i-1)k+j}$ of the magic polygon is labeled for a number

$$x_{(t-1)nk+(i-1)k+j}$$

where $t \in \{1, 2, \cdots, \frac{k}{2}\}, i \in \{1, 2, \cdots, n\}$ and $j \in \{1, \cdots, k\}$ and the root vertice is labeled by the number c, then we have

$$\frac{k^2n}{2} + 1 \tag{2.1}$$

point in the magic polygon.

The sum of values correspondig points on the i-th edge of t-th regular polygon is given by

$$x_{(t-1)nk+(i-1)k+1} + x_{(t-1)nk+(i-1)k+2} + \cdots + x_{(t-1)nk+ik} + x_{(t-1)nk+ik+1} = u, \tag{2.2}$$

where u is the magic sum.

The sum of values correspondig points on the segments determined by the points P_{j+ik} and $P_{j+(i-1)k+\frac{kn}{2}}$ is

$$\sum_{t=1}^{\frac{k}{2}} x_{(t-1)nk+(i-1)k+j} + \sum_{t=1}^{\frac{k}{2}} x_{(t-1)nk+(i-1)k+j+\frac{kn}{2}} + c = u, \tag{2.3}$$

where c is the value assigned to the root vertice and $j + (i-1)k \leq \frac{kn}{2} - 1$.

Let

$$S_j = \sum_{i=1}^{n} \sum_{t=1}^{\frac{k}{2}} x_{(t-1)nk+(i-1)k+j} \tag{2.4}$$

for $j \in \{1, 2, \cdots, k\}$.

By (2.3) and (2.4),

$$S_j + \frac{nc}{2} = \frac{nu}{2}, \tag{2.5}$$

which implies that

$$S_j = \frac{n(u-c)}{2} \tag{2.6}$$

Adding equations involving points on perimeters of the polygons, we obtain

$$2S_1 + S_2 + \cdots + S_k = \frac{knu}{2}. \tag{2.7}$$

Adding equations involving the root vertex of the magic polygon, we obtain

$$S_1 + S_2 + \cdots + S_k + \frac{knc}{2} = \frac{knu}{2} \tag{2.8}$$

which implies that

$$S_1 + S_2 + \cdots + S_k = \frac{kn(u-c)}{2} \tag{2.9}$$

Subtracting (2.9) from (2.7), we have

$$S_1 = \frac{knc}{2} \tag{2.10}$$

By (2.6) and (2.10),

$$\frac{n(u-c)}{2} = \frac{knc}{2}. \tag{2.11}$$

Therefore

$$u = (k+1)c \tag{2.12}$$

As values corresponding to the points of the magic polygon are distinct values in $\{1, 2, \cdots, \frac{k^2n}{2} + 1\}$, it follows that

$$S_1 + S_2 + \cdots + S_k + c = \sum_{i=1}^{\frac{k^2n}{2}+1} i = \frac{\left(\frac{k^2n}{2}+1\right)\left(\frac{k^2n}{2}+2\right)}{2} \tag{2.13}$$

By (2.9) and (2.13), it follows that

$$\frac{kn(u-c)}{2} + c = \frac{\left(\frac{k^2n}{2}+1\right)\left(\frac{k^2n}{2}+2\right)}{2} \tag{2.14}$$

By (2.12) and (2.14), it follows that

$$\left(\frac{k^2n}{2}+1\right)c = \frac{\left(\frac{k^2n}{2}+1\right)\left(\frac{k^2n}{2}+2\right)}{2}. \tag{2.15}$$

Therefore

$$c = \frac{k^2n+4}{4} \tag{2.16}$$

Moreover,

$$S_j \overset{(2.6)}{=} \frac{n(u-c)}{2} \overset{(2.12)}{=} \frac{nkc}{2} \overset{(2.16)}{=} \frac{nk\left[k^2n+4\right]}{8} \tag{2.17}$$

□

2.1 Constructing examples for $P(n,4)$

The following result affords us a construction for $P(n,4)$, provided that some conditions are satisfied:

Theorem 2.2. *Let two regular polygons with n sides of distinct sizes centered on a central point C whose sides are partitioned into 4 segments by points such that $2n$ segments passing through the center point and cutting the larger polygon intercept the sides of the polygons at these points, satisfying the following conditions:*

(i) *Each point $P_{4(t-1)n+4(i-1)+j}$ on partitions is labeled by $x_{4(t-1)n+4(i-1)+j} \in \{1,2,\cdots,8n+1\}$;*

(ii) *a central point is labeled by $c = 4n+1$;*

(iii) $x_{4(i-1)+j} + x_{4n+4(i-1)+j} = 2(4n+1)$;

(iv) $x_{4(t-1)n+4(i_1-1)+j_1} + x_{4(t-1)n+4(i_2-1)+j_2} \neq 2(4n+1)$, *if* $(i_1,j_1) \neq (i_2,j_2)$;

(v) $x_{4(i-1)+1} + x_{4(i-1)+2} + x_{4(i-1)+3} + x_{4(i-1)+4} + x_{4(i-1)+5} = 5(4n+1)$;
where $t \in \{1,2\}, i_1, i_2, i \in \{1,2,\cdots,n\}$ *e* $j_1, j_2, j \in \{1,2,3,4\}$.

In this conditions, we obtain $8n+1$ points that define a magic polygon $P(n,4)$.

Proof. Let $i \in \{1,2,\cdots,n-1\}$.

By (iii), we have $x_{4(i-1)+j} + x_{4n+4(i-1)+j} = 2(4n+1)$ for $j \in \{1,2,3,4\}$.

Therefore

$$x_{4(i-1)+1} + x_{4n+4(i-1)+1} + \cdots + x_{4(i-1)+4} + x_{4n+4(i-1)+4} + x_{4(i-1)+5} + x_{4n+4(i-1)+5} = 5 \cdot 2(4n+1) \tag{2.18}$$

By (2.18) and (v), we get

$$5(4n+1) + x_{4n+4(i-1)+1} + x_{4n+4(i-1)+2} + x_{4n+4(i-1)+3} + x_{4n+4(i-1)+4} + x_{4n+4(i-1)+5} = 10(4n+1) \tag{2.19}$$

Thus

$$x_{4n+4(i-1)+1} + x_{4n+4(i-1)+2} + x_{4n+4(i-1)+3} + x_{4n+4(i-1)+4} + x_{4n+4(i-1)+5} = 5(4n+1) \tag{2.20}$$

Consequently, the sum of values corresponding to the points on edges of the polygons is $5(4n+1)$.

Therefore, by (iii) and (ii), we get

$$x_{4(i-1)+j} + x_{4n+4(i-1)+j} + c + x_{4(i-1)+2n+j} + x_{4n+4(i-1)+2n+j} = 2(4n+1) + 4n+1 + 2(4n+1) = 5(4n+1) \tag{2.21}$$

Consequently, the sum of values corresponding to the points on the segments lying by central point is also $5(4n+1)$.

Hence, by definition, we get $P(n,4)$. ☐

In Figure 3 we have an example of Magic Polygon $P(4,4)$ constructed by Theorem 2.2. An example of Magic Polygon $P(4,4)$ that can not be obtained by this construction can be seen in the figure 1.

2.2 The particular case $P(n,2)$

Although it is a particular case of the general case seen in the previous section, we will use another reasoning for a demonstration of the properties of the magic polygons $P(n,2)$.

A **magic polygon** $P(n,2)$ is formed by $2n+1$ points, consisting of the vertices, the midpoints of the edges, and the geometric center of a regular polygon of n sides, which are labeled by numerical values from 1 to $2n+1$, so that the sum of the values assigned to any three collinear pointes is constant, called **magic sum.**

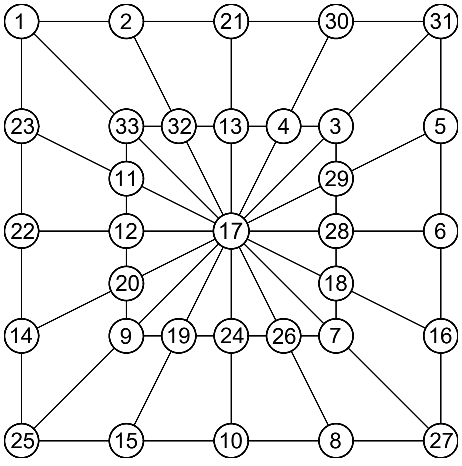

Figure 3: Example of Magic Polygon $P(4,4)$ using construction

Theorem 2.3. *If n is even, then a magic polygon $P(n,2)$ has the following properties:*

(i) the magic sum is é $3(n+1)$;

(ii) the value corresponding to the central point is é $n+1$;

(iii) the sum S_1 of the values assigned to vertices of the magic polygon and the sum S_2 of the values assigned to midpoints of edges of the magic polygon satisfies $S_1 = S_2 = n(n+1)$.

Proof. As a polygon with n sides has n vertex and n midpoints, including a central point, we obtain $2n+1$ points labeled using distinct integers numbers from 1 to $2n+1$.

Let x_n be the value assign to vertex V_n and let be y_n the values assign to the midpoint M_n whose endpoints are V_n and V_1. For each $i \in \{1, \cdots, n-1\}$, let x_i be the value assigned to the vertex V_i and let y_i be the value assigned to midpoint M_i whose endpoints are V_i e V_{i+1}.

For each $i \in \{1, \cdots, \frac{n}{2}\}$, let x_i^* be the value assigned to the point diametrically opposite to vertice V_i and y_i^* be the value assigned to the point diametrically opposite to the midpoint M_i.

Denoting by c the value assigned by the central point C of the magic polygon and denoting by u the value of the magic sum, we obtain two sets of equations that define a magic polygon with n sides, for an even number n: the equations involving points on the same side and equations involving points on simmetry axes of the magic polygon.

Analyzing equation involving points on the same side of a magic polygon, we obtain

$$\begin{cases} x_1 + y_1 + x_2 = u \\ x_2 + y_2 + x_3 = u \\ x_3 + y_3 + x_4 = u \\ \quad\vdots \\ x_n + y_n + x_1 = u \end{cases} \tag{2.22}$$

Analyzing equations involving points on the same segment in the definition of a magic polygon, we obtain

$$\begin{cases} x_1 + c + x_{1^*} = u \\ y_1 + c + y_{1^*} = u \\ \vdots \\ x_{\frac{n}{2}} + c + x_{\frac{n}{2}^*} = u \\ y_{\frac{n}{2}} + c + y_{\frac{n}{2}^*} = u \end{cases} \tag{2.23}$$

Let

$$S_1 = \sum_{i=1}^{n} x_i \text{ and } S_2 = \sum_{i=1}^{n} y_i$$

Adding the equations of (2.22), we get

$$2S_1 + S_2 = nu \tag{2.24}$$

Adding the equations of (2.23), we get

$$S_1 + S_2 + nc = nu \tag{2.25}$$

Subtracting (2.25) from (2.24), we get

$$S_1 = nc \tag{2.26}$$

By (2.26) and (2.25), we get

$$S_2 = nu - 2nc \tag{2.27}$$

As the values assigned to the n midpoints of the magic polygon with n sides are distinct values of the set $\{1, \cdots, 2n+1\}$ and the sum of this values is S_2, then

$$\sum_{i=1}^{n} i \le S_2 \le \sum_{i=n+2}^{2n+1} i \tag{2.28}$$

In addition,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{2.29}$$

and

$$\sum_{i=n+2}^{2n+1} i = \frac{3n(n+1)}{2} \tag{2.30}$$

Therefore, by (2.28), (2.29) and (2.30), we get

$$\frac{n(n+1)}{2} \le S_2 \le \frac{3n(n+1)}{2} \tag{2.31}$$

By (2.26) and (2.27), we obtain

$$S_1 + S_2 + c = nu - c(n-1) \tag{2.32}$$

Moreover,

$$S_1 + S_2 + c = \sum_{i=1}^{2n+1} i = \frac{(2n+1)(2n+2)}{2} = (2n+1)(n+1). \tag{2.33}$$

Therefore, by (2.32) and (2.33), we obtain the follow diophantine equation

$$nu - c(n-1) = (2n+1)(n+1) \tag{2.34}$$

whose general solution is

$$(u,c) = ((2n+1)(n+1) + (n-1)t, (2n+1)(n+1) + nt)\,, \forall t \in \mathbb{Z}. \tag{2.35}$$

By (2.35) and (2.27), we obtain

$$\begin{aligned} S_2 &= nu - 2nc \\ &= n\left[(2n+1)(n+1) + (n-1)t\right] - 2n\left[(2n+1)(n+1) + nt\right] \\ &= -n(2n+1)(n+1) - n(n+1)t \\ &= -(n+1)n(2n+1+t) \end{aligned} \tag{2.36}$$

By (2.36) and (2.31), we get

$$\frac{n(n+1)}{2} \leq -(n+1)n(2n+1+t) \leq \frac{3n(n+1)}{2}. \tag{2.37}$$

Simplyfing (2.37), we obtain

$$\frac{1}{2} \leq -(2n+1+t) \leq \frac{3}{2} \tag{2.38}$$

It follows from $-(2n+1+t) \in \mathbb{Z}$ and (2.38) that $-(2n+1+t) = 1$; hence

$$t = -2(n+1) \tag{2.39}$$

By (2.39) and (2.35),

$$\begin{aligned} (u,c) &= ((2n+1)(n+1) + (n+1)t, (2n+1)(n+1) + nt) \\ &= ((2n+1)(n+1) - 2(n+1)(n-1), (2n+1)(n+1) - 2n(n+1)) \\ &= (3(n+1), (n+1)) \end{aligned} \tag{2.40}$$

Therefore, the magic sum is $u = 3(n+1)$ and the value assigned to the central point is $c = n+1$. Replacing u e c from (2.40) in (2.26) and (2.27), we obtain

$$S_1 = S_2 = n(n+1). \tag{2.41}$$

□

Theorem 2.4. *If n is odd, then there is no magic polygon $P(n,2)$.*

Proof. Let n be odd and suppose that there is magic polygons with n sides.

If $x_1, \cdots, x_n$ are integer numbers corresponding vertices and $y_1, \cdots, y_n$, are integer numbers corresponding midpoints of the magic polygon, where y_i is the midpoint between the vertices x_i and x_{i+1}, we have

$$x_i + y_i + x_{i+1} = u \tag{2.42}$$

and

$$x_i + y_{m+i-1} + c = u \tag{2.43}$$

where $m = \frac{n+1}{2}, c$ is the value assigned to the root vertex of the magic polygon and $x_a = x_b$ or $y_a = y_b$ if $a \equiv b \bmod n$.

By (2.43), we obtain

$$x_i = u - c - y_{m+i-1} \tag{2.44}$$

Substituting (2.26) into (2.42), we obtain

$$y_i = 2c - u + y_{m+i-1} + y_{m+i} \tag{2.45}$$

and

$$y_{i+1} = 2c - u + y_{m+i} + y_{m+i+1} \tag{2.46}$$

By (2.45) and (2.46), we obtain

$$\begin{aligned} y_{i+1} - y_i &= y_{m+i+1} - y_{m+i-1} \\ &= y_{m+i+1} - y_{m+i} + y_{m+i} - y_{m+i-1} \\ &= y_{m+(m+i)+1} - y_{m+(m+i)-1} + y_{m+(m+i-1)+1} - y_{m+(m+i-1)-1} \\ &= y_{i+2} - y_i + y_{i+1} - y_{i-1}, \end{aligned} \tag{2.47}$$

Taking $n = 3$ and $i = 1$ in the first equality of (2.47), we obtain $y_2 - y_1 = y_1 - y_2$, which implies $y_1 = y_2$.

Taking $n \geq 5$ and $i = 2$ in (2.47), we get $y_1 = y_4$.

Therefore, we can not have a magic polygons with n sides, for n odd.

□

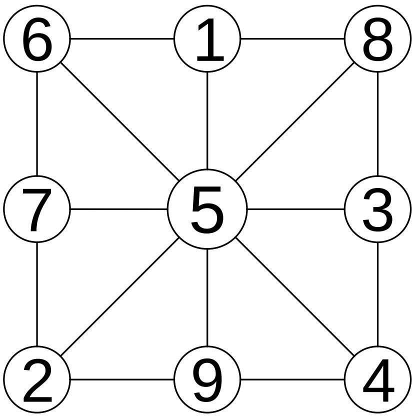

Figure 4: Example of Magic Polygon $P(4, 2)$

Theorem 2.5. *If n is an even number greater than or equal to 4, then there is a magic polygon $P(n, 2)$.*

Proof. Figure 4 shows the existence of Magic Polygons $P(4, 2)$. Let n be even greater than or equal to 6 and consider a regular polygon with n sides whose perimeter is indicated by the sequence of n vertices clockwise $V_1, V_2, \cdots, V_n$. Thus, if for each $j \in \{1, \cdots, n\}, x_j$ is the value assigned to vertex V_j and y_j is the value assigned to midpoint of the side whose ends are the vertices V_j and V_{j+1} of the magic polygon, then we obtain a magic polygon with n sides such that, for $i \in \{1, 2, \cdots, \frac{n}{2} - 2\}$, the values assigned to vertices of the magic polygon satisfy

$$\begin{cases} x_i &= \begin{cases} i + 1, & \text{if } i \text{ is odd} \\ n + i + 2, & \text{if } i \text{ is even} \end{cases} \\ \\ x_{i+\frac{n}{2}} &= \begin{cases} 2(n+1) - i - 1, & \text{if } i \text{ is odd} \\ n - i, & \text{if } i \text{ is even} \end{cases} \\ x_{\frac{n}{2}-1} &= n + 3 \\ x_{\frac{n}{2}} &= 1 \end{cases}$$

and the values assigned to midpoints of magic polygon satisfy

$$y_j = \begin{cases} 2(n+1) - x_j - x_{j+1}, & \text{if } j \in \{1,2,3,\cdots,n-1\} \\ 2(n+1) - x_1 - x_n, & \text{if } j = n \end{cases}$$

The verification that this construction defines a magic polygon can be seen in Jakicic and Bouchat (2018). □

In Figure 5 we have an example of Magic Polygon $P(4,2)$ using construction in the proof of the Theorem 2.5.

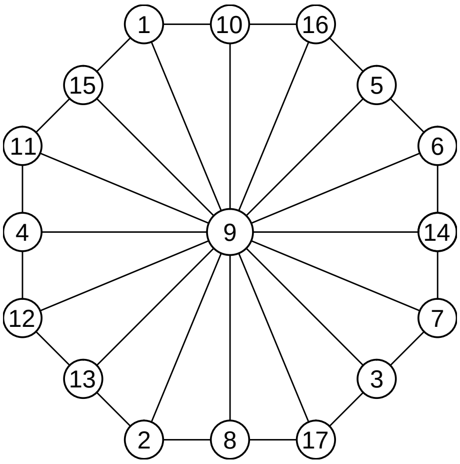

Figure 5: Magic Polygon $P(4,2)$ using construction

Proposition 2.1. *There are no magic polygons whose values assigned to all vertices have odd parity.*

Proof. If the values assigned to all vertices are odd numbers, then, by definition of Magic Polygon, the values assigned to midpoints are even numbers. This contradicts the fact that the magic sum is an odd number, by Theorems 2.3 and 2.4. □

Corollary 2.6. *There is no Magic Polygon $P(n,2)$ whose values assigned to all midpoints are even numbers.*

Proof. If all values assigned to midpoints of the Magic Polygon are even numbers, then, by definition of Magic Polygon, the values assigned to vertices of the Magic Polygon are odd numbers. This contradicts the Proposition 2.1. □

3 Degenerated Magic Polygons $D(n,k)$

Let Ω be a set of k regular polygons with distinct sizes and with a common vertex C, called the root vertex.

A degenerated magic polygon $D(n,k)$ with n sides and order $k+1$ is a set of $k^2(n-2)+k+1$ points that satisfies the following conditions:

(i) $k^2(n-2)+k+1$ points of degenerated magic polygon are assigned by distinct numbers from 1 to $k^2(n-2)+k+1$;

(ii) One point is the root vertex C;

(iii) $kn-k+1$ points are the vertices of k regular polygons of Ω;

(iv) each of the edges of the regular polygons not adjacent to the root vertices of Ω have $k-1$ intermediate points of the magic polygon beyond the vertices of the polygons, which gives a total of $(n-2)(k-1)k$ intermediate points;

(v) Segments with one end in points at the border of the larger polygon of Ω and other end on root vertex have $k+1$ points of degenerated magic polygon;

(vi) Segments with ends on adjacent vertices of polygons of Ω that do not contain the root vertex have $k+1$ points of degenerated magic polygon;

(vii) The sum of values assigned to the $k+1$ points of the degenerated magic polygon in each of the segments defined in (v) and (vi) is a fixed value u, called the magic sum.

Figures 6 and 7 illustrate the existence of Degenerated Magic Polygons.

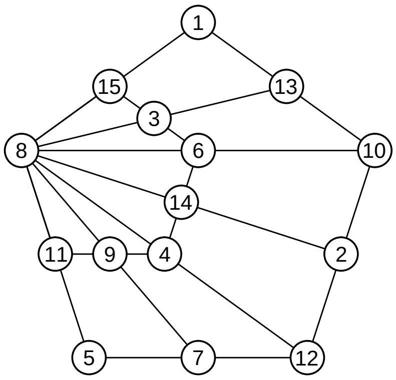

Figure 6: Example of Degenerated Magic Polygon $D(5,2)$

Theorem 3.1. *A degenerated magic polygon $D(n,k)$ has the following properties:*

(i) the magic sum is $(k+1)\frac{k^2(n-2)+k+2}{2}$;

(ii) the value that corresponds to the root vertex is $c=\frac{k^2(n-2)+k+2}{2}$;

(iii) the sum S_j of the values assigns to the j-th points on the edges in the representation of the degenerated magic polygon satisfies

$$S_j=(n-2)k\frac{k^2(n-2)+k+2}{2}$$

Proof. Let $P_{(t-1)k(n-2)+k(i-1)+j}$ be the j-th point of the i-th edge of the t-th largest polygon that represents the degenerated magic polygon $D(n,k)$, considering the clockwise direction, let

$$x_{(t-1)k(n-2)+k(i-1)+j}$$

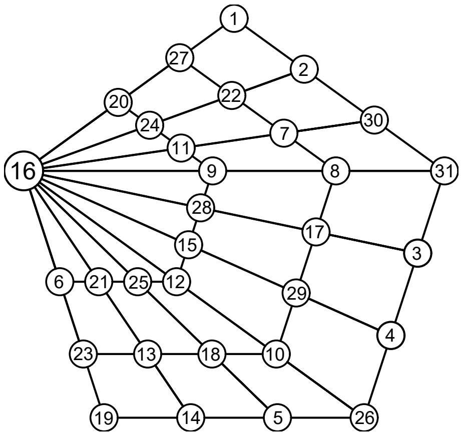

Figure 7: Example of Degenerated Magic Polygon $D(5,3)$

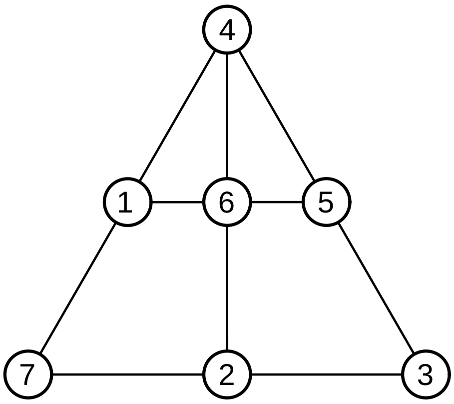

Figure 8: Example of Degenerated Magic Polygon $D(3,2)$

be the value assigned to the point $P_{(t-1)k(n-2)+k(i-1)+j}$,

$$x_{tk(n-2)+1}$$

be the value assigned to the point $P_{tk(n-2)+1}$, where $j,t \in \{1,2,\cdots,k\}$ and $i \in \{1,2,\cdots,(n-2)\}$; the root vertex is labeled by the number c. So we have

$$k^2(n-2)+k+1 \tag{3.1}$$

points in the degenerated magic polygon.

The equations involving segments that don't contain the root vertex are

$$\begin{cases} x_{(t-1)k(n-2)+1} + x_{(t-1)k(n-2)+2} + \cdots + x_{(t-1)k(n-2)+k+1} = u \\ x_{(t-1)k(n-2)+k+1} + x_{(t-1)k(n-2)+k+2} + \cdots + x_{(t-1)k(n-2)+2k+1} = u \\ \vdots \\ x_{(t-1)k(n-2)+(i-1)k+1} + x_{(t-1)k(n-2)+(i-1)k+2} + \cdots + x_{(t-1)k(n-2)+ik+1} = u \\ \vdots \\ x_{(t-1)k(n-2)+(n-3)k+1} + x_{(t-1)k(n-2)+(n-3)k+2} + \cdots + x_{(t-1)k(n-2)+(n-2)k+1} = u \end{cases} \tag{3.2}$$

The equations involving segments containing the root vertex are

$$\sum_{t=1}^{k} (x_{(t-1)k(n-2)+(i-1)k+j}) + c = u, \tag{3.3}$$

where $i \in \{1, 2, \cdots, n-2\}$ e $j \in \{1, 2, \cdots, k\}$ and

$$\sum_{t=1}^{k} x_{tk(n-2)+1} + c = u \tag{3.4}$$

Let

$$S_1 = \sum_{i=1}^{n-1} \sum_{t=1}^{k} x_{(t-1)k(n-2)+k(i-1)+1} = (n-1)(u-c) \tag{3.5}$$

and

$$S_j = \sum_{i=1}^{n-2} \sum_{t=1}^{k} x_{(t-1)k(n-2)+k(i-1)+j} = (n-2)(u-c) \tag{3.6}$$

for $j \in \{2, \cdots, k\}$.

By adding equations involving perimeters in the degenerated magic polygon, we obtain

$$2S_1 + S_2 + \cdots + S_k + 2c = [k(n-2)+2]u, \tag{3.7}$$

or, equivalently,

$$2S_1 + S_2 + \cdots + S_k = [k(n-2)+2]u - 2c \tag{3.8}$$

By adding equations involving the root vertex of the degenerated magic polygon, we obtain

$$S_1 + S_2 + \cdots + S_k + (k(n-2)+1)c = (k(n-2)+1)u \tag{3.9}$$

or, equivalently,

$$S_1 + S_2 + \cdots + S_k = (k(n-2)+1)(u-c) \tag{3.10}$$

By subtracting (3.10) from (3.8), we obtain

$$S_1 = u + (k(n-2)-1)c \tag{3.11}$$

By (3.5) and (3.11),

$$(n-1)(u-c) = u + (k(n-2)-1)c. \tag{3.12}$$

Hence

$$u = (k+1)c \tag{3.13}$$

As the values assigned to the points of the degenerated magic polygon are distinct values of the set $\{1, 2, \cdots, k^2(n-2)+k+1\}$, we obtain

$$S_1 + S_2 + \cdots + S_k + c = \sum_{i=1}^{k^2(n-2)+k+1} i = \frac{(k^2(n-2)+k+1)(k^2(n-2)+k+2)}{2} \tag{3.14}$$

By (3.10) and (3.14),

$$[k(n-2)+1](u-c)+c=\frac{(k^2(n-2)+k+1)(k^2(n-2)+k+2)}{2} \tag{3.15}$$

By (3.13) and (3.15),

$$[k^2(n-2)+k+1]c=\frac{(k^2(n-2)+k+1)(k^2(n-2)+k+2)}{2}, \tag{3.16}$$

that implies

$$c=\frac{k^2(n-2)+k+2}{2} \tag{3.17}$$

□

Corollary 3.2. *If k and n are positive integer numbers, such that k is odd and n is even, then there is no degenerated magic polygon $D(n,k)$.*

Proof. By Theorem 3.1,

$$c=\frac{k^2(n-2)+k+2}{2}.$$

Therefore, if k is odd and n is even, then c is not a integer number. Hence, there is no degenerated magic polygon of order $k+1$ with n vertex for k odd and n even.

□

Theorem 3.3. *If n is an integer number greater than or equal to 3, then there is a degenerated magic polygon $D(n,2)$.*

Proof. Let C be the root vertex of a degenerated magic polygon $D(n,2)$, $(C,V_1,\cdots,V_{n-1})$ be the sequence of vertices of the largest polygon and $(C,V_1^*,\cdots,V_{n-1}^*)$ be the sequence of vertices of smallest polygon, both on clockwise direction.

For each $j\in\{1,2,\cdots,n-2\}$, we consider M_j the point of degenerated magic polygon between V_j and V_{j+1} and M_j^* the point of the degenerated magic polygon between V_j^* and V_{j+1}^*.

Thus, if c is the value assigned to the root vertex C and for each $j\in\{1,2,\cdots,n-2\}, z_j$ is the value assigned to the vertex V_j, z_j^* is the value assigned to the vertex V_j^*, m_j is the value assigned to the point M_j and m_j^* is the value assigned to the point M_j^*, then, the following conditions are satisfied:

$$z_j+m_j+z_{j+1}=u \tag{3.18}$$

$$z_j^*+m_j^*+z_{j+1}^*=u \tag{3.19}$$

and

$$z_j+z_j^*+c=u \tag{3.20}$$

Setting, for each $j\in\{1,2,\cdots,n-1\}$,

$$\begin{cases} z_j &= \begin{cases} j, & \text{if } j \text{ is odd} \\ 2n+j-3, & \text{if } j \text{ is even} \end{cases} \\ \\ z_j^* &= \begin{cases} 4(n-1)-j, & \text{if } j \text{ is odd} \\ 2n-j-1, & \text{if } j \text{ is even} \end{cases} \\ m_j &= 4(n-1)-2j \\ m_j^* &= 2j \end{cases}$$

then, we obtain a degenerated magic polygon $P(n,2)$.

In fact, the condictions on segments are satisfied, because

$$z_j + m_j + z_{j+1} = \begin{cases} j + 2n + j + 1 - 3 + 4(n-1) - 2j = 6(n-1), & \text{if } j \text{ is odd} \\ 2n + j - 3 + 4(n-1) - 2j + j + 1 = 6(n-1), & \text{if } j \text{ is even} \end{cases}$$

$$z_j^* + m_j^* + z_{j+1}^* = \begin{cases} 4(n-1) - j + 2j + 2n - j - 1 - 1 = 6(n-1), & \text{if } j \text{ is odd} \\ 2n - j - 1 + 2j + 4(n-1) - j - 1 = 6(n-1), & \text{if } j \text{ is even} \end{cases}$$

$$z_j + z_j^* + c = \begin{cases} j + (4(n-1) - j) + 2(n-1) = 6(n-1), & \text{if } j \text{ is odd} \\ (2n + j - 3) + (2n - j - 1) + 2(n-1) = 6(n-1) & \text{if } j \text{ is even} \end{cases}$$

$$m_j + m_j^* + c = 4(n-1) - 2j + 2j + 2(n-1) = 6(n-1)$$

In addition, all values assigned to the points are different because

$$\begin{aligned} A_1 &= \{z_j \mid \ j \text{ is odd and } j \in \{1, 2, \cdots, n-1\}\} \\ &= \begin{cases} \{1, 3, 5, \cdots, n-2\}, & \text{if } n \text{ is odd} \\ \{1, 3, 5, \cdots, n-1\}, & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

$$\begin{aligned} B_1 &= \{z_j \mid \ j \text{ is even and } j \in \{1, 2, \cdots, n-1\}\} \\ &= \begin{cases} \{2n-1, 2n+1, 2n+3, \cdots, 3n-4\}, & \text{if } n \text{ is odd} \\ \{2n-1, 2n+1, 2n+3, \cdots, 3n-5\}, & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

$$\begin{aligned} A_2 &= \{z_j^* \mid \ j \text{ is odd and } j \in \{1, 2, \cdots, n-1\}\} \\ &= \begin{cases} \{4n-5, 4n-7, 4n-9, \cdots, 3n-2\}, & \text{if } n \text{ is odd} \\ \{4n-5, 4n-7, 4n-9, \cdots, 3n-3\}, & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

$$\begin{aligned} B_2 &= \{z_j^* \mid \ j \text{ is even and } j \in \{1, 2, \cdots, n-1\}\} \\ &= \begin{cases} \{2n-3, 2n-5, 2n-7, \cdots, n\}, & \text{if } n \text{ is odd} \\ \{2n-3, 2n-5, 2n-7, \cdots, n+1\}, & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

$$\begin{aligned} C_1 &= \{m_j \mid j \in \{1, 2, \cdots, n-2\}\} \\ &= \{4n-6, 4n-8, 4n-10, \cdots, 2n\} \end{aligned}$$

$$\begin{aligned} C_2 &= \{m_j^* \mid j \in \{1, 2, \cdots, n-2\}\} \\ &= \{2, 4, 6, 8, 10, 12, \cdots, 2n-4\} \end{aligned}$$

$$D = \{c\} = \{2(n-1)\}$$

satisfy

$$|A_1| = |A_2| = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

$$|B_1| = |B_2| = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd} \\ \frac{n-2}{2}, & \text{if } n \text{ is even} \end{cases}$$

$$|C_1| = |C_2| = n-2,$$

$$|A_1| + |A_2| + |B_1| + |B_2| + |C_1| + |C_2| + |D| = 4n-5$$

$$\begin{aligned} & A_1 \cup A_2 \cup B_1 \cup B_2 \cup C_1 \cup C_2 \cup D \\ = \quad & \{j \in \mathbb{Z} \mid 1 \leq j \leq 4n-5\} \\ = \quad & \{1, 2, \cdots, 4n-5\} \end{aligned}$$

and

$$|A_1 \cup A_2 \cup B_1 \cup B_2 \cup C_1 \cup C_2 \cup D| = 4n-5$$

□

In Figure 6 we have an example of Magic Polygon $D(5, 2)$ constructed in the proof of the Theorem 3.3. An example of Magic Polygon $D(5, 2)$ that can not be obtained by this construction can be seen in the figure 9.

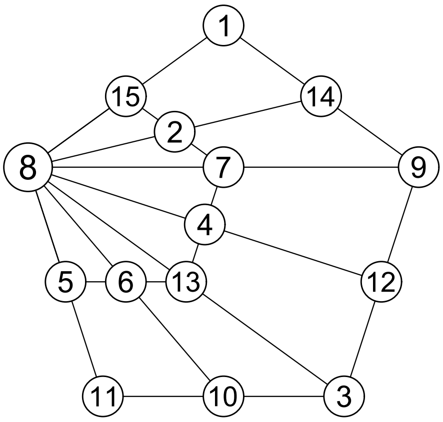

Figure 9: Example of Degenerated Magic Polygon $D(5, 2)$ using construction

4 Conclusions

The Magic Polygons P (n, 2) had already been studied in [Jakicic and Bouchat (2018)] where sufficient and necessary conditions for the existence of polygons $P(n, 2)$ such as those presented in this article. At the However, we do not yet have the necessary and sufficient conditions for existence of Magic Polygons $P(n, k)$ and for Magic Polygons Degenerates $D(n, k)$ presented in this work, for all values of n and k, although we discuss the existence and non-existence of examples for n and k satisfying some conditions. In this way, this subject constitutes a fecund field for future works.

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