Central conditions for a class of five-degree rigid differential systems

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ABSTRACT

The problem of determining necessary and sufficient conditions on P and Q for system $\dot{x}=-y+P(x,y),\ \dot{y}=x+Q(x,y)$,to have a center at the origin is known as the Poincaré center-focus problem.In this paper, we use the Poincaré and alwash Lloyed methods[1,2,3] to study the center focus problem of the five periodic differential equation ,and derive the center conditions for this differential system.

Keywords: Central focus, Center conditions, Periodic solutions,Composition condition.

LEMMA 1.Let $\tilde{P}(\theta)=\int P(\theta)d\theta,\ \bar{P}(\theta)=\tilde{P}(\theta)-\tilde{P}(0)$. If $\int_0^{2\pi}\bar{P}^k(\theta)g(\theta)d\theta=0$, then

$$\int_0^{2\pi}\tilde{P}^k(\theta)g(\theta)d\theta=0,(k=0,1,2\cdots).$$

Proof. $\bar{P}(\theta)=\int_0^{\theta}P(\theta)d\theta=\tilde{P}(\theta)-\tilde{P}(0)$, then $\tilde{P}(\theta)=\bar{P}(\theta)+\tilde{P}(0)$, then

$$\int_0^{2\pi}\tilde{P}^k(\theta)g(\theta)d\theta=\int_0^{2\pi}(\bar{P}(\theta)+\tilde{P}(0))^k g(\theta)d\theta=\int_0^{2\pi}\sum_{i=0}^{k}C_k^i\bar{P}^i(\theta)(-\tilde{P}(0))^{k-i}g(\theta)d\theta$$

$$=\sum_{i=0}^{k}(-\tilde{P}(0))^{k-i}\int_0^{2\pi}\bar{P}^i(\theta)g(\theta)d\theta=0.$$

LEMMA2. Let $P_k=\sum_{i+j=k}P_{ij}\cos^i\theta\sin^j\theta,(k=1,2,4)$ and $P_{10}^2+P_{01}^2\neq 0$. If $\int_0^{2\pi}\bar{P}_1^{2i}P_2d\theta=0(i=0,1),\ \int_0^{2\pi}\bar{P}_1^{2j}P_4d\theta=0(j=0,1,2)$, then $P_2=P_1(\lambda_0+\lambda_1\bar{P})$, $P_4=P_1(\mu_0+\mu_1\bar{P}_1+\mu_2\bar{P}_1^2+\mu_3\bar{P}_1^3),\ \lambda_i(i=0,1),\ \mu_i(i=0,1,2,3)$ are constants.

Proof. Set $P_1=A_1\cos\theta+B_1\sin\theta$,and $A_1=P_{10},B_1=P_{01},A_1^2+B_1^2\neq 0$,

$$\bar{P}_1=\tilde{P}_1+B_1, \quad (1)$$

$$\tilde{P}_1=A_1\sin\theta-B_1\cos\theta$$

we know from the **LEMMA 1**

$$\int_0^{2\pi}\bar{P}_1^{2i}P_2d\theta=\int_0^{2\pi}\tilde{P}_1^{2i}P_2d\theta=0(i=0,1),\ \int_0^{2\pi}\bar{P}_1^{2j}P_4d\theta=\int_0^{2\pi}\tilde{P}_1^{2j}P_4d\theta=0(j=0,1,2),$$

Because P_2 and P_4 are quadratic and quartic homogeneous polynomial,then

$$P_2=a_2\cos 2\theta+b_2\sin 2\theta,a_2=\frac{P_{20}-P_{02}}{2},b_2=\frac{P_{11}}{4},$$

$$P_4=d_0+d_2\cos 2\theta+e_2\sin 2\theta+d_4\cos 4\theta+e_4\sin 4\theta,\ d_2=\frac{P_{40}-P_{04}}{2},$$

$$e_2 = \frac{P_{31} + P_{13}}{4},\ d_4 = \frac{P_{40} - P_{22} + P_{04}}{8},\ e_4 = \frac{P_{31} - P_{13}}{8}.$$

From the condition $\int_0^{2\pi} \tilde{P}_1^2 P_2 d\theta = 0$, we know $A_2 b_2 - B_2 a_2 = 0$, $A_2 = -A_1 B_1$,

$$B_2 = \frac{1}{2}(A_1^2 - B_1^2),$$

Then

$$P_2 = \frac{a_2}{A_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) = \frac{b_2}{B_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) = \lambda_1 P_1 \tilde{P}_1,\ A_2^2 + B_2^2 \neq 0. \tag{2}$$

Substituting (1)into(2),we have

$$P_2 = P_1(\lambda_0 + \lambda_1 \overline{P}),\ \lambda_0 = \lambda_1 B_1,\ \lambda_1 = \frac{a_2}{A_2}\ or\ \frac{b_2}{B_2}.$$

From the condition $\int_0^{2\pi} P_4 = 0$,we know $d_0 = 0$,

From the condition $\int_0^{2\pi} \tilde{P}_1^2 P_4 = 0$,we know $A_2 e_2 - B_2 d_2 = 0$,

From the condition $\int_0^{2\pi} \tilde{P}_1^4 P_4 = 0$, and

$$P_1 \tilde{P}_1^3 = A_{31}(A_2 \cos 2\theta + B_2 \sin 2\theta) + A_4 \cos 4\theta + B_4 \sin 4\theta,\ A_{31} = \frac{1}{2}(A_1^2 + B_1^2),$$

we know

$$A_4 e_4 - B_4 d_4 = 0,\ A_4 = -\frac{1}{2}(B_1^2 - A_1^2) A_1 B_1,\ B_4 = -\frac{1}{8}(A_1^2 + B_1^2)^2.$$

Then

$$d_2 \cos 2\theta + e_2 \sin 2\theta = \frac{e_2}{B_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) = \frac{d_2}{A_2}(A_2 \cos 2\theta + B_2 \sin 2\theta),\ A_2^2 + B_2^2 \neq 0,$$

$$d_4 \cos 4\theta + e_4 \sin 4\theta = \frac{e_4}{B_4}(A_4 \cos 4\theta + B_4 \sin 4\theta) = \frac{d_4}{A_4}(A_4 \cos 4\theta + B_4 \sin 4\theta),\ A_4^2 + B_4^2 \neq 0,$$

$$\begin{aligned} P_4 &= d_0 + d_2 \cos 2\theta + e_2 \sin 2\theta + d_4 \cos 4\theta + e_4 \sin 4\theta \\ &= \frac{d_2}{A_2}(A_2 \cos 2\theta + B_2 \sin 2\theta) + \frac{d_4}{A_4}(A_4 \cos 4\theta + B_4 \sin 4\theta) \\ &= \frac{d_2}{A_2} P_1 \tilde{P}_1 + \frac{d_4}{A_4}(P_1 \tilde{P}_1^3 - A_{31} P_1 \tilde{P}_1). \end{aligned} \tag{3}$$

Substituting(1)into(3),we have $P_4 = P_1(\mu_0 + \mu_1 \overline{P_1} + \mu_2 \overline{P_1}^2 + \mu_3 \overline{P_1}^3)$,

$$\mu_0 = \frac{d_4}{A_4} B_1^3 - \frac{d_2}{A_2} B_1 + \frac{d_4}{A_4} A_{31} B_1,\ \mu_1 = \frac{d_2}{A_2} - \frac{d_4}{A_4} A_{31} + 3\frac{d_4}{A_4} B_1^2,\ \mu_2 = -3\frac{d_4}{A_4} B_1,\ \mu_3 = \frac{d_4}{A_4}.$$

Consider the fifth polynomial

$$\begin{cases} \dot{x} = -y + x(P_1(x,y) + P_3(x,y) + P_4(x,y)), \\ \dot{y} = x + y(P_1(x,y) + P_3(x,y) + P_4(x,y)), \end{cases} \tag{4}$$

with $P_n(x,y) = \sum_{i+j=n} P_{ij} x^i y^j$, P_{ij} are real constants.In this paper ,we give a short proof to the following theorem.

THEOREM. Let $\int_0^{2\pi} P_4 d\theta = 0$,then the origin is a center for (5) if and only if

$$\int_0^{2\pi} \overline{P_1}^{2i} P_2 d\theta = 0 (i = 0,1), \int_0^{2\pi} \overline{P_1}^{2j} P_4 d\theta = 0 (j = 1,2),$$

and the center is composition condition .

Proof. The system (4) in polar coordinates r and θ becomes

$$\begin{cases} \dot{r} = r^2 P_1(\cos\theta, \sin\theta) + r^3 P_2(\cos\theta, \sin\theta) + r^5 P_4(\cos\theta, \sin\theta), \\ \dot{\theta} = 1, \end{cases}$$

With,

$$P_1 = A_1 \cos\theta + B_1 \sin\theta,$$

$$P_2 = a_2 \cos 2\theta + b_2 \sin 2\theta,$$

$$P_4 = d_0 + d_2 \cos 2\theta + e_2 \sin 2\theta + d_4 \cos 4\theta + e_4 \sin 4\theta.$$

The origin is a center for (1) if and only if the polynomial differential equation

$$\frac{dr}{d\theta} = r^2 P_1(\cos\theta, \sin\theta) + r^3 P_2(\cos\theta, \sin\theta) + r^5 P_4(\cos\theta, \sin\theta), \tag{5}$$

Have $2\pi - periodic$ solution in a neighborhood of $r = 0$.

Let $r(\theta, c)$ be solution of (5) with $r(0,c) = c, 0 \prec |c| \prec\prec 1$ .We write

$$r(\theta, c) = \sum_{n=1}^{\infty} a_n(\theta) c^n, \tag{6}$$

Where $a_1(0) = 1$ and $a_n(0) = 0$ for $n \geq 1$.

The origin is a center if and only if $a_1(2\pi) = 1$ and $a_n(2\pi) = 0$ for all $n \geq 2, n \in Z^+$.

Substituting (6) into (5)

$$a_0' + a_1'c + \cdots + a_n'c^n + \cdots = P_1 c(a_0 + a_1 c + \cdots + a_n c^n + \cdots)^2 + P_2 c^2 (a_0 + a_1 c + \cdots + a_n c^n + \cdots)^3$$
$$+ P_4 c^4 (a_0 + a_1 c + \cdots + a_n c^n + \cdots)^5$$

Equating the coefficients of c yield

$$\dot{a}_n = P_1 \sum_{i+j=n-1} a_i a_j + P_2 \sum_{i+j+k=n-2} a_i a_j a_l + P_4 \sum_{i+j+k+l+m=n-4} a_i a_j a_k a_l a_m, \quad a_n(0) = 0. \tag{7}$$

Solving (7) gives

$$a_0 = 1,$$
$$a_1 = \overline{P_1},$$
$$a_2 = \overline{P_1}^2 + \overline{P_2},$$
$$a_3 = \overline{P_1}^3 + 2\overline{P_1}\,\overline{P_2} + \overline{\overline{P_1}P_2},$$

$$a_4 = \overline{P_4}^4 + 3\overline{P_1}^2\overline{P_2} + 2\overline{P_1}\,\overline{\overline{P_1}P_2} + \overline{\overline{P_1}^2P_2} + \frac{3}{2}\overline{P_2}^2 + \overline{P_4},$$

$$a_5 = \overline{P_1}^5 + 4\overline{P_1}^3\overline{P_2} + 4\overline{P_1}\,\overline{P_2}^2 + 3\overline{P_1}^2\overline{\overline{P_1}P_2} + 2\overline{P_1}\,\overline{\overline{P_1}^2P_2} + 2\overline{P_1}\,\overline{P_4} + 3\overline{\overline{P_1}P_2}\,\overline{P_2} + \overline{\overline{P_1}^3P_2} + \overline{\overline{P_1}P_2\overline{P_2}} + 3\overline{\overline{P_1}P_4},$$

$$a_6 = \overline{P_1}^6 + 5\overline{P_1}^4\overline{P_2} + 4\overline{P_1}^3\overline{\overline{P_1}P_2} + \frac{15}{2}\overline{P_1}^2\overline{P_2}^2 + 8\overline{P_1}\,\overline{\overline{P_1}P_2}\,\overline{P_2} + 3\overline{P_1}^2\overline{\overline{P_1}^2P_2} + 3\overline{P_1}^2\overline{P_4} + 2\overline{P_1}\,\overline{\overline{P_1}^3P_2} + 2\overline{P_1}\,\overline{\overline{P_1}P_2\overline{P_2}}$$
$$+ 6\overline{P_1}\,\overline{\overline{P_1}P_4} + \frac{5}{2}\overline{P_2}^3 + 3\overline{\overline{P_1}^2P_2}\,\overline{P_2} + 3\overline{P_2}\,\overline{P_4} + \overline{\overline{P_1}^4P_2} + 2\overline{\overline{P_1}^2P_2\overline{P_2}} + 2\overline{\overline{P_1}P_2}^2 + 6\overline{\overline{P_1}^2P_4} + 2\overline{\overline{P_2}P_4},$$

$$a_7 = \overline{P_1}^7 + 6\overline{P_1}^5\overline{P_2} + 5\overline{P_1}^4\overline{\overline{P_1}P_2} + 12\overline{P_1}^3\overline{P_2}^2 + 15\overline{P_1}^2\overline{\overline{P_1}P_2}\,\overline{P_2} + 5\overline{P_1}\,\overline{\overline{P_1}P_2}^2 + 4\overline{P_1}^3\overline{\overline{P_1}^2P_2} + 4\overline{P_1}^3\overline{P_4}$$
$$+ 8\overline{P_1}\,\overline{\overline{P_1}^2P_2}\,\overline{P_2} + 8\overline{P_1}\,\overline{P_2}^3 + 7\overline{P_1}\,\overline{P_2}\,\overline{P_4} + 3\overline{P_1}^2\overline{\overline{P_1}^3P_2} + 3\overline{P_1}^2\overline{\overline{P_1}P_2\overline{P_2}} + 9\overline{P_1}^2\overline{\overline{P_1}P_4} + 2\overline{P_1}\,\overline{\overline{P_1}^4P_2} + 4\overline{P_1}\,\overline{\overline{P_1}^2P_2\overline{P_2}}$$
$$+ 12\overline{P_1}\,\overline{\overline{P_1}^2P_4} + 4\overline{P_1}\,\overline{\overline{P_2}P_4} + \frac{15}{2}\overline{\overline{P_1}P_2}\,\overline{P_2}^2 + 3\overline{\overline{P_1}^3P_2}\,\overline{P_2} + 3\overline{\overline{P_1}P_2\overline{P_2}}\,\overline{P_2} + 9\overline{\overline{P_1}P_4}\,\overline{P_2} + 5\overline{\overline{P_1}P_2}\,\overline{P_4} + \overline{\overline{P_1}^5P_2} + 3\overline{\overline{P_1}^3P_2\overline{P_2}}$$
$$+ \frac{3}{2}\overline{\overline{P_1}P_2\overline{P_2}^2} + 4\overline{\overline{P_1}P_2}\,\overline{\overline{P_1}^2P_2} + \overline{\overline{P_1}^2\,\overline{\overline{P_1}P_2}\,P_2} + 10\overline{\overline{P_1}\,\overline{P_2}P_4} + 10\overline{\overline{P_1}^3P_4} + \overline{P_1\overline{P_2}\,\overline{P_4}},$$

$$a_8 = \overline{P_1}^8 + 7\overline{P_1}^6\overline{P_2} + 6\overline{P_1}^5\overline{\overline{P_1}P_2} + 5\overline{P_1}^4\overline{\overline{P_1}^2P_2} + \frac{35}{2}\overline{P_1}^4\overline{P_2}^2 + 5\overline{P_1}^4\overline{P_4} + 24\overline{P_1}^3\overline{\overline{P_1}P_2}\,\overline{P_2} + 15\overline{P_1}^2\overline{\overline{P_1}^2P_2}\,\overline{P_2}$$
$$+ \frac{35}{2}\overline{P_1}^2\overline{P_2}^3 + 9\overline{P_1}^2\overline{\overline{P_1}P_2}^2 + 10\overline{P_1}\,\overline{\overline{P_1}P_2}\,\overline{\overline{P_1}^2P_2} + 24\overline{P_1}\,\overline{\overline{P_1}P_2}\,\overline{P_2}^2 + 13\overline{P_1}^2\overline{P_2}\,\overline{P_4} + 12\overline{P_1}\,\overline{\overline{P_1}P_2}\,\overline{P_4} + 4\overline{P_1}^3\overline{\overline{P_1}^3P_2}$$
$$+ 4\overline{P_1}^3\overline{\overline{P_1}P_2\overline{P_2}} + 12\overline{P_1}^3\overline{\overline{P_1}P_4} + 8\overline{P_1}\,\overline{\overline{P_1}^3P_2}\,\overline{P_2} + 8\overline{P_1}\,\overline{\overline{P_1}P_2\overline{P_2}}\,\overline{P_2} + 24\overline{P_1}\,\overline{\overline{P_1}P_4}\,\overline{P_2} + 3\overline{P_1}^2\overline{\overline{P_1}^4P_2} + 6\overline{P_1}^2\overline{\overline{P_1}^2P_2\overline{P_2}}$$
$$+ 18\overline{P_1}^2\overline{\overline{P_1}^2P_4} + 6\overline{P_1}^2\overline{\overline{P_2}P_4} + 2\overline{P_1}\,\overline{\overline{P_1}^5P_2} + 6\overline{P_1}\,\overline{\overline{P_1}^3P_2\overline{P_2}} + 3\overline{P_1}\,\overline{\overline{P_1}P_2\overline{P_2}^2} + 2\overline{P_1}\,\overline{\overline{P_1}^2\,\overline{\overline{P_1}P_2}\,P_2} + 20\overline{P_1}\,\overline{\overline{P_1}\,\overline{P_2}P_4}$$
$$+ 20\overline{P_1}\,\overline{\overline{P_1}^3P_4} + \frac{35}{8}\overline{P_2}^4 + 9\overline{\overline{P_1}P_2}^2\overline{P_2} + \frac{15}{2}\overline{\overline{P_1}^2P_2}\,\overline{P_2}^2 + \frac{15}{2}\overline{P_2}^2\overline{P_4} + 3\overline{\overline{P_1}^4P_2}\,\overline{P_2} + 6\overline{\overline{P_1}^2P_2\overline{P_2}}\,\overline{P_2} + 18\overline{\overline{P_1}^2P_4}\,\overline{P_2}$$
$$+ 6\overline{P_2}\,\overline{\overline{P_2}P_4} + 5\overline{\overline{P_1}^2P_2}\,\overline{P_4} + 5\overline{P_4}^2 + \overline{\overline{P_1}^6P_2} + 4\overline{\overline{P_1}^4P_2\overline{P_2}} + 4\overline{\overline{P_1}^2\overline{P_2}^2P_2} + 4\overline{\overline{P_1}P_2}\,\overline{\overline{P_1}^3P_2} + 2\overline{\overline{P_1}^3\,\overline{\overline{P_1}P_2}\,P_2}$$
$$+ 4\overline{\overline{P_1}P_2}\,\overline{\overline{P_1}P_2\overline{P_2}} + 2\overline{\overline{P_1}P_2\,\overline{\overline{P_1}P_2}\,\overline{P_2}} + \frac{5}{2}\overline{\overline{P_1}^2P_2}^2 + 12\overline{\overline{P_1}P_2}\,\overline{\overline{P_1}P_4} + 6\overline{\overline{P_1}\,\overline{\overline{P_1}P_2}\,P_4} + 15\overline{\overline{P_1}^4P_4}$$
$$+ 24\overline{\overline{P_1}^2\overline{P_2}P_4} + 4\overline{\overline{P_2}^2P_4} + 2\overline{P_1}\,\overline{\overline{P_1}\,\overline{P_2}\,\overline{P_4}}.$$

We know $a_1(2\pi) = a_3(2\pi) = a_5(2\pi) = a_7(2\pi) = 0$.

A bar over a function denotes its indefinite integral.

The three necessary conditions for a center are $a_4(2\pi) = 0, a_6(2\pi) = 0, a_8(2\pi) = 0$.

Be equivalent to

$$\int_0^{2\pi}(\overline{P_1}^2P_2 + P_4\,d\theta) = 0, \tag{8}$$

$$\int_0^{2\pi}(\overline{P_1}^4P_2 + 2\overline{P_1}^2P_2\overline{P_2} + 6\overline{P_1}^2P_4 + 2\overline{P_2}P_4\,d\theta) = 0, \tag{9}$$

$$\int_0^{2\pi}(\overline{P_1}^6 P_2 + 4\overline{P_1}^4 P_2\overline{P_2} + 4\overline{P_1}^2\overline{P_2}^2 P_2 + 2\overline{P_1}^2\overline{P_1}P_2\overline{\overline{P_1}P_2} + 2\overline{\overline{P_1}P_2}\,\overline{P_1}P_2\overline{P_2} + 6\overline{P_1}P_4\overline{\overline{P_1}P_2} + 15\overline{P_1}^4 P_4 + 24\overline{P_1}^2\overline{P_2}P_4 + 4\overline{P_2}^2 P_4\, d\theta) = 0. \quad (10)$$

From the formula (8),We have Condition(I): $A_2 b_2 - B_2 a_2 = 0$, and from the **LEMMA2**,

$$P_2 = P_1(\lambda_0 + \lambda_1\overline{P}),\ \lambda_0 = \lambda_1 B_1,\ \lambda_1 = \frac{a_2}{A_2}\ or\ \frac{b_2}{B_2}.$$

From the formula (9)(10),we have Condition(II): $(6+\lambda_1)(A_2e_2 - B_2d_2) = 0$,

Condition(III): $(\lambda_1^2 + 14\lambda_1 + 15)(A_4e_4 - B_4d_4) + 4\lambda_0^2(A_2e_2 - B_2d_2) = 0$.

Now,we prove that these conditions are also sufficient.

If $(6+\lambda_1)(\lambda_1^2 + 14\lambda_1 + 15) \neq 0$, from the **LEMMA2** we know

$$\int_0^{2\pi}\overline{P_1}^2 P_2 = 0,\ \int_0^{2\pi}\overline{P_1}^2 P_4 = 0,\ \int_0^{2\pi}\overline{P_1}^4 P_4 = 0.$$

then

$$P_2 = P_1(\lambda_0 + \lambda_1\overline{P}),\ P_4 = P_1(\mu_0 + \mu_1\overline{P_1} + \mu_2\overline{P_1}^2 + \mu_3\overline{P_1}^3),\ \lambda_i(i=0,1),\ \mu_i(i=0,1,2,3)$$

are constants.

If $(6+\lambda_1)(\lambda_1^2 + 14\lambda_1 + 15) = 0$, we calculate the fourth necessary condition $a_{10}(2\pi) = 0$,

Be equivalent to

$$\int_0^{2\pi}(\overline{P_1}^8 P_2 + \overline{P_1}^6 P_2\overline{P_2} + 4\overline{P_1}^5\overline{\overline{P_1}P_2}P_2 + 12\overline{P_1}^4 P_2\overline{P_2}^2 + 12\overline{P_1}^3\overline{\overline{P_1}P_2}P_2\overline{P_2} + 2\overline{P_1}^2\overline{\overline{P_1}P_2}^2 P_2 + 8\overline{P_1}^2 P_2\overline{P_2}^3 + 6\overline{P_1}\,\overline{\overline{P_1}P_2}P_2\overline{P_2}^2$$
$$+ 2\overline{P_1}^4\overline{\overline{P_1}^2P_2}P_2 + 2\overline{P_1}^4 P_2\overline{P_4} + 4\overline{P_1}^2\overline{\overline{P_1}^2P_2}P_2\overline{P_2} + 4\overline{P_1}P_2\overline{\overline{P_1}P_2}\,\overline{P_4} + 20\overline{P_1}P_2\overline{P_2}\,\overline{\overline{P_1}P_4} + 12\overline{P_1}^2\overline{\overline{P_1}^2P_2}P_4 + 32\overline{P_1}\,\overline{\overline{P_1}P_2}\,\overline{P_2}P_4$$
$$+ 12\overline{P_1}^2 P_4\overline{P_4} + 4\overline{P_2}P_4\overline{P_4} + 28\overline{P_1}^6 P_4 + 88\overline{P_1}^4\overline{P_2}P_4 + 85\overline{P_1}^2\overline{P_2}^2 P_4 + 40\overline{P_1}^3\overline{\overline{P_1}P_2}P_4 + 8\overline{P_2}^3 P_4 + 4\overline{P_1}\,\overline{P_1\overline{P_2}\,\overline{P_4}}P_2$$
$$+ 31\overline{\overline{P_1}^2P_2}P_2\overline{P_4} + 26P_1\overline{\overline{P_1}P_2}^2\overline{P_4})d\theta = 0$$

then we have condition(IV):

$$(\lambda_1^3 + \frac{1669}{72}\lambda_1^2 + 60\lambda_1 + 28)\int_0^{2\pi}\overline{P_1}^6 P_4 d\theta + (\frac{367}{4}\lambda_0^2 + 12\lambda_0^2\lambda_1)\int_0^{2\pi}\overline{P_1}^4 P_4 d\theta + (12 + 2\lambda_1)\int_0^{2\pi}\overline{P_1}^2 P_4\overline{P_4}d\theta = 0$$

when $(6+\lambda_1)(\lambda_1^2 + 14\lambda_1 + 15) = 0$, we can obtain $\int_0^{2\pi}\overline{P_1}^6 P_4 = 0,\ \int_0^{2\pi}\overline{P_1}^4 P_4 = 0.$

Then from the **LEMMA2** sufficiency is demonstrated.

Competing Interests

Author has declared that no competing interests exist.

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