Original Research Article

On Summing Formulas of Generalized Hexanacci and Gaussian Generalized Hexanacci Numbers

Abstract. In this paper, we present linear summation formulas for generalized Hexanacci numbers and generalized Gaussian Hexanacci numbers. Also, as special cases, we give linear summation formulas of Hexanacci and Hexanacci-Lucas numbers; Gaussian Hexanacci and Gaussian Hexanacci-Lucas numbers.

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1. Introduction and Preliminaries

In this work, we investigate linear summation formulas of generalized Hexanacci numbers and generalized Gaussian Hexanacci numbers.

Some summing formulas of the Pell and Pell-Lucas numbers are well known and given in [2, 3], see also [1]. For linear sums of Tribonacci and Tetranacci and Pentanacci numbers, see [5], [6, 8] and [7], respectively.

First, in this section, we present some background about generalized Hexanacci numbers. There have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these type of sequences are the sequences of Hexanacci and Hexanacci-Lucas which are special case of generalized Hexanacci numbers. A generalized Hexanacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4, V_5)\}_{n\geq 0}$ is defined by the sixth-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} + V_{n-6} \tag{1.1}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4, V_5 = c_5$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} - V_{-(n-4)} - V_{-(n-5)} + V_{-(n-6)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The first few generalized Hexanacci numbers with positive subscript and negative subscript are given in the following Table 1:

Table 1. A few generalized Hexanacci numbers

n	V_n	V_{-n}
0	c_0	c_0
1	c_1	$-c_0 - c_1 - c_2 - c_3 - c_4 + c_5$
2	c_2	$2c_4 - c_5$
3	c_3	$2c_3 - c_4$
4	c_4	$2c_2 - c_3$
5	c_5	$2c_1 - c_2$
6	$c_0 + c_1 + c_2 + c_3 + c_4 + c_5$	$2c_0 - c_1$
7	$c_0 + 2c_1 + 2c_2 + 2c_3 + 2c_4 + 2c_5$	$-3c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 + 2c_5$
8	$2c_0 + 3c_1 + 4c_2 + 4c_3 + 4c_4 + 4c_5$	$c_0 + c_1 + c_2 + c_3 + 5c_4 - 3c_5$
9	$4c_0 + 6c_1 + 7c_2 + 8c_3 + 8c_4 + 8c_5$	$4c_3 - 4c_4 + c_5$
10	$8c_0 + 12c_1 + 14c_2 + 15c_3 + 16c_4 + 16c_5$	$4c_2 - 4c_3 + c_4$
11	$16c_0 + 24c_1 + 28c_2 + 30c_3 + 31c_4 + 32c_5$	$4c_1 - 4c_2 + c_3$

We consider two special cases of $\{V_n\}_{n\geq 0}$. Hexanacci sequence $\{H_n\}_{n\geq 0}$ and Hexanacci-Lucas sequence $\{E_n\}_{n\geq 0}$ (also called as Esanacci or 6-anacci sequence) are defined by the sixth-order recurrence relations (1.2)

$$H_n = H_{n-1} + H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6}, \qquad H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$$

and

(1.3) $$E_n = E_{n-1} + E_{n-2} + E_{n-3} + E_{n-4} + E_{n-5} + E_{n-6}, \qquad E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$$

respectively. Note that H_n is the sequence $A001592$ in [4] and E_n is the sequence $A074584$ in [4].

Next, we present the first few values of the Hexanacci and Hexanacci-Lucas numbers with positive and negative subscripts in the following Table 2:

Table 2. A few Hexanacci and Hexanacci-Lucas Numbers

n	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
H_n	2	0	0	0	0	-1	1	0	0	0	0	0	1	1	2	4	8	16	32	63
E_n	-1	-1	-1	-1	-8	11	-1	-1	-1	-1	-1	6	1	3	7	15	31	63	120	239

2. Linear Sums of Generalized Hexanacci Numbers

The following Theorem present some summation formulas of generalized Hexanacci numbers.

THEOREM 2.1. *For $n \geq 0$, we have the following lineer sum identities:*

(a): $\sum_{k=0}^{n} V_k = \frac{1}{5}(V_{n+5} - V_{n+3} - 2V_{n+2} - 3V_{n+1} + V_n - V_5 + V_3 + 2V_2 + 3V_1 + 4V_0)$

(b): $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{5}(3V_{2n+2} + 2V_{2n} - V_{2n-1} + V_{2n-2} - 2V_{2n-3} + 2V_5 - 5V_4 + 3V_3 - 4V_2 + 4V_1 - 3V_0)$,

(c): $\sum_{k=0}^{n} V_{2k} = \frac{1}{5}(-2V_{2n+2} + 5V_{2n+1} + 2V_{2n} + 4V_{2n-1} + V_{2n-2} + 3V_{2n-3} - 3V_5 + 5V_4 - 2V_3 + 6V_2 - V_1 + 7V_0)$,

(d): $\sum_{k=0}^{n} V_{3k} = \frac{1}{5}(-3V_{3n+3} + 5V_{3n+2} + 3V_{3n+1} + V_{3n} + 4V_{3n-1} + 2V_{3n-2} - 2V_5 + 7V_3 - V_2 + V_1 + 8V_0)$,

(e): $\sum_{k=0}^{n} V_{3k+1} = \frac{1}{5}(2V_{3n+3} - 2V_{3n+1} + V_{3n} - V_{3n-1} - 3V_{3n-2} - 2V_5 + 5V_4 - 3V_3 - V_2 + 6V_1 - 2V_0)$

(f): $\sum_{k=0}^{n} V_{3k+2} = \frac{1}{5}(2V_{3n+3} + 3V_{3n+1} + V_{3n} - V_{3n-1} + 2V_{3n-2} + 3V_5 - 5V_4 - 3V_3 + 4V_2 - 4V_1 - 2V_0)$

(g): $\sum_{k=0}^{n} V_{4k} = \frac{1}{5}(-2V_{4n+4} + 3V_{4n+3} + V_{4n+2} + 3V_{4n+1} + 2V_{4n} + 2V_{4n-1} - 2V_5 + 4V_4 - V_3 + V_2 - V_1 + 5V_0)$

(h): $\sum_{k=0}^{n} V_{4k+1} = \frac{1}{5}(V_{4n+4} - V_{4n+3} + V_{4n+2} - 2V_{4n-1} + 2V_5 - 3V_4 - V_3 - 3V_2 + 3V_1 - 2V_0)$

(i): $\sum_{k=0}^{n} V_{4k+2} = \frac{1}{5}(2V_{4n+3} + V_{4n+2} + V_{4n+1} - V_{4n} + V_{4n-1} - V_5 + V_4 - V_3 + 5V_2 + 2V_0)$

(j): $\sum_{k=0}^{n} V_{4k+3} = \frac{1}{5}(2V_{4n+4} + V_{4n+3} + V_{4n+2} - V_{4n+1} + V_{4n} - 2V_4 + 4V_3 - V_2 + V_1 - V_0)$

(k): $\sum_{k=0}^{n} V_{5k} = \frac{1}{25}(-9V_{5n+5} + 5V_{5n+4} + 14V_{5n+3} + 18V_{5n+2} + 17V_{5n+1} + 11V_{5n} + 9V_5 - 5V_4 - 14V_3 - 18V_2 - 17V_1 + 14V_0)$

(l): $\sum_{k=0}^{n} V_{5k+1} = \frac{1}{25}(-4V_{5n+5} + 5V_{5n+4} + 9V_{5n+3} + 8V_{5n+2} + 2V_{5n+1} - 9V_{5n} + 4V_5 - 5V_4 - 9V_3 - 8V_2 + 23V_1 + 9V_0)$

(m): $\sum_{k=0}^{n} V_{5k+2} = \frac{1}{25}(V_{5n+5} + 5V_{5n+4} + 4V_{5n+3} - 2V_{5n+2} - 13V_{5n+1} - 4V_{5n} - V_5 - 5V_4 - 4V_3 + 27V_2 + 13V_1 + 4V_0)$

(n): $\sum_{k=0}^{n} V_{5k+3} = \frac{1}{25}(6V_{5n+5} + 5V_{5n+4} - V_{5n+3} - 12V_{5n+2} - 3V_{5n+1} + V_{5n} - 6V_5 - 5V_4 + 26V_3 + 12V_2 + 3V_1 - V_0)$

(o): $\sum_{k=0}^{n} V_{5k+4} = \frac{1}{25}(11V_{5n+5} + 5V_{5n+4} - 6V_{5n+3} + 3V_{5n+2} + 7V_{5n+1} + 6V_{5n} - 11V_5 + 20V_4 + 6V_3 - 3V_2 - 7V_1 - 6V_0)$

Proof. (a), (b), (c) can be proved as in the case of (d),(e),(f) so we omit their proof.

(d),(e),(f): Using the recurrence relation

$$V_k = V_{k-1} + V_{k-2} + V_{k-3} + V_{k-4} + V_{k-5} + V_{k-6}$$

i.e.

$$V_{k-1} = V_k - V_{k-2} - V_{k-3} - V_{k-4} - V_{k-5} - V_{k-6}$$

we write the obvious equations

$$
\begin{aligned}
V_0 &= V_1 - V_{-1} - V_{-2} - V_{-3} - V_{-4} - V_{-5} \\
V_3 &= V_4 - V_2 - V_1 - V_0 - V_{-1} - V_{-2} \\
V_6 &= V_7 - V_5 - V_4 - V_3 - V_2 - V_1 \\
V_9 &= V_{10} - V_8 - V_7 - V_6 - V_5 - V_4 \\
V_{12} &= V_{13} - V_{11} - V_{10} - V_9 - V_8 - V_7 \\
V_{15} &= V_{16} - V_{14} - V_{13} - V_{12} - V_{11} - V_{10} \\
V_{18} &= V_{19} - V_{17} - V_{16} - V_{15} - V_{14} - V_{13} \\
V_{21} &= V_{22} - V_{20} - V_{19} - V_{18} - V_{17} - V_{16} \\
V_{24} &= V_{25} - V_{23} - V_{22} - V_{21} - V_{20} - V_{19} \\
V_{27} &= V_{28} - V_{26} - V_{25} - V_{24} - V_{23} - V_{22} \\
&\vdots \\
V_{3n-6} &= V_{3n-5} - V_{3n-7} - V_{3n-8} - V_{3n-9} - V_{3n-10} - V_{3n-11} \\
V_{3n-3} &= V_{3n-2} - V_{3n-4} - V_{3n-5} - V_{3n-6} - V_{3n-7} - V_{3n-8} \\
V_{3n} &= V_{3n+1} - V_{3n-1} - V_{3n-2} - V_{3n-3} - V_{3n-4} - V_{3n-5}
\end{aligned}
$$

Now, adding these equations, we have

$$
\begin{aligned}
\sum_{k=0}^{n} V_{3k} &= \left(\sum_{k=0}^{n} V_{3k+1}\right) + \left(-\sum_{k=0}^{n} V_{3k+2} - V_{-1} + V_{3n+2}\right) + \left(-\sum_{k=0}^{n} V_{3k+1} - V_{-2} + V_{3n+1}\right) \\
&+ \left(-\sum_{k=0}^{n} V_{3k} - V_{-3} + V_{3n}\right) + \left(-\sum_{k=0}^{n} V_{3k+2} - V_{-4} - V_{-1} + V_{3n-1} + V_{3n+2}\right) \\
&+ \left(-\sum_{k=0}^{n} V_{3k+1} - V_{-5} - V_{-2} + V_{3n-2} + V_{3n+1}\right) \\
&\Rightarrow \\
2\sum_{k=0}^{n} V_{3k} &= 2V_{3n+2} + 2V_{3n+1} + V_{3n} + V_{3n-1} + V_{3n-2} - V_{-5} - V_{-4} - V_{-3} - 2V_{-2} - 2V_{-1} \\
&-2\sum_{k=0}^{n} V_{3k+2} - \sum_{k=0}^{n} V_{3k+1}
\end{aligned}
$$

Similarly, we write the obvious equations

$$\begin{aligned}
V_{-1} &= V_0 - V_{-2} - V_{-3} - V_{-4} - V_{-5} - V_{-6} \\
V_2 &= V_3 - V_1 - V_0 - V_{-1} - V_{-2} - V_{-3} \\
V_5 &= V_6 - V_4 - V_3 - V_2 - V_1 - V_0 \\
V_8 &= V_9 - V_7 - V_6 - V_5 - V_4 - V_3 \\
V_{11} &= V_{12} - V_{10} - V_9 - V_8 - V_7 - V_6 \\
V_{14} &= V_{15} - V_{13} - V_{12} - V_{11} - V_{10} - V_9 \\
V_{17} &= V_{18} - V_{16} - V_{15} - V_{14} - V_{13} - V_{12} \\
V_{20} &= V_{21} - V_{19} - V_{18} - V_{17} - V_{16} - V_{15} \\
V_{23} &= V_{24} - V_{22} - V_{21} - V_{20} - V_{19} - V_{18} \\
V_{26} &= V_{27} - V_{25} - V_{24} - V_{23} - V_{22} - V_{21} \\
&\vdots \\
V_{3n-4} &= V_{3n-3} - V_{3n-5} - V_{3n-6} - V_{3n-7} - V_{3n-8} - V_{3n-9} \\
V_{3n-1} &= V_{3n} - V_{3n-2} - V_{3n-3} - V_{3n-4} - V_{3n-5} - V_{3n-6} \\
V_{3n+2} &= V_{3n+3} - V_{3n+1} - V_{3n} - V_{3n-1} - V_{3n-2} - V_{3n-3}
\end{aligned}$$

Now, adding these equations, we obtain

$$\begin{aligned}
V_{-1} + \sum_{k=0}^{n} V_{3k+2} &= \left(V_{3n+3} + \sum_{k=0}^{n} V_{3k}\right) + \left(-V_{-2} - \sum_{k=0}^{n} V_{3k+1}\right) + \left(-V_{-3} - \sum_{k=0}^{n} V_{3k}\right) \\
&\quad + \left(V_{3n+2} - V_{-1} - V_{-4} - \sum_{k=0}^{n} V_{3k+2}\right) + \left(V_{3n+1} - V_{-2} - V_{-5} - \sum_{k=0}^{n} V_{3k+1}\right) \\
&\quad + \left(-V_{-6} - V_{-3} + V_{3n} - \sum_{k=0}^{n} V_{3k}\right) \\
&\Rightarrow \\
2\sum_{k=0}^{n} V_{3k+2} &= -V_{-6} - V_{-5} - V_{-4} - 2V_{-3} - 2V_{-2} - 2V_{-1} + V_{3n+3} + V_{3n+2} + V_{3n+1} + V_{3n} \\
&\quad -2\sum_{k=0}^{n} V_{3k+1} - \sum_{k=0}^{n} V_{3k}.
\end{aligned}$$

Similarly, we write the obvious equations

$$\begin{aligned}
V_{-2} &= V_{-1} - V_{-3} - V_{-4} - V_{-5} - V_{-6} - V_{-7} \\
V_{1} &= V_{2} - V_{0} - V_{-1} - V_{-2} - V_{-3} - V_{-4} \\
V_{4} &= V_{5} - V_{3} - V_{2} - V_{1} - V_{0} - V_{-1} \\
V_{7} &= V_{8} - V_{6} - V_{5} - V_{4} - V_{3} - V_{2} \\
V_{10} &= V_{11} - V_{9} - V_{8} - V_{7} - V_{6} - V_{5} \\
V_{13} &= V_{14} - V_{12} - V_{11} - V_{10} - V_{9} - V_{8} \\
V_{16} &= V_{17} - V_{15} - V_{14} - V_{13} - V_{12} - V_{11} \\
V_{19} &= V_{20} - V_{18} - V_{17} - V_{16} - V_{15} - V_{14} \\
V_{22} &= V_{23} - V_{21} - V_{20} - V_{19} - V_{18} - V_{12} \\
V_{25} &= V_{26} - V_{24} - V_{23} - V_{22} - V_{21} - V_{20} \\
&\vdots \\
V_{3n-5} &= V_{3n-4} - V_{3n-6} - V_{3n-7} - V_{3n-8} - V_{3n-9} - V_{3n-8} \\
V_{3n-2} &= V_{3n-1} - V_{3n-3} - V_{3n-4} - V_{3n-5} - V_{3n-6} - V_{3n-7} \\
V_{3n+1} &= V_{3n+2} - V_{3n} - V_{3n-1} - V_{3n-2} - V_{3n-3} - V_{3n-4}
\end{aligned}$$

Now, adding these equations, we obtain

$$\begin{aligned}
V_{-2} + \sum_{k=0}^{n} V_{3k+1} &= \left(V_{-1} + \sum_{k=0}^{n} V_{3k+2}\right) + \left(-V_{-3} - \sum_{k=0}^{n} V_{3k}\right) + \left(V_{3n+2} - V_{-4} - V_{-1} - \sum_{k=0}^{n} V_{3k+2}\right) \\
&\quad + \left(V_{3n+1} - V_{-5} - V_{-2} - \sum_{k=0}^{n} V_{3k+1}\right) + \left(V_{3n.} - V_{-6} - V_{-3} - \sum_{k=0}^{n} V_{3k}\right) \\
&\quad + \left(V_{3n-1} + V_{3n+2} - V_{-7} - V_{-4} - V_{-1} - \sum_{k=0}^{n} V_{3k+2}\right) \\
&\Rightarrow \\
2\sum_{k=0}^{n} V_{3k+1} &= -V_{-7} - 2V_{-2} - V_{-6} - V_{-5} - 2V_{-4} - 2V_{-3} - V_{-1} + 2V_{3n+2} + V_{3n+1} + V_{3n.} + V_{3n-1} \\
&\quad -2\sum_{k=0}^{n} V_{3k} - \sum_{k=0}^{n} V_{3k+2}.
\end{aligned}$$

Solving the following system

$$\begin{aligned}
2\sum_{k=0}^{n}V_{3k} &= 2V_{3n+2}+2V_{3n+1}+V_{3n}+V_{3n-1}+V_{3n-2}-V_{-5}-V_{-4}-V_{-3}-2V_{-2}-2V_{-1} \\
&\quad -2\sum_{k=0}^{n}V_{3k+2}-\sum_{k=0}^{n}V_{3k+1} \\
2\sum_{k=0}^{n}V_{3k+2} &= -V_{-6}-V_{-5}-V_{-4}-2V_{-3}-2V_{-2}-2V_{-1}+V_{3n+3}+V_{3n+2}+V_{3n+1}+V_{3n} \\
&\quad -2\sum_{k=0}^{n}V_{3k+1}-\sum_{k=0}^{n}V_{3k} \\
2\sum_{k=0}^{n}V_{3k+1} &= -V_{-7}-2V_{-2}-V_{-6}-V_{-5}-2V_{-4}-2V_{-3}-V_{-1}+2V_{3n+2}+V_{3n+1}+V_{3n}.+V_{3n-1} \\
&\quad -2\sum_{k=0}^{n}V_{3k}-\sum_{k=0}^{n}V_{3k+2}
\end{aligned}$$

we find

$$\begin{aligned}
\textstyle\sum_{k=0}^{n}V_{3k} &= \tfrac{1}{5}(-3V_{3n+3}+5V_{3n+2}+3V_{3n+1}+V_{3n}+4V_{3n-1}+2V_{3n-2}-2V_5+7V_3-V_2+V_1+8V_0), \\
\textstyle\sum_{k=0}^{n}V_{3k+1} &= \tfrac{1}{5}(2V_{3n+3}-2V_{3n+1}+V_{3n}-V_{3n-1}-3V_{3n-2}-2V_5+5V_4-3V_3-V_2+6V_1-2V_0), \\
\textstyle\sum_{k=0}^{n}V_{3k+2} &= \tfrac{1}{5}(2V_{3n+3}+3V_{3n+1}+V_{3n}-V_{3n-1}+2V_{3n-2}+3V_5-5V_4-3V_3+4V_2-4V_1-2V_0).
\end{aligned}$$

(g),(h),(i),(j): As in the cases (d),(e),(f), solving the following system

$$\begin{aligned}
2\sum_{k=0}^{n}V_{4k} &= 2V_{4n+3}+V_{4n+2}+V_{4n+1}+V_{4n}+V_{4n-1}-2V_{-1}-V_{-2}-V_{-3}-V_{-4}-V_{-5}- \\
&\quad \sum_{k=0}^{n}V_{4k+2}-2\sum_{k=0}^{n}V_{4k+3} \\
2\sum_{k=0}^{n}V_{4k+1} &= -V_{-1}-V_{-2}-V_{-3}-V_{-4}+V_{4n+3}+V_{4n+2}+V_{4n+1}+V_{4n}.-\sum_{k=0}^{n}V_{4k+3}-2\sum_{k=0}^{n}V_{4k} \\
2\sum_{k=0}^{n}V_{4k+2} &= V_{4n+3}+V_{4n+2}+V_{4n+1}-V_{-1}-V_{-2}-V_{-3}-2\sum_{k=0}^{n}V_{4k+1}-\sum_{k=0}^{n}V_{4k} \\
2\sum_{k=0}^{n}V_{4k+3} &= V_{4n+4}+V_{4n+3}+V_{4n+2}-V_0-V_{-1}-V_{-2}-2\sum_{k=0}^{n}V_{4k+2}-\sum_{k=0}^{n}V_{4k+1}
\end{aligned}$$

we find

$$\sum_{k=0}^{n} V_{4k} = \frac{1}{5}(-2V_{4n+4} + 3V_{4n+3} + V_{4n+2} + 3V_{4n+1} + 2V_{4n} + 2V_{4n-1} - 2V_5 + 4V_4 - V_3 + V_2 - V_1 + 5V_0)$$

$$\sum_{k=0}^{n} V_{4k+1} = \frac{1}{5}(V_{4n+4} - V_{4n+3} + V_{4n+2} - 2V_{4n-1} + 2V_5 - 3V_4 - V_3 - 3V_2 + 3V_1 - 2V_0)$$

$$\sum_{k=0}^{n} V_{4k+2} = \frac{1}{5}(2V_{4n+3} + V_{4n+2} + V_{4n+1} - V_{4n} + V_{4n-1} - V_5 + V_4 - V_3 + 5V_2 + 2V_0)$$

$$\sum_{k=0}^{n} V_{4k+3} = \frac{1}{5}(2V_{4n+4} + V_{4n+3} + V_{4n+2} - V_{4n+1} + V_{4n} - 2V_4 + 4V_3 - V_2 + V_1 - V_0)$$

(k),(l),(m),(n),(o): As in the cases (d),(e),(f), solving the following system

$$\begin{aligned}
2\sum_{k=0}^{n} V_{5k} &= V_{5n+4} + V_{5n+3} + V_{5n+2} + V_{5n+1} + V_{5n} - V_{-1} - V_{-2} - V_{-3} - V_{-4} - V_{-5} \\
&\quad - \sum_{k=0}^{n} V_{5k+4} - \sum_{k=0}^{n} V_{5k+3} - \sum_{k=0}^{n} V_{5k+2} \\
2\sum_{k=0}^{n} V_{5k+1} &= V_{5n+4} + V_{5n+3} + V_{5n+2} + V_{5n+1} - V_{-1} - V_{-2} - V_{-3} - V_{-4} \\
&\quad - \sum_{k=0}^{n} V_{5k+4} - \sum_{k=0}^{n} V_{5k+3} - \sum_{k=0}^{n} V_{5k} \\
2\sum_{k=0}^{n} V_{5k+2} &= V_{5n+4} + V_{5n+3} + V_{5n+2} - V_{-1} - V_{-2} - V_{-3} - \sum_{k=0}^{n} V_{5k+4} - \sum_{k=0}^{n} V_{5k+1} - \sum_{k=0}^{n} V_{5k} \\
2\sum_{k=0}^{n} V_{5k+3} &= V_{5n+4} + V_{5n+3} - V_{-1} - V_{-2} - \sum_{k=0}^{n} V_{5k+2} - \sum_{k=0}^{n} V_{5k+1} - \sum_{k=0}^{n} V_{5k} \\
2\sum_{k=0}^{n} V_{5k+4} &= V_{5n+5} + V_{5n+4} - V_0 - V_{-1} - \sum_{k=0}^{n} V_{5k+3} - \sum_{k=0}^{n} V_{5k+2} - \sum_{k=0}^{n} V_{5k+1}
\end{aligned}$$

we find

$$\begin{aligned}
\sum_{k=0}^{n} V_{5k} &= \frac{1}{25}(-9V_{5n+5} + 5V_{5n+4} + 14V_{5n+3} + 18V_{5n+2} + 17V_{5n+1} + 11V_{5n} + 9V_5 - 5V_4 - 14V_3 \\
&\quad -18V_2 - 17V_1 + 14V_0)
\end{aligned}$$

$$\sum_{k=0}^{n} V_{5k+1} = \frac{1}{25}(-4V_{5n+5} + 5V_{5n+4} + 9V_{5n+3} + 8V_{5n+2} + 2V_{5n+1} - 9V_{5n} + 4V_5 - 5V_4 - 9V_3 - 8V_2 + 23V_1 + 9V_0)$$

$$\sum_{k=0}^{n} V_{5k+2} = \frac{1}{25}(V_{5n+5} + 5V_{5n+4} + 4V_{5n+3} - 2V_{5n+2} - 13V_{5n+1} - 4V_{5n} - V_5 - 5V_4 - 4V_3 + 27V_2 + 13V_1 + 4V_0)$$

$$\sum_{k=0}^{n} V_{5k+3} = \frac{1}{25}(6V_{5n+5} + 5V_{5n+4} - V_{5n+3} - 12V_{5n+2} - 3V_{5n+1} + V_{5n} - 6V_5 - 5V_4 + 26V_3 + 12V_2 + 3V_1 - V_0)$$

$$\sum_{k=0}^{n} V_{5k+4} = \frac{1}{25}(11V_{5n+5} + 5V_{5n+4} - 6V_{5n+3} + 3V_{5n+2} + 7V_{5n+1} + 6V_{5n} - 11V_5 + 20V_4 + 6V_3 - 3V_2 - 7V_1 - 6V_0).$$

As special cases of above Theorem, we have the following two Corollaries. First one present some summation formulas of Hexanacci numbers.

COROLLARY 2.2. *For $n \geq 0$, we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} H_k = \frac{1}{5}(H_{n+5} - H_{n+3} - 2H_{n+2} - 3H_{n+1} + H_n - 1)$
- **(b):** $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{5}(3H_{2n+2} + 2H_{2n} - H_{2n-1} + H_{2n-2} - 2H_{2n-3} + 2)$
- **(c):** $\sum_{k=0}^{n} H_{2k} = \frac{1}{5}(-2H_{2n+2} + 5H_{2n+1} + 2H_{2n} + 4H_{2n-1} + H_{2n-2} + 3H_{2n-3} - 3)$
- **(d):** $\sum_{k=0}^{n} H_{3k} = \frac{1}{5}(-3H_{3n+3} + 5H_{3n+2} + 3H_{3n+1} + H_{3n} + 4H_{3n-1} + 2H_{3n-2} - 2)$
- **(e):** $\sum_{k=0}^{n} H_{3k+1} = \frac{1}{5}(2H_{3n+3} - 2H_{3n+1} + H_{3n} - H_{3n-1} - 3H_{3n-2} + 3)$
- **(f):** $\sum_{k=0}^{n} H_{3k+2} = \frac{1}{5}(2H_{3n+3} + 3H_{3n+1} + H_{3n} - H_{3n-1} + 2H_{3n-2} - 2)$
- **(g):** $\sum_{k=0}^{n} H_{4k} = \frac{1}{5}(-2H_{4n+4} + 3H_{4n+3} + H_{4n+2} + 3H_{4n+1} + 2H_{4n} + 2H_{4n-1} - 2)$
- **(h):** $\sum_{k=0}^{n} H_{4k+1} = \frac{1}{5}(H_{4n+4} - H_{4n+3} + H_{4n+2} - 2H_{4n-1} + 2)$
- **(i):** $\sum_{k=0}^{n} H_{4k+2} = \frac{1}{5}(2H_{4n+3} + H_{4n+2} + H_{4n+1} - H_{4n} + H_{4n-1} - 1)$
- **(j):** $\sum_{k=0}^{n} H_{4k+3} = \frac{1}{5}(2H_{4n+4} + H_{4n+3} + H_{4n+2} - H_{4n+1} + H_{4n})$
- **(k):** $\sum_{k=0}^{n} H_{5k} = \frac{1}{25}(-9H_{5n+5} + 5H_{5n+4} + 14H_{5n+3} + 18H_{5n+2} + 17H_{5n+1} + 11H_{5n} - 11)$
- **(l):** $\sum_{k=0}^{n} H_{5k+1} = \frac{1}{25}(-4H_{5n+5} + 5H_{5n+4} + 9H_{5n+3} + 8H_{5n+2} + 2H_{5n+1} - 9H_{5n} + 9)$
- **(m):** $\sum_{k=0}^{n} H_{5k+2} = \frac{1}{25}(H_{5n+5} + 5H_{5n+4} + 4H_{5n+3} - 2H_{5n+2} - 13H_{5n+1} - 4H_{5n} + 4)$
- **(n):** $\sum_{k=0}^{n} H_{5k+3} = \frac{1}{25}(6H_{5n+5} + 5H_{5n+4} - H_{5n+3} - 12H_{5n+2} - 3H_{5n+1} + H_{5n} - 1)$
- **(o):** $\sum_{k=0}^{n} H_{5k+4} = \frac{1}{25}(11H_{5n+5} + 5H_{5n+4} - 6H_{5n+3} + 3H_{5n+2} + 7H_{5n+1} + 6H_{5n} - 6)$.

Next Corollary gives some summation formulas of Hexanacci-Lucas numbers.

COROLLARY 2.3. *For $n \geq 0$, we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} E_k = \frac{1}{5}(E_{n+5} - E_{n+3} - 2E_{n+2} - 3E_{n+1} + E_n + 9)$
- **(b):** $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{5}(3E_{2n+2} + 2E_{2n} - E_{2n-1} + E_{2n-2} - 2E_{2n-3} - 18)$,
- **(c):** $\sum_{k=0}^{n} E_{2k} = \frac{1}{5}(-2E_{2n+2} + 5E_{2n+1} + 2E_{2n} + 4E_{2n-1} + E_{2n-2} + 3E_{2n-3} + 27)$,
- **(d):** $\sum_{k=0}^{n} E_{3k} = \frac{1}{5}(-3E_{3n+3} + 5E_{3n+2} + 3E_{3n+1} + E_{3n} + 4E_{3n-1} + 2E_{3n-2} + 33)$,
- **(e):** $\sum_{k=0}^{n} E_{3k+1} = \frac{1}{5}(2E_{3n+3} - 2E_{3n+1} + E_{3n} - E_{3n-1} - 3E_{3n-2} - 17)$
- **(f):** $\sum_{k=0}^{n} E_{3k+2} = \frac{1}{5}(2E_{3n+3} + 3E_{3n+1} + E_{3n} - E_{3n-1} + 2E_{3n-2} + 3E_5 - 7)$
- **(g):** $\sum_{k=0}^{n} E_{4k} = \frac{1}{5}(-2E_{4n+4} + 3E_{4n+3} + E_{4n+2} + 3E_{4n+1} + 2E_{4n} + 2E_{4n-1} + 23)$
- **(h):** $\sum_{k=0}^{n} E_{4k+1} = \frac{1}{5}(E_{4n+4} - E_{4n+3} + E_{4n+2} - 2E_{4n-1} - 8)$
- **(i):** $\sum_{k=0}^{n} E_{4k+2} = \frac{1}{5}(2E_{4n+3} + E_{4n+2} + E_{4n+1} - E_{4n} + E_{4n-1} + 4)$
- **(j):** $\sum_{k=0}^{n} E_{4k+3} = \frac{1}{5}(2E_{4n+4} + E_{4n+3} + E_{4n+2} - E_{4n+1} + E_{4n} - 10)$
- **(k):** $\sum_{k=0}^{n} E_{5k} = \frac{1}{25}(-9E_{5n+5} + 5E_{5n+4} + 14E_{5n+3} + 18E_{5n+2} + 17E_{5n+1} + 11E_{5n} + 119)$
- **(l):** $\sum_{k=0}^{n} E_{5k+1} = \frac{1}{25}(-4E_{5n+5} + 5E_{5n+4} + 9E_{5n+3} + 8E_{5n+2} + 2E_{5n+1} - 9E_{5n} + 39)$
- **(m):** $\sum_{k=0}^{n} E_{5k+2} = \frac{1}{25}(E_{5n+5} + 5E_{5n+4} + 4E_{5n+3} - 2E_{5n+2} - 13E_{5n+1} - 4E_{5n} - 16)$
- **(n):** $\sum_{k=0}^{n} E_{5k+3} = \frac{1}{25}(6E_{5n+5} + 5E_{5n+4} - E_{5n+3} - 12E_{5n+2} - 3E_{5n+1} + E_{5n} - 46)$

(o): $\sum_{k=0}^{n} E_{5k+4} = \frac{1}{25}(11E_{5n+5} + 5E_{5n+4} - 6E_{5n+3} + 3E_{5n+2} + 7E_{5n+1} + 6E_{5n} - 51)$

3. Linear Sums of Generalized Gaussian Hexanacci Numbers

A Gaussian integer z is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib$, $a, b \in \mathbb{Z}$. If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers. Gaussian generalized Hexanacci numbers $\{GV_n\}_{n\geq 0} = \{GV_n(GV_0, GV_1, GV_2, GV_3, GV_4)\}_{n\geq 0}$ are defined by

$$GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3} + GV_{n-4} + GV_{n-5}, \tag{3.1}$$

with the initial conditions

$$\begin{aligned} GV_0 &= c_0 + (-c_0 - c_1 - c_2 - c_3 + c_4)i, GV_1 = c_1 + c_0 i, GV_2 = c_2 + c_1 i, \\ GV_3 &= c_3 + c_2 i, GV_4 = c_4 + c_3 i \end{aligned}$$

not all being zero. The sequences $\{GV_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GV_{-n} = -GV_{-(n-1)} - GV_{-(n-2)} - GV_{-(n-3)} - GV_{-(n-4)} + GV_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (3.1) hold for all integer n. Note that for $n \geq 0$

$$GV_n = V_n + iV_{n-1} \tag{3.2}$$

and

$$GV_{-n} = V_{-n} + iV_{-n-1}.$$

The first few generalized Gaussian Hexanacci numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4:

Table 3. A few Gaussian generalized Hexanacci numbers with positive subscript

n	GV_n
0	$c_0+(-c_0-c_1-c_2-c_3-c_4+c_5)i$
1	c_1+c_0i
2	c_2+c_1i
3	c_3+c_2i
4	c_4+c_3i
5	c_5+c_4i
6	$(c_0+c_1+c_2+c_3+c_4+c_5)+c_5i$
7	$(c_0+2c_1+2c_2+2c_3+2c_4+2c_5)+(c_0+c_1+c_2+c_3+c_4+c_5)i$
8	$(2c_0+3c_1+4c_2+4c_3+4c_4+4c_5)+(c_0+2c_1+2c_2+2c_3+2c_4+2c_5)i$
9	$(4c_0+6c_1+7c_2+8c_3+8c_4+8c_5)+(2c_0+3c_1+4c_2+4c_3+4c_4+4c_5)i$
10	$(8c_0+12c_1+14c_2+15c_3+16c_4+16c_5)+(4c_0+6c_1+7c_2+8c_3+8c_4+8c_5)i$
11	$(16c_0+24c_1+28c_2+30c_3+31c_4+32c_5)+(8c_0+12c_1+14c_2+15c_3+16c_4+16c_5)i$

Table 4. A few Gaussian generalized Hexanacci numbers with negative subscript

n	GV_{-n}
0	$c_0+(-c_0-c_1-c_2-c_3-c_4+c_5)i$
1	$(-c_0-c_1-c_2-c_3-c_4+c_5)+(2c_4-c_5)i$
2	$(2c_4-c_5)+(2c_3-c_4)i$
3	$(2c_3-c_4)+(2c_2-c_3)i$
4	$(2c_2-c_3)+(2c_1-c_2)i$
5	$(2c_1-c_2)+(2c_0-c_1)i$
6	$(2c_0-c_1)+(-3c_0-2c_1-2c_2-2c_3-2c_4+2c_5)i$
7	$(-3c_0-2c_1-2c_2-2c_3-2c_4+2c_5)+(c_0+c_1+c_2+c_3+5c_4-3c_5)i$
8	$(c_0+c_1+c_2+c_3+5c_4-3c_5)+(4c_3-4c_4+c_5)i$
9	$(4c_3-4c_4+c_5)+(4c_2-4c_3+c_4)i$
10	$(4c_2-4c_3+c_4)+(4c_1-4c_2+c_3)i$
11	$(4c_1-4c_2+c_3)+(4c_0-4c_1+c_2)i$

We consider two special cases of $GV_n : GV_n(0,1,1+i,2+i,4+2i) = GH_n$ is the sequence of Gaussian Hexanacci numbers and $GV_n(5-i,1+5i,3+i,7+3i,15+7i) = GE_n$ is the sequence of Gaussian Hexanacci-Lucas numbers. We formally define them as follows:

Gaussian Hexanacci numbers are defined by

$$GH_n = GH_{n-1} + GH_{n-2} + GH_{n-3} + GH_{n-4} + GH_{n-5}, \tag{3.3}$$

with the initial conditions

$$GH_0 = 0, GH_1 = 1, GH_2 = 1+i, GH_3 = 2+i, GH_4 = 4+2i$$

and Gaussian Hexanacci-Lucas numbers are defined by

(3.4) $$GE_n = GE_{n-1} + GE_{n-2} + GE_{n-3} + GE_{n-4} + GE_{n-5}$$

with the initial conditions

$$GE_0 = 5 - i, GE_1 = 1 + 5i, GE_2 = 3 + i, GE_3 = 7 + 3i, GE_4 = 15 + 7i.$$

Note that for $n \geq 0$

$$GH_n = M_n + iM_{n-1},\ GE_n = R_n + iR_{n-1}$$

and

$$GH_{-n} = M_{-n} + iM_{-n-1}, GE_{-n} = R_{-n} + iR_{-n-1}.$$

Next, we present the first few values of the Gaussian Hexanacci and Hexanacci-Lucas numbers with positive and negative subscripts in the following Table 5:

Table 5. A few Gaussian Hexanacci and Hexanacci-Lucas Numbers

n	0	1	2	3	4	5	6	7	8
GH_n	0	1	$1+i$	$2+i$	$4+2i$	$8+4i$	$16+8i$	$32+16i$	$63+32i$
GH_{-n}	0	0	0	0	i	$1-i$	-1	0	0
GE_n	$6-i$	$1+6i$	$3+i$	$7+3i$	$15+7i$	$31+15i$	$63+31i$	$120+63i$	$239+120i$
GE_{-n}	$6-i$	$-1-i$	$-1-i$	$-1-i$	$-1-i$	$-1+11i$	$11-8i$	$-8-i$	$-1-i$

The following Theorem present some summation formulas of Gaussian generalized Hexanacci numbers.

THEOREM 3.1. *For $n \geq 0$, we have the following lineer sum identities:*

(a): $\sum_{k=0}^{n} GV_k = \frac{1}{5}(GV_{n+5} - GV_{n+3} - 2GV_{n+2} - 3GV_{n+1} + GV_n - GV_5 + GV_3 + 2GV_2 + 3GV_1 + 4GV_0)$

(b): $\sum_{k=0}^{n} GV_{2k+1} = \frac{1}{5}(3GV_{2n+2} + 2GV_{2n} - GV_{2n-1} + GV_{2n-2} - 2GV_{2n-3} + 2GV_5 - 5GV_4 + 3GV_3 - 4GV_2 + 4GV_1 - 3GV_0)$,

(c): $\sum_{k=0}^{n} GV_{2k} = \frac{1}{5}(-2GV_{2n+2} + 5GV_{2n+1} + 2GV_{2n} + 4GV_{2n-1} + GV_{2n-2} + 3GV_{2n-3} - 3GV_5 + 5GV_4 - 2GV_3 + 6GV_2 - GV_1 + 7GV_0)$,

(d): $\sum_{k=0}^{n} GV_{3k} = \frac{1}{5}(-3GV_{3n+3} + 5GV_{3n+2} + 3GV_{3n+1} + GV_{3n} + 4GV_{3n-1} + 2GV_{3n-2} - 2GV_5 + 7GV_3 - GV_2 + GV_1 + 8GV_0)$,

(e): $\sum_{k=0}^{n} GV_{3k+1} = \frac{1}{5}(2GV_{3n+3} - 2GV_{3n+1} + GV_{3n} - GV_{3n-1} - 3GV_{3n-2} - 2GV_5 + 5GV_4 - 3GV_3 - GV_2 + 6GV_1 - 2GV_0)$

(f): $\sum_{k=0}^{n} GV_{3k+2} = \frac{1}{5}(2GV_{3n+3} + 3GV_{3n+1} + GV_{3n} - GV_{3n-1} + 2GV_{3n-2} + 3GV_5 - 5GV_4 - 3GV_3 + 4GV_2 - 4GV_1 - 2GV_0)$

(g): $\sum_{k=0}^{n} GV_{4k} = \frac{1}{5}(-2GV_{4n+4} + 3GV_{4n+3} + GV_{4n+2} + 3GV_{4n+1} + 2GV_{4n} + 2GV_{4n-1} - 2GV_5 + 4GV_4 - GV_3 + GV_2 - GV_1 + 5GV_0)$

(h): $\sum_{k=0}^{n} GV_{4k+1} = \frac{1}{5}(GV_{4n+4} - GV_{4n+3} + GV_{4n+2} - 2GV_{4n-1} + 2GV_5 - 3GV_4 - GV_3 - 3GV_2 + 3GV_1 - 2GV_0)$

- **(i):** $\sum_{k=0}^{n} GV_{4k+2} = \frac{1}{5}(2GV_{4n+3}+GV_{4n+2}+GV_{4n+1}-GV_{4n}+GV_{4n-1}-GV_5+GV_4-GV_3+5GV_2+2GV_0)$
- **(j):** $\sum_{k=0}^{n} GV_{4k+3} = \frac{1}{5}(2GV_{4n+4}+GV_{4n+3}+GV_{4n+2}-GV_{4n+1}+GV_{4n}-2GV_4+4GV_3-GV_2+GV_1-GV_0)$
- **(k):** $\sum_{k=0}^{n} GV_{5k} = \frac{1}{25}(-9GV_{5n+5}+5GV_{5n+4}+14GV_{5n+3}+18GV_{5n+2}+17GV_{5n+1}+11GV_{5n}+9GV_5-5GV_4-14GV_3-18GV_2-17GV_1+14GV_0)$
- **(l):** $\sum_{k=0}^{n} GV_{5k+1} = \frac{1}{25}(-4GV_{5n+5}+5GV_{5n+4}+9GV_{5n+3}+8GV_{5n+2}+2GV_{5n+1}-9GV_{5n}+4GV_5-5GV_4-9GV_3-8GV_2+23GV_1+9GV_0)$
- **(m):** $\sum_{k=0}^{n} GV_{5k+2} = \frac{1}{25}(GV_{5n+5}+5GV_{5n+4}+4GV_{5n+3}-2GV_{5n+2}-13GV_{5n+1}-4GV_{5n}-GV_5-5GV_4-4GV_3+27GV_2+13GV_1+4GV_0)$
- **(n):** $\sum_{k=0}^{n} GV_{5k+3} = \frac{1}{25}(6GV_{5n+5}+5GV_{5n+4}-GV_{5n+3}-12GV_{5n+2}-3GV_{5n+1}+GV_{5n}-6GV_5-5GV_4+26GV_3+12GV_2+3GV_1-GV_0)$
- **(o):** $\sum_{k=0}^{n} GV_{5k+4} = \frac{1}{25}(11GV_{5n+5}+5GV_{5n+4}-6GV_{5n+3}+3GV_{5n+2}+7GV_{5n+1}+6GV_{5n}-11GV_5+20GV_4+6GV_3-3GV_2-7GV_1-6GV_0)$.

Proof. (a)-(o) can be proved exactly as in the proof of Theorem 2.1.

As special cases of the above Theorem, we have the following two Corollaries. First one present summation formulas of Gaussian Hexanacci numbers.

COROLLARY 3.2. *For $n \geq 0$, we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} GH_k = \frac{1}{5}(GH_{n+5}-GH_{n+3}-2GH_{n+2}-3GH_{n+1}+GH_n-1-i)$
- **(b):** $\sum_{k=0}^{n} GH_{2k+1} = \frac{1}{5}(3GH_{2n+2}+2GH_{2n}-GH_{2n-1}+GH_{2n-2}-2GH_{2n-3}+2-3i)$
- **(c):** $\sum_{k=0}^{n} GH_{2k} = \frac{1}{5}(-2GH_{2n+2}+5GH_{2n+1}+2GH_{2n}+4GH_{2n-1}+GH_{2n-2}+3GH_{2n-3}-3+2i)$
- **(d):** $\sum_{k=0}^{n} GH_{3k} = \frac{1}{5}(-3GH_{3n+3}+5GH_{3n+2}+3GH_{3n+1}+GH_{3n}+4GH_{3n-1}+2GH_{3n-2}-2-2i)$
- **(e):** $\sum_{k=0}^{n} GH_{3k+1} = \frac{1}{5}(2GH_{3n+3}-2GH_{3n+1}+GH_{3n}-GH_{3n-1}-3GH_{3n-2}+3-2i)$
- **(f):** $\sum_{k=0}^{n} GH_{3k+2} = \frac{1}{5}(2GH_{3n+3}+3GH_{3n+1}+GH_{3n}-GH_{3n-1}+2GH_{3n-2}-2+3i)$
- **(g):** $\sum_{k=0}^{n} GH_{4k} = \frac{1}{5}(-2GH_{4n+4}+3GH_{4n+3}+GH_{4n+2}+3GH_{4n+1}+2GH_{4n}+2GH_{4n-1}-2)$
- **(h):** $\sum_{k=0}^{n} GH_{4k+1} = \frac{1}{5}(GH_{4n+4}-GH_{4n+3}+GH_{4n+2}-2GH_{4n-1}+2-2i)$
- **(i):** $\sum_{k=0}^{n} GH_{4k+2} = \frac{1}{5}(2GH_{4n+3}+GH_{4n+2}+GH_{4n+1}-GH_{4n}+GH_{4n-1}-1+2i)$
- **(j):** $\sum_{k=0}^{n} GH_{4k+3} = \frac{1}{5}(2GH_{4n+4}+GH_{4n+3}+GH_{4n+2}-GH_{4n+1}+GH_{4n}-i)$
- **(k):** $\sum_{k=0}^{n} GH_{5k} = \frac{1}{25}(-9GH_{5n+5}+5GH_{5n+4}+14GH_{5n+3}+18GH_{5n+2}+17GH_{5n+1}+11GH_{5n}-11-6i)$
- **(l):** $\sum_{k=0}^{n} GH_{5k+1} = \frac{1}{25}(-4GH_{5n+5}+5GH_{5n+4}+9GH_{5n+3}+8GH_{5n+2}+2GH_{5n+1}-9GH_{5n}+9-11i)$
- **(m):** $\sum_{k=0}^{n} GH_{5k+2} = \frac{1}{25}(GH_{5n+5}+5GH_{5n+4}+4GH_{5n+3}-2GH_{5n+2}-13GH_{5n+1}-4GH_{5n}+4+9i)$
- **(n):** $\sum_{k=0}^{n} GH_{5k+3} = \frac{1}{25}(6GH_{5n+5}+5GH_{5n+4}-GH_{5n+3}-12GH_{5n+2}-3GH_{5n+1}+GH_{5n}-1+4i)$
- **(o):** $\sum_{k=0}^{n} GH_{5k+4} = \frac{1}{25}(11GH_{5n+5}+5GH_{5n+4}-6GH_{5n+3}+3GH_{5n+2}+7GH_{5n+1}+6GH_{5n}-6-i)$

Next Corollary gives some summation formulas of Gaussian Hexanacci-Lucas numbers.

COROLLARY 3.3. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{k=0}^{n} GE_k = \frac{1}{5}(GE_{n+5} - GE_{n+3} - 2GE_{n+2} - 3GE_{n+1} + GE_n + 9 + 4i)$

(b): $\sum_{k=0}^{n} GE_{2k+1} = \frac{1}{5}(3GE_{2n+2} + 2GE_{2n} - GE_{2n-1} + GE_{2n-2} - 2GE_{2n-3} - 18 + 27i)$

(c): $\sum_{k=0}^{n} GE_{2k} = \frac{1}{5}(-2GE_{2n+2} + 5GE_{2n+1} + 2GE_{2n} + 4GE_{2n-1} + GE_{2n-2} + 3GE_{2n-3} + 27 - 23i)$

(d): $\sum_{k=0}^{n} GE_{3k} = \frac{1}{5}(-3GE_{3n+3} + 5GE_{3n+2} + 3GE_{3n+1} + GE_{3n} + 4GE_{3n-1} + 2GE_{3n-2} + 33 - 12i)$

(e): $\sum_{k=0}^{n} GE_{3k+1} = \frac{1}{5}(2GE_{3n+3} - 2GE_{3n+1} + GE_{3n} - GE_{3n-1} - 3GE_{3n-2} - 17 + 33i)$

(f): $\sum_{k=0}^{n} GE_{3k+2} = \frac{1}{5}(2GE_{3n+3} + 3GE_{3n+1} + GE_{3n} - GE_{3n-1} + 2GE_{3n-2} - 7 - 17i)$

(g): $\sum_{k=0}^{n} GE_{4k} = \frac{1}{5}(-2GE_{4n+4} + 3GE_{4n+3} + GE_{4n+2} + 3GE_{4n+1} + 2GE_{4n} + 2GE_{4n-1} + 23 - 15i)$

(h): $\sum_{k=0}^{n} GE_{4k+1} = \frac{1}{5}(GE_{4n+4} - GE_{4n+3} + GE_{4n+2} - 2GE_{4n-1} - 8 + 23i)$

(i): $\sum_{k=0}^{n} GE_{4k+2} = \frac{1}{5}(2GE_{4n+3} + GE_{4n+2} + GE_{4n+1} - GE_{4n} + GE_{4n-1} + 4 - 8i)$

(j): $\sum_{k=0}^{n} GE_{4k+3} = \frac{1}{5}(2GE_{4n+4} + GE_{4n+3} + GE_{4n+2} - GE_{4n+1} + GE_{4n} - 10 + 4i)$

(k): $\sum_{k=0}^{n} GE_{5k} = \frac{1}{25}(-9GE_{5n+5} + 5GE_{5n+4} + 14GE_{5n+3} + 18GE_{5n+2} + 17GE_{5n+1} + 11GE_{5n} + 119 - 76i)$

(l): $\sum_{k=0}^{n} GE_{5k+1} = \frac{1}{25}(-4GE_{5n+5} + 5GE_{5n+4} + 9GE_{5n+3} + 8GE_{5n+2} + 2GE_{5n+1} - 9GE_{5n} + 39 + 119i)$

(m): $\sum_{k=0}^{n} GE_{5k+2} = \frac{1}{25}(GE_{5n+5} + 5GE_{5n+4} + 4GE_{5n+3} - 2GE_{5n+2} - 13GE_{5n+1} - 4GE_{5n} - 16 + 39i)$

(n): $\sum_{k=0}^{n} GE_{5k+3} = \frac{1}{25}(6GE_{5n+5} + 5GE_{5n+4} - GE_{5n+3} - 12GE_{5n+2} - 3GE_{5n+1} + GE_{5n} - 46 - 16i)$

(o): $\sum_{k=0}^{n} GE_{5k+4} = \frac{1}{25}(11GE_{5n+5} + 5GE_{5n+4} - 6GE_{5n+3} + 3GE_{5n+2} + 7GE_{5n+1} + 6GE_{5n} - 11GE_5 + 20GE_4 + 6GE_3 - 3GE_2 - 7GE_1 - 6GE_0)$

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