Original Research Article

The Marshall-Olkin Inverse Lomax Distribution (MO-ILD) with Application on Cancer Stem Cell.

Abstract:

In this article, a new compound distribution called the Marshall Olkin Inverse Lomax distribution was proposed, extending the Inverse Lomax distribution. Its basic statistical properties were derived and model parameters estimated by the maximum likelihood estimate method. The Proposed distribution was applied on Cancer Stem Cell data and compared to the Marshall Olkin Flexible Weibull Extension Distribution, and the Marshall-Olkin exponential Weibull distribution (MO-EWD).

Keywords: marshall-olkin inverse Lomax distribution; inverse Lomax; distributions; quantile function, hazard functions, survival function, maximum likelihood estimate.

1. Introduction

The problem of finding an appropriate model for real-life data has been thoroughly researched in literature. But there are many scenarios in which standard models are not suitable or less indicative of actual data. Most traditional distributions don't fit the actual data well. In the literature, several distributions have been proposed to model lifetime data by combining some useful lifetime distributions with discreet distribution in order to overcome these problems. The Lomax distribution also referred to as “Pareto type II” distribution is a special case of the generalized Pareto distribution and can be seen in many fields of application such as actuarial science, economics and so on [1]. It is a heavy-tailed distribution and considered to be useful in survival and life testing problems in engineering and in survival analysis as an alternative distribution [2]. An inverted version of the Lomax distribution called the Inverse Lomax distribution has been investigated and found to be very flexible to analyze the situation where the non-monotonicity of the failure rate has been realized, see Singh et al. [3]. Uzma Jan and S.P Ahmad [4] investigating the behavior of Inverse Lomax Distribution's shape parameter by employing different approximation techniques, such as Normal approximation and Tierney and Kadane (T-K) approximation. Kleiber [5] used the Inverse Lomax distribution for Lorenz's ordered statistical relationship. This lifetime distribution was applied by McKenzie [6] to geophysical data particularly for land fibre sizes in the California United States. Rahman et al [7] studied in detail the estimated and predicted values calculated by the Bayesian approach using different loss functions. In addition, Rahman and Aslam [8] used two Inverse Lomax component mixture models to predict future ordered observations using predictive models in the Bayesian framework. Singh et. al. [9] examined the said model and obtained its survival estimates under Type II censoring using Markov Chain Monte Carlo method. Furthermore, Yadav et.al [10] applied the hybrid, censored Inverse Lomax distribution to the survival data.

A random variable X is said to have an Inverted Lomax Distribution with parameters α and β if its probability density function (pdf) is given by the following equation [10]:

$$f(x;\alpha,\beta) = \frac{\alpha\beta}{x^2}\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)} ;\ x \geq 0, \alpha, \beta > 0 \qquad (1)$$

While the cumulative distribution function (cdf) is given by the equation

$$F(x) = \left(1+\frac{\beta}{x}\right)^{-\alpha} \quad ;\ x \geq 0, \alpha, \beta > 0 \qquad (2)$$

The survival function is given by the equation

$$S(t) = 1-\left(1+\frac{\beta}{t}\right)^{-\alpha} ;\ t \geq 0, \alpha, \beta > 0 \qquad (3)$$

The hazard function is

$$H(t) = \frac{\alpha\beta\left(1+\frac{\beta}{t}\right)^{-(1+\alpha)}}{x^2\left[1-\left(1+\frac{\beta}{t}\right)^{-\alpha}\right]} \quad ;\ t \geq 0, \alpha, \beta > 0 \qquad (4)$$

And the reversed hazard rate function is

$$R(x) = \frac{\alpha\beta\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}}{x^2\left[\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]} ;\ x \geq 0, \alpha, \beta > 0 \qquad (5)$$

The addition of parameters to an existing distribution allows for more flexible classes [11]. Marshall and Olkin [12] introduced a new method for adding a new parameter to the existing distribution, leading to greater flexibility in modeling various data types. They considered a case of an arbitrary continuous distribution called a baseline distribution with a cumulative distribution function G(x) and a corresponding probability density function pdf g(x). The associated Marshall–Olkin extended family of distribution then has the cumulative distribution function given by

$$F_{MO}(x) = \frac{G(x)}{1-(1-\theta)S(x)} \qquad -\infty < x < \infty, \theta > 0 \quad (6)$$

The probability density function is given by

$$f_{MO}(x) = \frac{\theta g(x)}{[1-(1-\theta)S(x)]^2} \qquad -\infty < x < \infty \qquad (7)$$

The survival function is given by

$$S_{MO}(x) = \frac{\theta S(x)}{1-(1-\theta)S(x)} \qquad (8)$$

The hazard function is given by

$$h_{MO}(x) = \frac{h(x)}{1-(1-\theta)S(x)} \qquad (9)$$

The reversed hazard function is given by

$$r_{MO}(x) = \frac{\theta r(x)}{1-(1-\theta)S(x)} \qquad (10)$$

And the cumulative hazard function is given by

$$H_{MO}(x) = -\log\big(S_{MO}(x)\big) = -\log\frac{\theta S(x)}{1-(1-\theta)S(x)} \qquad (11)$$

Thus, in this paper we proposed a new generalization of the inverse Lomax distribution called the Marshall-Olkin Inverse Lomax distribution (MO-ILD). This paper is structured as follows; we derive the cumulative distribution, probability density, survival function, and hazard functions of the Marshall-Olkin Inverse Lomax distribution in section 2. In minute details, we introduced some of the asymptotic and statistical properties, which include the mode, quantile function, Skewness and kurtosis in section 3. In section 4, we determined the maximum likelihood estimate of the unknown parameters, and the order statistics in section 5. Application to cancer stem cell data and a final remark is introduced in section 6 and section 7 respectively.

2. Marshall-Olkin Inverse Lomax distribution

The three parameters of the Marshall- Olkin Inverse Lomax distribution have been examined in this section. Substitution of Equations (2), and (3) into equation (6), the Marshall- Olkin Inverse Lomax distribution (MO- ILD) cumulative distribution function is given by

$$F_{MO-ILD}(x) = \frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]} \quad x \geq 0, \alpha, \beta, \theta > 0 \qquad (12)$$

The corresponding probability density function is obtained by substituting Equations (1), and (3) into equation (7)

$$f_{MO-ILD}(x) = \frac{\theta\alpha\beta\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}}{x^2\left[1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\right]^2} \quad x \geq 0, \alpha, \beta, \theta > 0 \qquad (13)$$

The survival function is obtained by substituting equation (3) into equation (8)

$$S_{MO-ILD}(x) = \frac{\theta\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]}{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]} \tag{14}$$

The hazard function is obtained by substituting equation (3) and (4) into equation (9)

$$h_{MO-ILD}(x) = \frac{\alpha\beta\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}}{x^2\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\left\{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\right\}} \tag{15}$$

The reverse hazard function is obtained by substituting equation (3) and (5) into equation (10)

$$r_{MO-ILD}(x) = \frac{\theta\alpha\beta\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}}{x^2\left[\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\left\{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\right\}} \tag{16}$$

The cumulative hazard function is obtained by substituting equation (3) into equation (11)

$$H_{MO-ILD}(x) = -\log\frac{\theta\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]}{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]} \tag{17}$$

with $x \geq 0, \alpha, \beta, \theta > 0$ in equations (14), (15), (16), (17) respectively.

For some parameter values, figures (1-6) indicates the cumulative distribution function, probability density function, survival function, hazard function of the Marshall-Olkin inverse Lomax distribution.

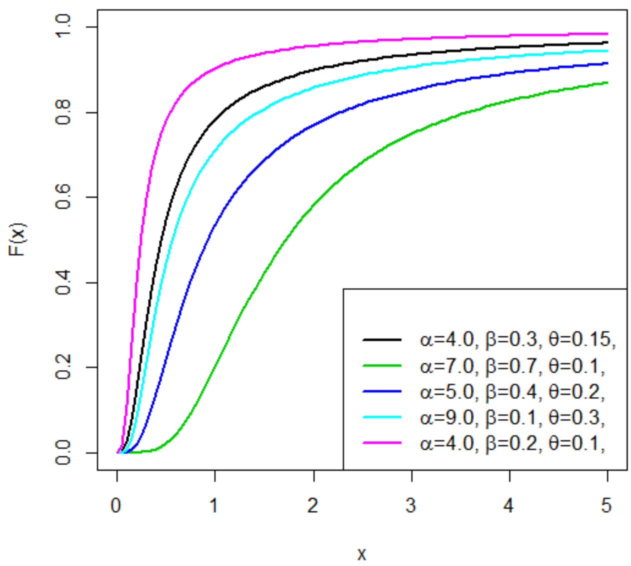

Figure 1: Cumulative distribution function of the MO-ILD

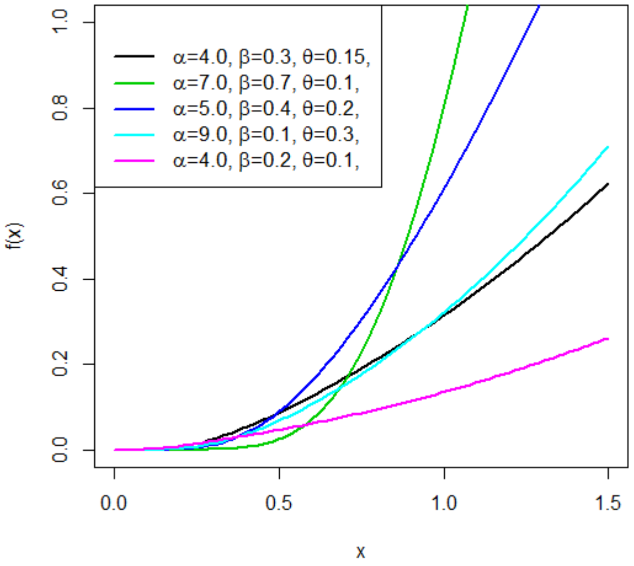

Figure 2: Probability density function of the MO-ILD

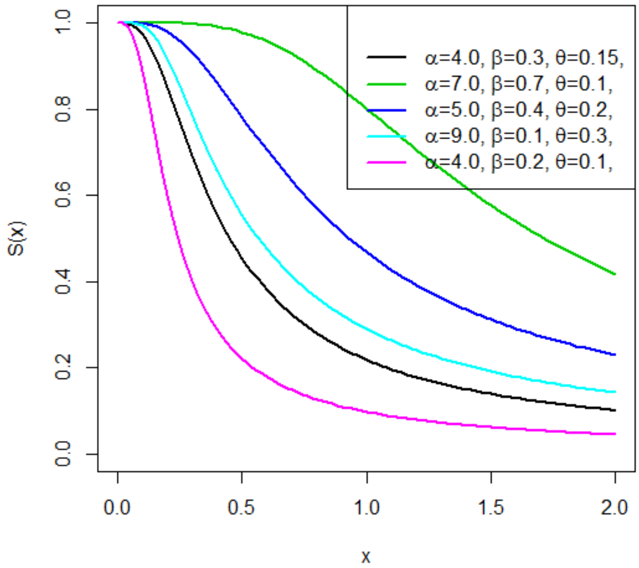

Figure 3: Survival function of the MO-ILD

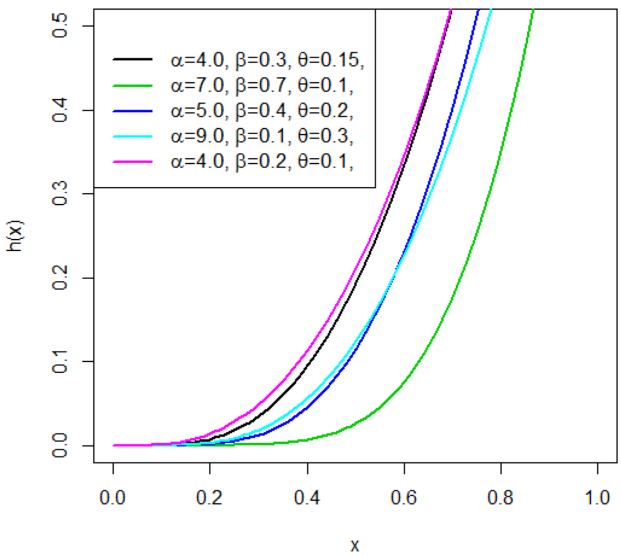

Figure 4: Hazard function of the MO-ILD

3. Some Statistical Properties of the MO-ILD

The statistical properties, especially asymptotic behavior, quantile, median, Skewness, and kurtosis of the Marshall-Olkin Inverse Lomax distribution are discussed in this section.

3.1 Asymptotic Behavior

The behavior of the Marshall-Olkin Inverse Lomax distribution model in equation (13) is investigated here as $x \to 0$

$$\lim_{x\to 0} f_{MO-ILD}(x) = \lim_{x\to 0} \frac{\theta\alpha\beta\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}}{x^2\left[1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\right]^2} = 0$$

Lemma 1: It can be shown that for the K-IL Distribution, $\lim_{x\to\infty} F_{MO-ILD}(x) =$

Proof

$$\lim_{x\to\infty} F_{MO-ILD}(x) = \frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]}$$

$$= \frac{\left(1+\frac{\beta}{\infty}\right)^{-\alpha}}{1-(1-\theta)\left[1-\left(1+\frac{\beta}{\infty}\right)^{-\alpha}\right]}$$

$$= \frac{(1+0)^{-\alpha}}{1-(1-\theta)[1-(1+0)^{-\alpha}]}$$

$$= \frac{1}{1-(1-\theta)[1-1]}$$

$$= \frac{1}{1-(1-\theta)[0]}$$

$$= 1$$

From which we conclude that the MO-ILD has a proper PDF, the shape of the MO-ILD could be increasing or decreasing, it is also positively or negatively skewed depending on the values of the parameter.

3.2 *Quantile function and Median*

Quantile function may be derived from

$$Q_{MO-ILD}(u) = F_{MO-ILD}{}^{-1}(u)$$

$$Let\ u = \frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]}$$

$$u\left[1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\right] = \left(1+\frac{\beta}{x}\right)^{-\alpha}$$

$$\left[u-u(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\right] = \left(1+\frac{\beta}{x}\right)^{-\alpha}$$

$$\left[u-u(1-\theta)+u(1-\theta)\left(1+\frac{\beta}{x}\right)^{-\alpha}\right] = \left(1+\frac{\beta}{x}\right)^{-\alpha}$$

$$\left[u\theta+u(1-\theta)\left(1+\frac{\beta}{x}\right)^{-\alpha}\right] = \left(1+\frac{\beta}{x}\right)^{-\alpha}$$

$$u\theta = \left(1+\frac{\beta}{x}\right)^{-\alpha} - u(1-\theta)\left(1+\frac{\beta}{x}\right)^{-\alpha}$$

$$u\theta = \left(1+\frac{\beta}{x}\right)^{-\alpha}[1-u+u\theta]$$

$$\frac{u\theta}{[1-u+u\theta]} = \left(1+\frac{\beta}{x}\right)^{-\alpha}$$

$$\left[\frac{u\theta}{[1-u+u\theta]}\right]^{-1/\alpha} = \left(1+\frac{\beta}{x}\right)$$

$$1-\left[\frac{u\theta}{[1-u+u\theta]}\right]^{-1/\alpha} = \frac{\beta}{x}$$

$$x = \beta\left[1-\left[\frac{u\theta}{[1-u+u\theta]}\right]^{-1/\alpha}\right]^{-1}$$

Hence, the Quantile function of the Marshall-Olkin Inverse Lomax distribution is given by;

$$Q(u) = \beta\left[1 - \left[\frac{u\theta}{[1-u+u\theta]}\right]^{-1/\alpha}\right]^{-1} \quad (18)$$

Where $u \sim Uniform(0,1)$.

Random numbers can be generated from the Marshall-Olkin Inverse Lomax distribution using

$$x = \beta\left[1 - \left[\frac{u\theta}{[1-u+u\theta]}\right]^{-1/\alpha}\right]^{-1}$$

The median of the Marshall-Olkin Inverse Lomax distribution is obtained by substituting $u = 0.5$ in equation (18) above, we obtain

$$Median = \beta\left[1 - \left[\frac{0.5\theta}{0.5(1+\theta)}\right]^{-1/\alpha}\right]^{-1}$$

3.2 Skewness and Kurtosis

The analysis of the Skewness and Kurtosis variability on the shape parameters can be examined on the basis of quantile action. The weaknesses of the conventional measure of kurtosis are well known. Kenney and Keeping [13] gives the Bowely Skewness based on quartiles as;

$$S_k = \frac{q_{(0.75)} - 2q_{(0.5)} + q_{(0.25)}}{q_{(0.75)} - q_{(0.25)}} \quad (19)$$

Moors [14] gave the Moors quantile based Kurtosis as;

$$K_u = \frac{q_{(0.875)} - q_{(0.625)} - q_{(0.375)} + q_{(0.125)}}{q_{(0.75)} - q_{(0.25)}} \quad (20)$$

With $q_{(.)}$ representing quantile function

4. Parameter Estimation

In this section, the unknown parameter of the Marshall Olkin Inverse Lomax distribution based on a complete sample is estimated using maximum estimation techniques. Let $X_1, \ldots X_n$ indicate a random sample of the complete MO-ILD data, and then the sample's likelihood function is given as;

$$L = \prod_{i=1}^{n} f(x_i; \theta, \alpha, \beta)$$

$$L = \prod_{i=1}^{n} \frac{\theta\alpha\beta\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha)}}{{x_i}^2\left[1-(1-\theta)\left[1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha}\right]\right]^2}$$

The log likelihood function may be expressed as;

$$\ln L\,(x_i; \theta\alpha, \beta) = n[\ln\theta + \ln\alpha + \ln\beta] - 2\sum_{i=1}^{n}\ln x_i - (1+\alpha)\sum_{i=1}^{n}\ln\left(1+\frac{\beta}{x_i}\right) - 2\sum_{i=1}^{n}\ln\left[1-(1-\theta)(1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha})\right] \quad (21)$$

By taking the derivative with respect to $\theta, \alpha, and\ \beta$, and fixing the outcome to zero, we have;

$$\frac{\partial \ln L(x)}{\partial\theta} = \frac{n}{\theta} - \frac{2\sum_{i=1}^{n}\left(1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha}\right)}{1-(1-\theta)(1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha})} = 0$$

$$\frac{\partial \ln L(x)}{\partial\alpha} = \frac{n}{\alpha} - \frac{2\sum_{i=1}^{n}\left(1+\frac{\beta}{x_i}\right)^{-\alpha}\left[\ln\theta\left(1+\frac{\beta}{x_i}\right)^{\theta-1}\right]}{1-(1-\theta)(1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha})} = 0$$

$$\frac{\partial \ln L(x)}{\partial\beta} = \frac{n}{\beta} - \frac{-(1+\alpha)\sum_{i=1}^{n}x_i^{-1}}{\left(1+\frac{\beta}{x_i}\right)} + \frac{2\alpha(1-\theta)\sum_{i=1}^{n}(x+\beta)^{-(\alpha+1)}x^{\alpha}}{1-(1-\theta)(1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha})} = 0$$

By solving the above derivatives iteratively, we obtain the parameter estimate of the Marshall Olkin Inverse Lomax distribution

5. Order Statistics

Let $X_1, \ldots X_n$ denote a random sample from the densities of a Marshall Olkin Inverse Lomax distribution as given in equation (12) and (13) respectively. Then the probability density function $f_{w:n}(x)$ of the jth order statistics of the MO-ILD is given by;

$$f_{j:n}(x) = \frac{n!}{(j-1)!\,(n-1)!} f_{MO-ILD}(x) F_{MO-ILD}(x)^{j-1}[1-F_{MO-ILD}(x)]^{n-j}$$

The pdf of the kth order statistics for the MO-ILD is therefore obtained as;

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!}\left(\frac{\theta\alpha\beta\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}}{x^2\left[1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\right]^2}\right)\left(\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]}\right)^{j-1}\left[1-\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]}\right]^{n-j} \quad (22)$$

When $j = 1$, the distribution of minimum order statistics for the Marshall Olkin Inverse Lomax distribution is given as;

$$f_{1:n}(x) = n\left(\frac{\theta\alpha\beta\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}}{x^2\left[1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\right]^2}\right)\left[1-\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]}\right]^{n-1} \quad (23)$$

And the maximum order statistics for the Marshall Olkin Inverse Lomax distribution obtained by substituting $j = n$ is given as;

$$f_{n:n}(x) = n\left(\frac{\theta\alpha\beta\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}}{x^2\left[1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]\right]^2}\right)\left(\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-(1-\theta)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]}\right)^{n-1} \quad (24)$$

6. Application to Cancer Stem Cell

The Marshall Olkin Inverse Lomax distribution was applied for illustration purposes to a real-life dataset and its performance was compared to the other fitted model like the Marshall Olkin Flexible Weibull Extension Distribution (MO-FWED) [15], and the Marshall-Olkin exponential Weibull distribution (MO-EWD) [11]. The selection criteria for the most appropriate was based on values of the Log-likelihood, Akaike Information Criterion (AIC), Constant Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) and Hannan-Quinn information criterion (HQIC). The Shapiro Wilks (SW) tests, Kolmogorov Smirnov (K-S) statistic, Anderson Darling (AD) statistic with its corresponding p-value are also recorded.

Data I: The Cancer Stem Cells Patients Data Set

A random sample of 128 patients with bladder cancers reported in Lee and Wang [16] is presented in the data set (in months). The observations are as follows

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51.

Table 1: Descriptive Statistics on Cancer Stem Cell Data

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis	Variance
0.080	3.250	5.620	9.258	11.250	79.050	3.31811	18.06265	119.1731

Table 2: MLEs, SW, AD and K–S of parameters for Cancer Stem Cell data

Model	Estimates			SW	K-S	AD	P-Value
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$				
MO-ILD (Proposed Model)	2.1024	0.6054	1.1804	0.1125	0.1758	0.4145	0.4120
MO-FWED	0.6251	1.1241	0.1145	0.3124	0.5124	0.7451	0.3315
MO-EWD	0.9715	2.0415	1.1874	0.3397	0.5429	0.9813	0.3218

For all competing distributions using the Cancer data set, Table 2 shows parameter estimate and the value for the Shapiro Wilk (S-W), Anderson Darling (AD), and the Kolmogorov Smirnov (K-S) statistic.

Table 3: Log-likelihood, AIC, AICC, BIC and HQIC values of models fitted for Cancer Stem Cell data

Model	l	AIC	CAIC	BIC	HQIC
MO-ILD (Proposed Model)	-130.5010	283.5141	283.789	289.0148	285.6514
MO-FWED	-161.2495	291.0147	291.9108	297.0936	296.5912
MO-EWD	-167.1963	293.9126	294.0018	302.1796	298.0145

From Table 3, the MO-ILD has the highest log-likelihood values and the lowest AIC, CAIC, BIC and HQIC values; hence, the MO-ILD is chosen as the most appropriate model amongst the considered distributions.

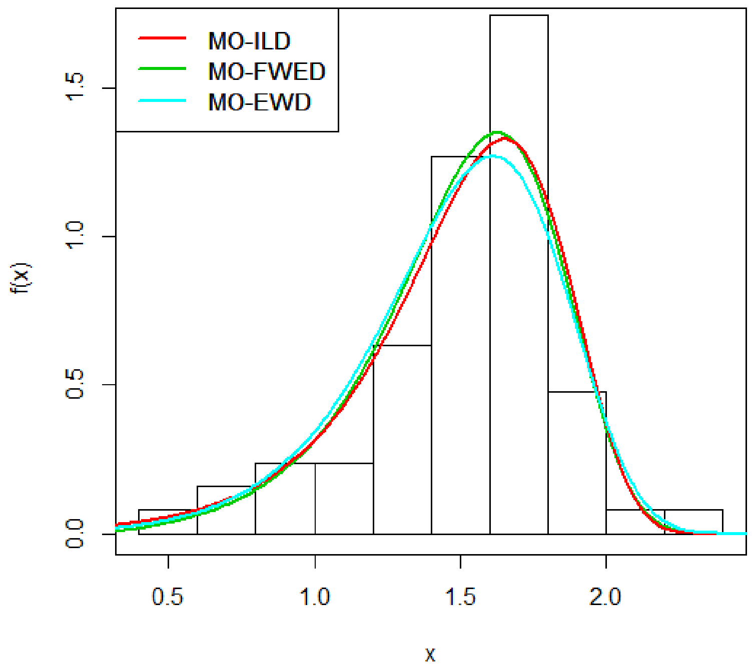

Figure 5: Histogram of the fitted distributions

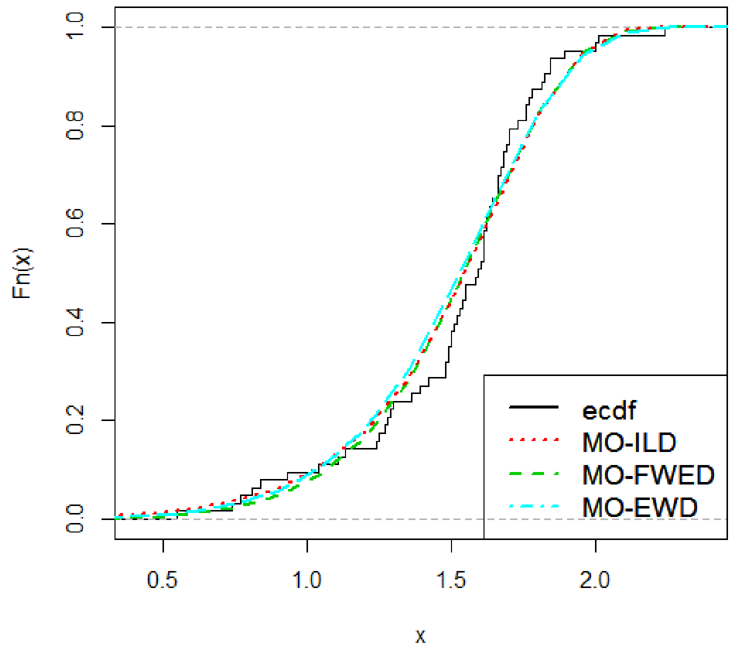

Figure 6: Empirical cdf of the fitted distributions

7. Conclusion

The Inverse Lomax distribution has been successfully extended in this research. Densities were carefully derived and expressions for the basic statistical characteristics for the Marshall-Olkin Inverse Lomax distribution have been developed. The Marshall Olkin Inverse Lomax distribution was applied to Cancer Stem data set and provided a better fit than the than the Marshall Olkin Flexible Weibull Extension Distribution, and the Marshall-Olkin exponential Weibull distribution based on log-likelihood AIC, CAIC, BIC and HQIC values. We therefore, conclude that the Marshall Olkin Inverse Lomax distribution is the most appropriate model amongst the considered distributions, and a very competitive model for describing life time phenomenon.

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