Determinants and Inverses of Skew Symmetric Generalized Foeplitz Matrices

Abstract

In this paper, explicit determinants and inverses of skew symmetric generalized Foeplitz matrices are given by constructing the special transformation matrices.

Keywords: generalized Foeplitz matrices; determinant; inverse; Fibonacci number.

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1 Introduction

Toeplitz matrices have important applications in various disciplines including image processing, signal processing, and solving least squares problems [Mukhexjee and Maiti (1988)]. It is an ideal research area and hot topic for the inverses of Toeplitz matrices and the special matrices with famous numbers. Due to the special structure, it is a major topic of research that the inversion of Toeplitz matrices can be reconstructed by use of a low number of its columns and the entries of the original Toeplitz matrix. The stability of the algorithms emerging from Toeplitz matrix inversion formulas was considered in [Wen et al. (2004)].

In addition, some researchers showed the explicit determinants and inverses of the special matrices involving famous numbers. In [Akbulak and Bozkurt (2008)], M. Akbulak and D. Bozkurt discussed originally Fibonacci and Lucas Toeplitz matrices with entries from Fibonacci and Lucas numbers, and they presented the upper and lower bounds for the spectral norms of the Fibonacci and Lucas Toeplitz matrix. The authors considered the determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [Bozkurt and Tam (2012)]. In [Jiang et al. (2014a)], circulant type matrices with the k-Fibonacci and k-Lucas numbers are presented and the explicit determinants and inverse matrices are presented by constructing the transformation matrices. Jiang et al. [Jiang et al. (2014b)] gave the invertibility of circulant type matrices with the sum and product of Fibonacci and Lucas numbers and provided the determinants and the inverses of these matrices. And for the RSFPLR circulant matrices and the RSLPFL circulant matrices involving Padovan, Perrin, Tribonacci and the generalized Lucas numbers by the inverse factorization of polynomial in [Jiang and Hong

(2014)].

It should be noted that Jiang and Zhou [Jiang and Zhou (2015)] obtained the explicit formula for spectral norm of an r-circulant matrix whose entries in the first row are alternately positive and negative, and the authors [Zhou and Jiang (2014)] investigated explicit formulas of spectral norms for g-circulant matrices with Fibonacci and Lucas numbers. The authors [Zheng and Shon (2015)] proposed the invertibility criterium of the generalized Lucas skew circulant type matrices and provided their determinants and the inverse matrices. Furthermore, in [Jiang and Hong (2015)] the determinants and inverses are discussed and evaluated for Tribonacci skew circulant type matrices.

In this paper, we will show the explicit determinants and inverses of the skew symmetric generalized Foeplitz matrices.

Here the Fibonacci sequence is defined by the following recurrence relation:

$$F_{n+1} = F_n + F_{n-1}\ (n \geqslant 1) \text{ where } F_0 = 0,\ F_1 = 1.$$

Definition 1.1. An $n \times n$ skew symmetric generalized Foeplitz matrix is meant a square matrix of the form

$$\mathbf{T}_{F_k,n} = \begin{pmatrix} 0 & F_k & F_{k+1} & \cdots & F_{k+n-3} & F_{k+n-2} \\ -F_k & 0 & F_k & \cdots & F_{k+n-4} & F_{k+n-3} \\ -F_{k+1} & -F_k & 0 & \cdots & F_{k+n-5} & F_{k+n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -F_{k+n-3} & -F_{k+n-4} & -F_{k+n-5} & \cdots & 0 & F_k \\ -F_{k+n-2} & -F_{k+n-3} & -F_{k+n-4} & \cdots & -F_k & 0 \end{pmatrix}_{n\times n}, \tag{1.1}$$

where $F_k,\ F_{k+1}, \cdots,\ F_{k+n-2}$ are the Fibonacci numbers, and $k \geq 2$.

Obviously, this matrix is completely determined by its first row, and $\mathbf{T}_{F_k,n}^{\mathrm{T}} = -\mathbf{T}_{F_k,n}$.

Specially, in the case of $k = 2$, explicit determinants and inverses of Fibonacci skew symmetric Toeplitz matrices are given in [Chen et al. (2016)].

2 Determinants and inverses of the skew symmetric generalized Foeplitz matrices

In this section, we will give the determinant and the inverse of the matrix $\mathbf{T}_{F_k,n}$.

Obviously, the determinant of an n-dimension skew symmetric matrix is zero, if n is an odd number. So in this essay we always assume that n is an even number.

Theorem 2.1. *Let $\mathbf{T}_{F_k,n}$ be a skew symmetric generalized Foeplitz matrix as the form of (1.1), we have*

$$\begin{aligned} \det \mathbf{T}_{F_k,n} =& F_{k+n-2}[\alpha_1 \det G_{n-2}([\beta_i]_{i=2}^{n-1}, -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) - \\ & \beta_1 \det G_{n-2}([F_i]_{i=k+n-3}^{k}, -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1})] \end{aligned} \tag{2.1}$$

where

$$G_{n-2}([\beta_i]_{i=2}^{n-1}, -F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1})$$
$$= \begin{pmatrix} \beta_2 & \beta_3 & \beta_4 & \cdots & \cdots & \cdots & \beta_{n-3} & \beta_{n-2} & \beta_{n-1} \\ 2F_k & F_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ F_{k+2} & 2F_k & F_{k-1} & \ddots & & & & & \vdots \\ F_{k-1}-2F_k & 0 & 2F_k-F_{k-1} & \ddots & \ddots & & & & \vdots \\ -F_{k-1} & F_{k-1}-2F_k & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -F_{k-1} & F_{k-1}-2F_k & 0 & 2F_k-F_{k-1} & F_{k-1} \end{pmatrix}_{(n-2)\times(n-2)},$$

$$G_{n-2}([F_i]_{i=k+n-3}^{k}, -F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1})$$
$$= \begin{pmatrix} F_{k+n-3} & F_{k+n-4} & F_{k+n-5} & \cdots & \cdots & \cdots & F_{k+2} & F_{k+1} & F_k \\ 2F_k & F_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ F_{k+2} & 2F_k & F_{k-1} & \ddots & & & & & \vdots \\ F_{k-1}-2F_k & 0 & 2F_k-F_{k-1} & \ddots & \ddots & & & & \vdots \\ -F_{k-1} & F_{k-1}-2F_k & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -F_{k-1} & F_{k-1}-2F_k & 0 & 2F_k-F_{k-1} & F_{k-1} \end{pmatrix}_{(n-2)\times(n-2)},$$

$$[\beta_i]_{i=2}^{n-1} = \beta_2, \beta_3, \cdots, \beta_{n-1}; \quad [F_i]_{i=k+n-3}^{k} = F_{k+n-3}, F_{k+n-4}, \cdots, F_k,$$

$$\alpha_1 = \sum_{i=0}^{n-2} F_{k+i}x_{n-i-1}, \quad \alpha_2 = -\sum_{i=0}^{n-3} F_{k+n-i-3}x_{n-i-1}, \tag{2.2}$$

$$\beta_1 = \sum_{i=0}^{n-4}(-F_{k+n-i-4} + \frac{F_{k+n-i-3}F_{k+n-3}}{F_{k+n-2}})x_{n-i-1} + \frac{F_kF_{k+n-3}}{F_{k+n-2}}x_2 + F_kx_1, \tag{2.3}$$

$$\beta_2 = \frac{F_kF_{k+n-3}}{F_{k+n-2}}, \quad \beta_i = -F_{k+i-3} + \frac{F_{k+i-2}F_{k+n-3}}{F_{k+n-2}}, \ (i = 3, 4, \cdots, n-1), \tag{2.4}$$

$$x_1 = 1, \quad x_3 = x_2^2, \quad F_{k-1}x_3 + 2F_kx_2 + F_{k+2} = 0, \tag{2.5}$$

$$x_i = -\frac{2F_kx_{i-1} + F_{k+2}x_{i-2} + \sum_{j=1}^{i-3} 2F_{k+j}x_{i-j-2}}{F_{k-1}}, \ (i = 4, 5, 6, \cdots, n-1), \tag{2.6}$$

$$
\begin{aligned}
&\det G_{n-2}([\beta_i]_{i=2}^{n-1}, -F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) = \\
&\qquad (-1)^{n-1}\beta_{n-1}\det L_{n-3}(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) + \\
&\qquad F_{k-1}\det G_{n-3}([\beta_i]_{i=2}^{n-2}, -F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}), \qquad (2.7)\\
&\det G_{n-2}([F_i]_{i=k+n-3}^{k}, -F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) = \\
&\qquad (-1)^{n-1}F_k\det L_{n-3}(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) + \\
&\qquad F_{k-1}\det G_{n-3}([F_i]_{i=k+n-3}^{k+1}, -F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}), \qquad (2.8)
\end{aligned}
$$

$$
L_i(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) =
$$

$$
\begin{pmatrix}
2F_k & F_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
F_{k+2} & 2F_k & F_{k-1} & \ddots & & & & \vdots \\
F_{k-1}-2F_k & 0 & 2F_k-F_{k-1} & \ddots & \ddots & & & \vdots \\
-F_{k-1} & F_{k-1}-2F_k & \ddots & \ddots & \ddots & \ddots & & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & F_{k-1} \\
0 & \cdots & \cdots & 0 & -F_{k-1} & F_{k-1}-2F_k & 0 & 2F_k-F_{k-1}
\end{pmatrix}_{i\times i},
$$

$$
\begin{aligned}
&\det L_i(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) = \\
&\qquad F_{k-1}^4\det L_{i-4}(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) + \\
&\qquad F_{k-1}^2(F_{k-1}-2F_k)\det L_{i-3}(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) + \\
&\qquad (2F_k-F_{k-1})\det L_{i-1}(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}), \quad (i=5,6,\cdots,n-3), \qquad (2.9)
\end{aligned}
$$

$$\det L_1(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) = 2F_k, \qquad (2.10)$$

$$\det L_2(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) = 4F_k^2-2F_kF_{k-1}-F_{k-1}^2, \qquad (2.11)$$

$$\det L_3(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) = 8F_k^3-8F_k^2F_{k-1}-2F_kF_{k-1}^2+2F_{k-1}^3, \qquad (2.12)$$

$$\det L_4(-F_{k-1}, F_{k-1}-2F_k, 0, 2F_k-F_{k-1}, F_{k-1}) = 16F_k^4-24F_k^3F_{k-1}+9F_kF_{k-1}^3-2F_{k-1}^4. \qquad (2.13)$$

Proof. Let $\mathbf{T}_{F_k,n}$ be an $n\times n$ skew symmetric generalized Foeplitz matrix. In the case $n\geq 4$, let

$$
\mathcal{C}_1 = \begin{pmatrix}
1 & & & & & & & 0 \\
 & & & & & & & 1 \\
 & & & & & & 1 & -\frac{F_{k+n-3}}{F_{k+n-2}} \\
 & & & & & 1 & 1 & -1 \\
 & & & & 1 & 1 & -1 & \\
 & & & \iddots & \iddots & \iddots & & \\
 & & 1 & 1 & -1 & & & \\
0 & 1 & 1 & -1 & & & &
\end{pmatrix}, \quad
\mathcal{D}_1 = \begin{pmatrix}
1 & 0 & & & & & & 0 \\
 & x_{n-1} & & & & & & 1 \\
 & x_{n-2} & & & & & 1 & \\
 & x_{n-3} & & & & 1 & & \\
 & \vdots & & & \iddots & & & \\
 & x_3 & & 1 & & & & \\
 & x_2 & 1 & & & & & \\
 & x_1 & & & & & &
\end{pmatrix},
$$

be two $n\times n$ matrices, which are invertible. And x_i $(i=1,2,\cdots,n-1)$ are the same as (2.5)-(2.6).

Multiplying $\mathbf{T}_{F_k,n}$ by $\mathcal{C}_1$ from the left, and multiplying $\mathcal{D}_1$ from the right, we obtain

$$\mathcal{C}_1\mathbf{T}_{F_k,n}\mathcal{D}_1 = \begin{pmatrix} 0 & a_1 & F_{k+n-3} & F_{k+n-4} & F_{k+n-5} & \cdots & \cdots & F_{k+2} & F_{k+1} & F_k \\ -F_{k+n-2} & \alpha_2 & -F_k & -F_{k+1} & -F_{k+2} & \cdots & \cdots & -F_{k+n-5} & -F_{k+n-4} & -F_{k+n-3} \\ 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots & \cdots & \beta_{n-3} & \beta_{n-2} & \beta_{n-1} \\ \vdots & 0 & 2F_k & F_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & F_{k+2} & 2F_k & \ddots & \ddots & & & & \vdots \\ \vdots & \vdots & 2F_{k+1} & F_{k+2} & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & 2F_{k+2} & 2F_{k+1} & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 2F_{k+n-6} & 2F_{k+n-7} & 2F_{k+n-8} & \cdots & \cdots & 2F_k & F_{k-1} & 0 \\ 0 & 0 & 2F_{k+n-5} & 2F_{k+n-6} & 2F_{k+n-7} & \cdots & \cdots & F_{k+2} & 2F_k & F_{k-1} \end{pmatrix},$$

where $\alpha_1,\ \alpha_2,\ \beta_i\ (i=1,2,\cdots,n-1)$ are the same as (2.2)-(2.4), and from the last matrix we can easily get,

$$\det(\mathcal{C}_1\mathbf{T}_{F_k,n}\mathcal{D}_1) = F_{k+n-2}\det G_{n-1}(\alpha_1,[F_i]_{i=k+n-3}^{k},[\beta_i]_{i=1}^{n-1}),$$

where

$$G_{n-1}(\alpha_1,[F_i]_{i=k+n-3}^{k},[\beta_i]_{i=1}^{n-1}) = \begin{pmatrix} \alpha_1 & F_{k+n-3} & F_{k+n-4} & F_{k+n-5} & \cdots & \cdots & F_{k+2} & F_{k+1} & F_k \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots & \cdots & \beta_{n-3} & \beta_{n-2} & \beta_{n-1} \\ 0 & 2F_k & F_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & F_{k+2} & 2F_k & \ddots & \ddots & & & & \vdots \\ \vdots & 2F_{k+1} & F_{k+2} & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & 2F_{k+2} & 2F_{k+1} & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 2F_{k+n-6} & 2F_{k+n-7} & 2F_{k+n-8} & \cdots & \cdots & 2F_k & F_{k-1} & 0 \\ 0 & 2F_{k+n-5} & 2F_{k+n-6} & 2F_{k+n-7} & \cdots & \cdots & F_{k+2} & 2F_k & F_{k-1} \end{pmatrix}_{(n-1)\times(n-1)}.$$

In order to simply compute the determinant of $G_{n-1}(\alpha_1,[F_i]_{i=k+n-3}^{k},[\beta_i]_{i=1}^{n-1})$, we apply methods of elementary row transformation to this matrix, then we can obtain $G_{n-1}^{'}(\alpha_1,[F_i]_{i=k+n-3}^{2},[\beta_i]_{i=1}^{n-1})$, it is the form as,

$$G'_{n-1}(\alpha_1,[F_i]_{i=k+n-3}^{k},[\beta_i]_{i=1}^{n-1})$$

$$=\begin{pmatrix}
\alpha_1 & F_{k+n-3} & F_{k+n-4} & F_{k+n-5} & \cdots & \cdots & \cdots & F_{k+2} & F_{k+1} & F_k \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots & \cdots & \cdots & \beta_{n-3} & \beta_{n-2} & \beta_{n-1} \\
0 & 2F_k & F_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & F_{k+2} & 2F_k & F_{k-1} & \ddots & & & & & \vdots \\
\vdots & F_{k-1}-2F_k & 0 & 2F_k-F_{k-1} & \ddots & \ddots & & & & \vdots \\
\vdots & -F_{k-1} & F_{k-1}-2F_k & \ddots & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & -F_{k-1} & F_{k-1}-2F_k & 0 & 2F_k-F_{k-1} & F_{k-1}
\end{pmatrix},$$

then we can obtain

$$\begin{aligned}\det G'_{n-1}(\alpha_1,&[F_i]_{i=k+n-3}^{k},[\beta_i]_{i=1}^{n-1}) = \\
&\alpha_1 \det G_{n-2}([\beta_i]_{i=2}^{n-1},-F_{k-1},F_{k-1}-2F_k,0,2F_k-F_{k-1},F_{k-1})- \\
&\beta_1 \det G_{n-2}([F_i]_{i=k+n-3}^{k},-F_{k-1},F_{k-1}-2F_k,0,2F_k-F_{k-1},F_{k-1}),\end{aligned}$$

where $G_{n-2}([\beta_i]_{i=2}^{n-1},-F_{k-1},F_{k-1}-2F_k,0,2F_k-F_{k-1},F_{k-1})$ and $G_{n-2}([F_i]_{i=k+n-3}^{k},-F_{k-1},F_{k-1}-2F_k,0,2F_k-F_{k-1},F_{k-1})$ are the forms as in the interpretation of Theorem 2.1.

To obtain the determinant of $G_{n-2}([\beta_i]_{i=2}^{n-1},-F_{k-1},F_{k-1}-2F_k,0,2F_k-F_{k-1},F_{k-1})$ and $G_{n-2}([F_i]_{i=k+n-3}^{k},-F_{k-1},F_{k-1}-2F_k,0,2F_k-F_{k-1},F_{k-1})$, developed them in accordance with the last column, and in turn it, we can get recursive formulas (2.7)-(2.13). And observe that

$$\det \mathcal{C}_1 = \det \mathcal{D}_1 = (-1)^{\frac{(n-1)(n-2)}{2}},$$

then we can obtain $\det \mathbf{T}_{F_k,n}$, which completes the proof. □

Theorem 2.2. *Let $\mathbf{T}_{F_k,n}$ be a skew symmetric generalized Foeplitz matrix as the form of (1.1). If $\mathbf{T}_{F_k,n}$ is a nonsingular matrix, then*

$$\mathbf{T}_{F_k,n}^{-1}=\begin{pmatrix}
0 & \gamma_{12} & \gamma_{13} & \gamma_{14} & \cdots & \gamma_{1,n-1} & \gamma_{1n} \\
-\gamma_{12} & 0 & \gamma_{23} & \gamma_{24} & \cdots & \gamma_{2,n-1} & \gamma_{1,n-1} \\
-\gamma_{13} & -\gamma_{23} & 0 & \gamma_{3,4} & \cdots & \gamma_{2,n-2} & \gamma_{1,n-2} \\
-\gamma_{14} & -\gamma_{24} & -\gamma_{3,4} & 0 & \cdots & \gamma_{2,n-3} & \gamma_{1,n-3} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
-\gamma_{1,n-1} & -\gamma_{2,n-1} & -\gamma_{2,n-2} & -\gamma_{2,n-3} & \cdots & 0 & \gamma_{12} \\
-\gamma_{1,n} & -\gamma_{1,n-1} & -\gamma_{1,n-2} & -\gamma_{1,n-3} & \cdots & -\gamma_{12} & 0
\end{pmatrix}, \qquad (2.14)$$

that is to say $\mathbf{T}_{F_k,n}^{-1}$ is skew symmetric about diagonal and symmetric about secondary diagonal as *well, where*

$$\gamma_{12}=\epsilon_{1n},\quad \gamma_{13}=\epsilon_{1,n-1}+\epsilon_{1n},\quad \gamma_{23}=\epsilon_{2,n-1}+\epsilon_{2n},\quad \gamma_{1n}=\epsilon_{11}-\frac{F_{k+n-3}}{F_{k+n-2}}\epsilon_{13}-\epsilon_{14},$$

$$\gamma_{ij}=\epsilon_{i,n+2-j}+\epsilon_{i,n+3-j}-\epsilon_{i,n+4-j},\ (i=1,2,\cdots,\frac{n}{2};j=4,5,\cdots,n-1),$$

$$\epsilon_{11}=-\frac{1}{F_{k+n-2}},\quad \epsilon_{12}=\frac{g_1}{\alpha_1},\quad \epsilon_{13}=\frac{g_2}{\zeta_1}+\sum_{i=1}^{n-3}g_{i+2}\vartheta_i^{'},$$

$$\epsilon_{1j}=g_2\eta_{j-3}^{'}+\sum_{i=1}^{n-3}g_{i+2}\varsigma_{i,j-3},\ (j=4,5,\cdots,n),$$

$$\epsilon_{i1}=0,\ (i=2,3,\cdots,n),\quad \epsilon_{i2}=\frac{x_{n+1-i}}{\alpha_1},\ (i=2,3,\cdots,n),$$

$$\epsilon_{i3}=\frac{h_2x_{n+1-i}}{\zeta_1}+\sum_{m=1}^{n-3}h_{m+2}x_{n+1-i}\vartheta_m^{'}+\vartheta_{n-1-i}^{'},\ (i=2,3,\cdots,n),$$

$$\epsilon_{ij}=h_2x_{n+1-i}\eta_{j-3}^{'}+\sum_{m=1}^{n-3}h_{m+2}x_{n+1-i}\varsigma_{m,j-3}+\varsigma_{n-1-i,j-3},\ (i=2,3,\cdots,n;\ j=4,5,\cdots,n),$$

$$g_1=\frac{\alpha_2}{F_{k+n-2}},\quad g_i=\frac{-\alpha_2F_{k+n-i-1}-\alpha_1F_{k+i-2}}{\alpha_1F_{k+n-2}},\ (i=2,3,\cdots,n-1),$$

$$h_i=\frac{-F_{k+n-i-1}}{\alpha_1},\ (i=2,3,\cdots,n-1),$$

$$\zeta_1=-\frac{\beta_1}{\alpha_1}F_{k+n-3}+\beta_2-\mathbb{V}_1\mathbb{W}_1^{-1}\mathbb{U}_1,\quad \mathbb{U}_1=(2F_k,F_{k+2},2F_{k+1},\cdots,2F_{k+n-6},2F_{k+n-5})^T,$$

$$\mathbb{V}_1=(-\frac{\beta_1}{\alpha_1}F_{k+n-4}+\beta_3,-\frac{\beta_1}{\alpha_1}F_{k+n-5}+\beta_4,\cdots,-\frac{\beta_1}{\alpha_1}F_{k+1}+\beta_{n-2},-\frac{\beta_1}{\alpha_1}F_k+\beta_{n-1}),$$

$$\mathbb{W}_1^{-1}=(\alpha_{i,j})_{i,j=1}^{n-3},\quad \alpha_{i,j}=\begin{cases}\mu_{i-j+1}, & i\geqslant j,\\ 0, & i<j,\end{cases}\quad (i,j=1,2,\cdots,n-3),$$

$$\mu_i=\frac{(-1)^{i-1}\lambda_i}{F_{k-1}^i},\quad \lambda_1=1,\quad \lambda_2=2F_k,$$

$$\lambda_i=2F_k\lambda_{i-1}-F_{k-1}F_{k+2}\lambda_{i-2}+\sum_{j=2}^{i-2}(-1)^jF_{k-1}^j2F_{k+j-1}\lambda_{i-j-1},\ (i=3,4,5,\cdots,n-3),$$

$$\vartheta_i^{'}=-\frac{1}{\zeta_1}\vartheta_i,\ (i=1,2,\cdots,n-3),\quad \varpi_i^{'}=-\frac{1}{\zeta_1}\varpi_i,\ (i=1,2,\cdots,n-3),$$

$$\vartheta_i=\sum_{j=i}^{n-3}(-\frac{\beta_1}{\alpha_1}F_{k+n-3-j}+\beta_{j+2})\mu_{j+1-i},\ (i=1,2,\cdots,n-3),$$

$$\varpi_1=\mu_1\cdot 2F_k,\qquad \varpi_2=\mu_2\cdot 2F_k+\mu_1\cdot F_{k+2},$$

$$\varpi_i=\mu_i\cdot 2F_k+\mu_{i-1}\cdot F_{k+2}+\sum_{j=1}^{i-2}\mu_{i-1-j}\cdot 2F_{k+j},\ (i=3,4,\cdots,n-3),$$

$$\varsigma_{i,j}=\mu_{i-j+1}+\frac{1}{\zeta_1}\varpi_i\vartheta_j,\ (i,j=1,2,\cdots,n-3),\ \text{if } \mathrm{i-j+1<1},\ \text{denote } \mu_{\mathrm{i-j+1}}=0$$

$\alpha_1,\ \alpha_2,\ \beta_i,\ x_i,\ (i=1,2,\cdots,n-1)$ are the same as in Theorem 2.1.

Proof. Let $\mathcal{C}_2$ and $\mathcal{D}_2$ be two $n \times n$ invertible matrices, defined by

$$\mathcal{C}_2 = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & & & & & & \\ -\frac{\beta_1}{\alpha_1} & 0 & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}_{n\times n}, \quad \mathcal{D}_2 = \begin{pmatrix} 1 & g_1 & g_2 & g_3 & \cdots & g_{n-2} & g_{n-1} \\ & 1 & h_2 & h_3 & \cdots & h_{n-2} & h_{n-1} \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}_{n\times n},$$

where $g_1 = \frac{\alpha_2}{F_{k+n-2}},\ g_i = \frac{-\alpha_2 F_{k+n-i-1} - \alpha_1 F_{k+i-2}}{\alpha_1 F_{k+n-2}},\ h_i = \frac{-F_{k+n-i-1}}{\alpha_1},\ (i = 2, 3, \cdots, n-1)$.
Let $\mathcal{C}_1$ and $\mathcal{D}_1$ be as in the proof of Theorem 2.1, multiplying $\mathcal{C}_1 \mathbf{T}_{F_k,n} \mathcal{D}_1$ by $\mathcal{C}_2$ from the left and by $\mathcal{D}_2$ from the right, we obtain

$$\mathcal{C}_2\mathcal{C}_1\mathbf{T}_{F_k,n}\mathcal{D}_1\mathcal{D}_2$$
$$= \begin{pmatrix} -F_{k+n-2} & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \alpha_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & -\frac{\beta_1}{\alpha_1}F_{k+n-3}+\beta_2 & -\frac{\beta_1}{\alpha_1}F_{k+n-4}+\beta_3 & \cdots & -\frac{\beta_1}{\alpha_1}F_{k+1}+\beta_{n-2} & -\frac{\beta_1}{\alpha_1}F_k+\beta_{n-1} \\ \vdots & \vdots & 2F_k & F_{k-1} & 0 & \cdots & 0 \\ \vdots & \vdots & F_{k+2} & 2F_k & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 2F_{k+n-5} & 2F_{k+n-6} & \cdots & 2F_k & F_{k-1} \end{pmatrix},$$

this matrix admits a block partition of the form as

$$\mathcal{C}_2\mathcal{C}_1\mathbf{T}_{F_k,n}\mathcal{D}_1\mathcal{D}_2 = \mathfrak{N} \oplus \mathfrak{M},$$

where $\mathfrak{N} \oplus \mathfrak{M}$ is the direct sum of $\mathfrak{N}$ and $\mathfrak{M}$. $\mathfrak{N} = \mathrm{diag}(-F_{k+n-2},\ \alpha_1)$ is a nonsingular diagonal matrix,

$$\mathfrak{M} = \begin{pmatrix} -\frac{\beta_1}{\alpha_1}F_{k+n-3}+\beta_2 & -\frac{\beta_1}{\alpha_1}F_{k+n-4}+\beta_3 & -\frac{\beta_1}{\alpha_1}F_{k+n-5}+\beta_4 & \cdots & -\frac{\beta_1}{\alpha_1}F_{k+1}+\beta_{n-2} & -\frac{\beta_1}{\alpha_1}F_k+\beta_{n-1} \\ 2F_k & F_{k-1} & 0 & \cdots & 0 & 0 \\ F_{k+2} & 2F_k & F_{k-1} & \ddots & & \vdots \\ 2F_{k+1} & F_{k+2} & 2F_k & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 2F_{k+n-5} & 2F_{k+n-6} & 2F_{k+n-7} & \cdots & 2F_k & F_{k-1} \end{pmatrix}.$$

Let $\mathcal{C} = \mathcal{C}_2\mathcal{C}_1$ and $\mathcal{D} = \mathcal{D}_1\mathcal{D}_2$, we can get,

$$\mathbf{T}_{F_k,n}^{-1} = \mathcal{D}(\mathfrak{N}^{-1} \oplus \mathfrak{M}^{-1})\mathcal{C},$$

where

$$\mathcal{C}=\begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ -\frac{\beta_1}{\alpha_1} & \vdots & & & & \iddots & 1 & -\frac{F_{k+n-3}}{F_{k+n-2}} \\ 0 & \vdots & & & \iddots & 1 & 1 & -1 \\ \vdots & \vdots & & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & 0 & 1 & 1 & -1 & \iddots & & \vdots \\ 0 & 1 & 1 & -1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

and

$$\mathcal{D}=\begin{pmatrix} 1 & g_1 & g_2 & g_3 & \cdots & g_{n-3} & g_{n-2} & g_{n-1} \\ 0 & x_{n-1} & h_2x_{n-1} & h_3x_{n-1} & \cdots & h_{n-3}x_{n-1} & h_{n-2}x_{n-1} & h_{n-1}x_{n-1}+1 \\ 0 & x_{n-2} & h_2x_{n-2} & h_3x_{n-2} & \cdots & h_{n-3}x_{n-2} & h_{n-2}x_{n-2}+1 & h_{n-1}x_{n-2} \\ 0 & x_{n-3} & h_2x_{n-3} & h_3x_{n-3} & \cdots & h_{n-3}x_{n-3}+1 & h_{n-2}x_{n-3} & h_{n-1}x_{n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & x_3 & h_2x_3 & h_3x_3+1 & \cdots & h_{n-3}x_3 & h_{n-2}x_3 & h_{n-1}x_3 \\ 0 & x_2 & h_2x_2+1 & h_3x_2 & \cdots & h_{n-3}x_2 & h_{n-2}x_2 & h_{n-1}x_2 \\ 0 & x_1 & h_2 & h_3 & \cdots & h_{n-3} & h_{n-2} & h_{n-1} \end{pmatrix}.$$

Observe that the inverse matrix of $\mathfrak{N}$ is of the form:

$$\mathfrak{N}^{-1}=\mathrm{diag}(-F_{k+n-2}^{-1},\ \alpha_1^{-1}).$$

Let $\mathfrak{M}=\begin{pmatrix} -\frac{\beta_1}{\alpha_1}F_{k+n-3}+\beta_2 & \mathbb{V}_1 \\ \mathbb{U}_1 & \mathbb{W}_1 \end{pmatrix}$ be an $(n-2)\times(n-2)$ matrix, where $\mathbb{U}_1=(2F_k, F_{k+2}, 2F_{k+1}, \cdots, 2F_{k+n-6}, 2F_{k+n-5})^T$, $\mathbb{V}_1=(-\frac{\beta_1}{\alpha_1}F_{k+n-4}+\beta_3, -\frac{\beta_1}{\alpha_1}F_{k+n-5}+\beta_4, \cdots, -\frac{\beta_1}{\alpha_1}F_{k+1}+\beta_{n-2}, -\frac{\beta_1}{\alpha_1}F_k+\beta_{n-1})$,

$$\mathbb{W}_1=\begin{pmatrix} F_{k-1} & & & & & & \\ 2F_k & F_{k-1} & & & & & \\ F_{k+2} & 2F_k & F_{k-1} & & & & \\ 2F_{k+1} & F_{k+2} & 2F_k & F_{k-1} & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ 2F_{k+n-7} & 2F_{k+n-8} & 2F_{k+n-9} & 2F_{k+n-10} & \cdots & F_{k-1} & \\ 2F_{k+n-6} & 2F_{k+n-7} & 2F_{k+n-8} & 2F_{k+n-9} & \cdots & 2F_k & F_{k-1} \end{pmatrix},$$

as $F_{k-1}\geq 1$, so $\mathbb{W}_1$ is an invertible matrix. Use Lemma 1.1 in [Chen et al. (2016)] we can obtain the inverse matrix of $\mathbb{W}_1$,

$$\mathbb{W}_1^{-1}=\begin{pmatrix} \mu_1 & 0 & \cdots & \cdots & 0 & 0 \\ \mu_2 & \mu_1 & \ddots & & & 0 \\ \mu_3 & \mu_2 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mu_{n-4} & \ddots & \ddots & \ddots & \ddots & 0 \\ \mu_{n-3} & \mu_{n-4} & \cdots & \mu_3 & \mu_2 & \mu_1 \end{pmatrix},$$

where μ_1 $(i = 1, 2, \cdots, n-3)$ are the forms as in the interpretation of Theorem 2.2.

Use Lemma 5 in Liu and Jiang (2015), we obtain,

$$\mathfrak{M}^{-1} = \begin{pmatrix} \frac{1}{\zeta_1} & -\frac{1}{\zeta_1}\mathbb{V}_1\mathbb{W}_1^{-1} \\ -\frac{1}{\zeta_1}\mathbb{W}_1^{-1}\mathbb{U}_1 & \mathbb{W}_1^{-1} + \frac{1}{\zeta_1}\mathbb{W}_1^{-1}\mathbb{U}_1\mathbb{V}_1\mathbb{W}_1^{-1} \end{pmatrix},$$

where $\zeta_1 = -\frac{\beta_1}{\alpha_1}F_{k+n-3} + \beta_2 - \mathbb{V}_1\mathbb{W}_1^{-1}\mathbb{U}_1$, and simply we can get $(\mathfrak{N} \oplus \mathfrak{M})^{-1}$,

$$(\mathfrak{N} \oplus \mathfrak{M})^{-1} = \begin{pmatrix} \frac{1}{-F_{k+n-2}} & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \frac{1}{\alpha_1} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{\zeta_{1,}} & \vartheta_1^{'} & \vartheta_2^{'} & \cdots & \cdots & \vartheta_{n-4}^{'} & \vartheta_{n-3}^{'} \\ 0 & 0 & \varpi_1^{'} & \varsigma_{11} & \varsigma_{12} & \cdots & \cdots & \varsigma_{1,n-4} & \varsigma_{1,n-3} \\ 0 & 0 & \varpi_2^{'} & \varsigma_{21} & \varsigma_{22} & \cdots & \cdots & \varsigma_{2,n-4} & \varsigma_{2,n-3} \\ 0 & 0 & \varpi_3^{'} & \varsigma_{31} & \varsigma_{32} & \cdots & \cdots & \varsigma_{3,n-4} & \varsigma_{3,n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & \varpi_{n-4}^{'} & \varsigma_{n-4,1} & \varsigma_{n-4,2} & \cdots & \cdots & \varsigma_{n-4,n-4} & \varsigma_{n-4,n-3} \\ 0 & 0 & \varpi_{n-3}^{'} & \varsigma_{n-3,1} & \varsigma_{n-3,2} & \cdots & \cdots & \varsigma_{n-3,n-4} & \varsigma_{n-3,n-3} \end{pmatrix},$$

where $\vartheta_i^{'},\ \varpi_i^{'},\ \varsigma_{i,j}$ $(i, j = 1, 2, \cdots, n-3)$ are the forms as in the interpretation of Theorem 2.2.

Then multiplying $(\mathfrak{N} \oplus \mathfrak{M})^{-1}$ by $\mathcal{D}$ from the left, we can obtain

$$\mathcal{D}(\mathfrak{N} \oplus \mathfrak{M})^{-1} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} & \cdots & \epsilon_{1,n-1} & \epsilon_{1,n} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} & \cdots & \epsilon_{2,n-1} & \epsilon_{2,n} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} & \cdots & \epsilon_{3,n-1} & \epsilon_{3,n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \epsilon_{n-1,1} & \epsilon_{n-1,2} & \epsilon_{n-1,3} & \cdots & \epsilon_{n-1,n-1} & \epsilon_{n-1,n} \\ \epsilon_{n,1} & \epsilon_{n,2} & \epsilon_{n,3} & \cdots & \epsilon_{n,n-1} & \epsilon_{n,n} \end{pmatrix},$$

where ϵ_{ij} $(i, j = 1, 2, \cdots, n)$ are the forms as in the interpretation of Theorem 2.2.

In the end, we can obtain $\mathbf{T}_{F_k,n}^{-1}$,

$$\mathbf{T}_{F_k,n}^{-1} = \mathcal{D}(\mathfrak{N} \oplus \mathfrak{M})^{-1}\mathcal{C} = \begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} & \gamma_{14} & \cdots & \gamma_{1,n-1} & \gamma_{1n} \\ -\gamma_{12} & 0 & \gamma_{23} & \gamma_{24} & \cdots & \gamma_{2,n-1} & \gamma_{1,n-1} \\ -\gamma_{13} & -\gamma_{23} & 0 & \gamma_{3,4} & \cdots & \gamma_{2,n-2} & \gamma_{1,n-2} \\ -\gamma_{14} & -\gamma_{24} & -\gamma_{3,4} & 0 & \cdots & \gamma_{2,n-3} & \gamma_{1,n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\gamma_{1,n-1} & -\gamma_{2,n-1} & -\gamma_{2,n-2} & -\gamma_{2,n-3} & \cdots & 0 & \gamma_{12} \\ -\gamma_{1,n} & -\gamma_{1,n-1} & -\gamma_{1,n-2} & -\gamma_{1,n-3} & \cdots & -\gamma_{12} & 0 \end{pmatrix},$$

where γ_{ij} $(i = 1, 2, \cdots, \frac{n}{2}; j = 2, 3, \cdots, n)$ are the forms as in the interpretation of Theorem 2.2, which completes the proof. □

3 CONCLUSION

In this paper, by constructing the special transformation matrices we get the determinant and inverse of the skew symmetric generalized Foeplitz matrices in section 2.

Competing Interest

The author declare that no competing interests exist.

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