On nonsingularity of RSFPLR circulant matrices

Abstract

In this paper, we discuss the nonsingularity of a row skew first-plus-last right (RSFPLR) circulant matrices with the first row $(a_1, a_2, \ldots, a_n)$, which is determined by entries of the first row. First, the sufficient condition for the matrix to be nonsingular is that, there exists an element a_{i_0} belonging to the first row, whose absolute value is greater than the sum of the corresponding power of 2 and the absolute values of the remaining $(n-1)$ elements, that is, $|a_{i_0}| > \sum_{i=1, i\neq i_0}^{n} 2^{i-i_0}|a_i|$. Moreover, we derive other sufficient conditions for judging the nonsingularity of the matrix.

Keywords: RSFPLR circulant matrix, nonsingularity, singularity.

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1 Introduction

Circulant matrix family have important applications in various disciplines including image processing, communications, signal processing and encoding. They have been put on firm basis with the work of P. Davis [1] and Z. L. Jiang [2].

The circulant matrices, long a fruitful subject of research, have in recent years been extended in many directions. The $f(x)$-circulant matrices are another natural extension of this well-studied class, and can be found in [3–13]. The $f(x)$-circulant matrix has a wide application, especially on the generalized cyclic codes [13]. The properties and structures of the $(x^n - x + 1)$- circulant matrices, which are called row skew first-plus-last right (RSFPLR) circulant matrices, are better than those of the general $f(x)$-circulant matrices, so there are good methods for discriminations its nonsingularity.

Firstly, we introduce the RSFPLR circulant matrix in the following definition.

Definition 1.1. Let A=RSFPLRcircfr$(a_1, a_2, \ldots, a_n)$ be a RSFPLR circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$, defined as follows

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\ -a_n & a_1 + a_n & a_2 & \ddots & \ddots & a_{n-1} \\ -a_{n-1} & -a_n + a_{n-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & a_3 \\ -a_3 & -a_4 + a_3 & \ddots & \ddots & \ddots & a_2 \\ -a_2 & -a_3 + a_2 & -a_4 + a_3 & \ldots & -a_n + a_{n-1} & a_1 + a_n \end{pmatrix}_{n\times n}.$$

Note that the RSFPLR circulant matrix is a $(x^n - x + 1)$-circulant matrix [13–17], and that is neither the extension of skew circulant matrix [1, 2] nor its special case and they are two different kinds of special matrices. Moreover, it is a FLS r-circulant matrix [3–5] with $r = -1$.

Let $\Theta_{(-1,1)}$ be the basic RSFPLR circulant matrix, denoted by

$$\Theta_{(-1,1)} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ -1 & 1 & 0 & \ldots & 0 \end{pmatrix}_{n\times n}. \tag{1.1}$$

It is easily verified that $g(x) = x^n - x + 1$ has no repeated roots in its splitting field and $g(x) = x^n - x + 1$ is both the minimal polynomial and the characteristic polynomial of the matrix $\Theta_{(-1,1)}$. In addition, $\Theta_{(-1,1)}$ is nonderogatory and satisfies $\Theta^j_{(-1,1)} = \mathrm{RSFPLRcircfr}(\underbrace{0, \cdots, 0}_{j}, 1, \underbrace{0, \cdots, 0}_{n-j-1})$ and $\Theta^n_{(-1,1)} = -I_n + \Theta_{(-1,1)}$. Therefore, a matrix A can be written in the form

$$A = f\left(\Theta_{(-1,1)}\right) = \sum_{i=1}^{n} a_i \Theta^{i-1}_{(-1,1)} \tag{1.2}$$

if and only if A is a RSFPLR circulant matrix, where the polynomial $f(x) = \sum_{i=1}^{n} a_i x^{i-1}$ is called the representer of the RSFPLR circulant matrix A. It is clear that A is a RSFPLR circulant matrix if and only if A commutes with the $\Theta_{(-1,1)}$, that is,

$$A\Theta_{(-1,1)} = \Theta_{(-1,1)}A. \tag{1.3}$$

Secondly, based on [4], we deduce the following lemma.

Lemma 1.1. *Let $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ be a RSFPLR circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$. Then A is singular if and only if there exists $j_0 (1 \le j_0 \le n)$ such that $f(\omega_{j_0}) = 0$, where $f(x) = \sum_{i=1}^{n} a_i x^{i-1}$.*

2 Main Results

Let $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ be a RSFPLR circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$. We discuss the nonsingularity on matrix A under different conditions in this section. At the same time, several corollaries are derived.

Theorem 2.1. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$*. If there exists an* $a_{i_0} \in \{a_1, a_2, \ldots, a_n\}$*, such that*

$$|a_{i_0}| > \sum_{i=1, i\neq i_0}^{n} 2^{i-i_0}|a_i|, i = 1, ..., n, i \neq i_0, \tag{2.1}$$

then A *is nonsingular.*

Proof. If A is singular, then by Lemma 1.1, there exists $j_0 (1 \leq j_0 \leq n)$, such that

$$f(\omega_{j_0}) = \sum_{i=1}^{n} a_i(\omega_{j_0})^i = 0.$$

So

$$a_{i_0}(\omega_{j_0})^{i_0} = -\sum_{i=1, i\neq i_0}^{n} a_i(\omega_{j_0})^i.$$

Taking the absolute value of the above equation

$$|a_{i_0}(\omega_{j_0})^{i_0}| = |\sum_{i=1, i\neq i_0}^{n} a_i(\omega_{j_0})^i| \leq \sum_{i=1, i\neq i_0}^{n} |a_i||\omega_{j_0}|^i,$$

we have

$$|a_{i_0}| \leq \sum_{i=1, i\neq i_0}^{n} |a_i||\omega_{j_0}|^{i-i_0}.$$

Note that ω_{j_0} are the roots of the characteristic polynomial $g(x) = x^n - x + 1$ for matrix $\Theta_{(-1,1)}$, i.e. $(\omega_{j_0})^n - \omega_{j_0} + 1 = 0$. So we get from [18, Corollary 6.1.5] that

$$|\omega_{j_0}| \leq 2.$$

Hence

$$|a_{i_0}| \leq \sum_{i=1, i\neq i_0}^{n} 2^{i-i_0}|a_i|,$$

which contradicts to inequality (2.1). Therefore, A is nonsingular. □

Corollary 2.2. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, ..., a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$*. If there exists an* $a_{i_0} \in \{a_1, a_2, ..., a_n\}$*, for any* $i \neq i_0$*,*$1 \leq i \leq n$*, such that*

$$|a_{i_0}| > (n-1)|a_i| \sqrt[n]{2^{i-i_0}}, \tag{2.2}$$

then A *is nonsingular.*

Proof. If A is singular, then by Lemma 1.1, there exists $j_0 (1 \leq j_0 \leq n)$, such that

$$f(\omega_{j_0}) = \sum_{i=1}^{n} a_i(\omega_{j_0})^i = 0.$$

So

$$a_{i_0}(\omega_{j_0})^{i_0} = -\sum_{i=1, i\neq i_0}^{n} a_i(\omega_{j_0})^i.$$

Taking the absolute value of the above equation

$$|a_{i_0}(\omega_{j_0})^{i_0}| = |\sum_{i=1, i\neq i_0}^{n} a_i(\omega_{j_0})^i| \leq \sum_{i=1, i\neq i_0}^{n} |a_i||\omega_{j_0}|^i,$$

we get

$$|a_{i_0}| \leq \sum_{i=1, i\neq i_0}^{n} |a_i||\omega_{j_0}|^{i-i_0}.$$

Note that ω_{j_0} are the roots of the characteristic polynomial $g(x) = x^n - x + 1$ for matrix $\Theta_{(-1,1)}$, i.e. $(\omega_{j_0})^n - \omega_{j_0} + 1 = 0$. So we get from [18, Corollary 6.1.5] that

$$|\omega_{j_0}| \leq 2.$$

Thus

$$|a_{i_0}| \leq \sum_{i=1, i\neq i_0}^{n} 2^{i-i_0}|a_i|.$$

Hence there exists k_0, such that

$$|a_{i_0}| \leq (n-1)|a_{k_0}| \sqrt[n]{2^{k_0-i_0}},$$

which contradicts to inequality (2.2). Therefore, A is nonsingular. □

Corollary 2.3. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, ..., a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$*. If there exists an* $a_{i_0} \in \{a_1, a_2, ..., a_n\}$*, for any* $i \neq i_0$*,* $1 \leq i \leq n$*, such that*

$$\frac{|a_i|}{|a_{i_0}|} < \frac{1}{(n-1)\sqrt[n]{2^{i-i_0}}},$$

then A *is nonsingular.*

Theorem 2.4. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, ..., a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$*. If*

$$|a_M| > \sum_{i=1, i\neq M}^{n} 2^{i-M}|a_i|, \tag{2.3}$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. The proof process similar to Theorem 2.1 □

Corollary 2.5. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, ..., a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$*. If for any* $i \neq M$*,*$1 \leq i \leq n$
such that

$$|a_M| > (n-1)2^{i-M}|a_i|, \tag{2.4}$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. If A is singular, then by Lemma 1.1, there exists $j_0 (1 \leq j_0 \leq n)$, such that

$$f(\omega_{j_0}) = \sum_{i=1}^{n} a_i(\omega_{j_0})^i = 0.$$

So

$$a_M(\omega_{j_0})^M = -\sum_{i=1, i\neq M}^{n} a_i(\omega_{j_0})^i.$$

Taking the absolute value of the above equation

$$|a_M(\omega_{j_0})^M| = |\sum_{i=1 i\neq M}^{n} a_i(\omega_{j_0})^i| \leq \sum_{i=1, i\neq M}^{n} |a_i||\omega_{j_0}|^i,$$

we have

$$|a_M| \le \sum_{i=1,i\neq M}^{n} |a_i||\omega_{j_0}|^{i-M}.$$

Note that ω_{j_0} are the roots of the characteristic polynomial $g(x) = x^n - x + 1$ for matrix $\Theta_{(-1,1)}$, i.e. $(\omega_{j_0})^n - \omega_{j_0} + 1 = 0$. So we get from [18, Corollary 6.1.5] that

$$|\omega_{j_0}| \le 2.$$

Thus

$$|a_M| \le \sum_{i=1,i\neq M}^{n} 2^{i-M}|a_i|$$

Hence there exists k_0, such that

$$|a_M| \le (n-1)2^{k_0-M}|a_{k_0}|,$$

which contradicts to inequality (2.4). Therefore A is nonsingular. □

Corollary 2.6. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, ..., a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$*. If for any* $i \neq M$,$1 \le i \le n$
such that

$$\sum_{i=1 i\neq M}^{n} \frac{|a_i|}{|a_M|} 2^{i-M},$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Corollary 2.7. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, ..., a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$*. If for any* $i \neq M$,$1 \le i \le n$
such that

$$\frac{|a_i|}{|a_M|} < \frac{1}{(n-1)\sqrt[n]{2^{i-M}}},$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Theorem 2.8. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, ..., a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$*. If there exists an* $a_{i_0} \in (a_1, a_2, ..., a_n)$*, such that*

$$|1 - a_{i_0}| < \frac{1}{n}, 2|a_i| < \frac{1}{n}, i = 1, ..., n, i \neq i_0,$$

then A *is nonsingular.*

Proof. By adding the both sides of the n inequalities, we have

$$|1 - a_{i_0}| + \sum_{i=1,i\neq i_0}^{n} 2^{i-i_0}|a_i| < 1.$$

Since

$$|1 - a_{i_0}| \ge 1 - |a_{i_0}|,$$

we have

$$|a_{i_0}| > \sum_{i=1,i\neq i_0}^{n} 2^{i-i_0}|a_i|. \tag{2.5}$$

Therefore,the conclusion is clearly established based on Theorem 2.1 and (2.5). □

Corollary 2.9. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, ..., a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$. *If*

$$|1 - a_M| < \frac{1}{n},\ 2|a_i| < \frac{1}{n},\ i = 1, ..., n,\ i \neq M,$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. By adding the both sides of the n inequalities, we have

$$|1 - a_M| + \sum_{i=1, i\neq M}^{n} 2^{i-M}|a_i| < 1.$$

Since

$$|1 - a_M| \geq 1 - |a_M|,$$

we have

$$|a_M| > \sum_{i=1, i\neq M}^{n} 2^{i-M}|a_i|.$$

According to Theorem 2.4, A is nonsingular. □

Theorem 2.10. *Let* $A = \mathrm{RSFPLRcircfr}(a_1, a_2, ..., a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$. *If*

$$\sqrt{n[(1 - a_M)^2 + \sum_{i=1, i\neq M}^{n} |a_i|^2 2^{2(i-M)}]} < 1, \tag{2.6}$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. Since

$$\sqrt{\frac{(1 - a_M)^2 + \sum_{i=1, i\neq M}^{n} |a_i|^2 2^{2(i-M)}}{n}} \geq \frac{|1 - a_M| + \sum_{i=1, i\neq M}^{n} |a_i| 2^{i-M}}{n},$$

we have

$$\begin{aligned}\sqrt{n[(1 - a_M)^2 + \sum_{i=1, i\neq M}^{n} |a_i|^2 2^{2(i-M)}]} &\geq |1 - a_M| + \sum_{i=1, i\neq M}^{n} |a_i| 2^{i-M} \\ &\geq 1 - |a_M| + \sum_{i=1, i\neq M}^{n} |a_i| 2^{i-M}\end{aligned}$$

By the inequality (2.6), we get

$$|a_M| \geq \sum_{i=1, i\neq M}^{n} |a_i| 2^{i-M}.$$

According to Theorem 2.4, A is nonsingular. □

3 Conclusion

In this paper, based on the properties and structures of the RSFPLR circulant matrices, we discuss their nonsingularities under different conditions by utilizing only entries of the first row of themselves. In Theorem 2.1, the sufficient condition for the RSFPLR circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$ to be nonsingular is that, there exists an element a_{i_0} belonging to the first row, whose absolute value is greater than the sum of the corresponding power of 2 and the absolute values of the remaining $(n-1)$ elements, that is, $|a_{i_0}| > \sum_{i=1,i\neq i_0}^{n} 2^{i-i_0}|a_i|$. Besides, we further derive other sufficient conditions for judging the nonsingularity of the matrix, which enriches our understanding of the RSFPLR circulant matrix in Section 2.

Competing Interest

The authors declare that no competing interests exist.

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