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A CRITICAL ANALYSIS OF MULTI-REGIME FUNDAMENTAL EQUATIONS

Original Research Article

Abstract

Multi-regime fundamental models use two or more equations to describe the association among the main macroscopic traffic variables encountered in traffic analysis. The paper investigates specific properties of some multi-phase speed-density equations. It first compares the characteristics of each of these equations by solving the nonlinear continuity traffic equation. It was observed that predicting vehicular trajectories with these model equations could lead to misinformation. The kinematic wave and stable shockwave properties of these models were also ascertained. Based on the results, it was concluded that it would be more cumbersome to explain nonlinear traffic characteristics when these two and three regime models are adopted.

Keywords: Speed-Density Equation, Characteristic Curves, Multi-Regime Models, Continuity Equation
2010 Mathematics Subject Classification: 65M25; 35F10; 34G20

1 Introduction

Vehicular traffic flow are often dichotomized into two regimes; the free flow regime and the congested regime. In the free-flow regime there is a positive relationship between flow and density, while the relationship is negative during a traffic cluster. Single-regime models use a single equation to characterize this physical process[7, 6, 9, 12, 10]. However, some authors postulated the use of two or more equations to describe this same phenomenon, hence the name multi-phase equations [3, 2]. More often one equation is used to describe freeway traffic and the other for jam formation. Multi-regime models use piecewise curves to describe the functional relationship among macroscopic traffic variables. In general, a speed-density relationship is easy to explain when compared to other fundamental relationships. There is a one-to-one relationship between driver behavior and the number of vehicles present on the road. Speed-density relationship is a part of traffic dynamic studies to explore traffic flow patterns such as shock waves and queue lengths on highways and urban bypass.

Comparison of the one-equation model has explored extensively in the literature[13, 14, 1]. In earlier research, [4] presented an analysis of the characteristic profiles of some single regime models. All selected speed-density equation had similar characteristic curves exhibiting wave-fans. The forward movement of trajectories authenticated that any of the speed-density functions could be coupled with the LWR equation to achieve the required simulation results. More recently, [5] also presented on the static and dynamic properties of one-equation speed-density-flow models. This was to determine an appropriate single-regime speed-density equation suitable for mixed traffic in the cities. The static properties are the usual flow characteristics invariant to time. This include the following: the speed $u(k)$ converges to it maximum as density reduces to near empty road; $u'(0) = 0$, that is vehicles can attain maximum speed when there is minimal interplay among vehicles; and the speed $u(k)$ get to zero as density reaches its maximum.

Hither, multi-phase density-speed functionals are examined for their suitability for microscopic and macroscopic traffic modeling. Four of these models namely: Edie's two-regime model [3], modified Greenberg's two-phase equation, and Drakes two and three-regime models [2]. Each of this equation together with the continuity is solved using the method of characteristics to determine vehicle trajectory paths. The method of characteristics is employed for solving these first order nonlinear partial differential equations. The method is used to formulate equations that defines a family of lines in the (x,t) plane along which information travels. We further determine some time-dependent properties of each equation. The dynamic properties crosschecked for each model are tabled below:

- The derivative $q'(k)$ should be less than zero when density converges to jam density. $q'(k)$ characterizes the kinematic wave speed.
- The second derivative $q''(k)$ should rather be greater than zero. This is the stable shockwave property. A stable shock will be observed moving from a jam traffic to a free-flowing traffic. For the converse, then $q''(k)$ should be greater than zero.

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2 The Model

2.1 Edie's Multi-Regime Model

Edie in 1961 first proposed the idea of a two-regime model following the disadvantage of single-regime models. The Underwood model was used for the free-flow regime and the Greenberg model was used for the congested-flow regime. The speed-density equation is expressed as:

$$u = \begin{cases} 54.9e^{-\frac{\rho}{163.9}}, & \rho \leq 50 \\ 26.8 \ln\left(\frac{162.5}{\rho}\right), & \rho \geq 50 \end{cases} \tag{2.1}$$

u is the vehicles speed and ρ is traffic density.

The LWR macroscopic model will be coupled with all the multi-regime models to determine their flow characteristics. The LWR equation is a one-dimensional continuity equation used for modeling traffic flow [8, 11]. It is of the form:

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0$$

From the calculus of chain rule, the one-dimensional continuity equation can also be expressed as

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{dq(x,t)}{d\rho}\frac{\partial \rho(x,t)}{\partial x} = 0 \tag{2.2}$$

where $q(x,t) = \rho(x,t)u(x,t)$ is the flow function. The flow function for Edie's model is classified into two; the freeway regime and cluster regime. In the free-flow regime

$$q(x,t) = 54.9e^{-\frac{\rho}{163.9}} \text{ with } \frac{dq}{d\rho} = -0.335e^{-\frac{\rho}{163.9}}$$

As it similitude to equation (2.2), the LWR model reduces to

$$\frac{\partial \rho}{\partial t} - 0.335e^{-\frac{\rho}{163.9}}\frac{\partial \rho}{\partial x} = 0 \tag{2.3}$$

Again, it can be established by the method of characteristics that

$$\frac{dx}{dt} = -0.335e^{-\frac{\rho}{163.9}} = \frac{dq}{d\rho}$$

Hence the characteristic curve for the free flow regime is given by the equation:

$$x(t) = -0.335te^{-\frac{\rho_0}{163.9}} + x_0 \tag{2.4}$$

For the congested regime, the flow equation is given by

$$q(\rho) = 26.8\rho \ln\left(\frac{162.5}{\rho}\right)$$

with

$$\frac{dq}{d\rho} = 26.8\left[\ln\left(\frac{162.5}{\rho}\right) - 1\right] = \frac{\partial x}{\partial t}$$

Hence the characteristic is

$$x(t) = 26.8\left[\ln\left(\frac{162.5}{\rho_0}\right) - 1\right]t + x_0 \tag{2.5}$$

Figure 1 shows a diagrammatic representation of equations (2.4) and (2.5). Left is the freeway regime and right is the obstructed regime. A linear initial density profiles ρ_0 is chosen for plotting these lines of characteristics.

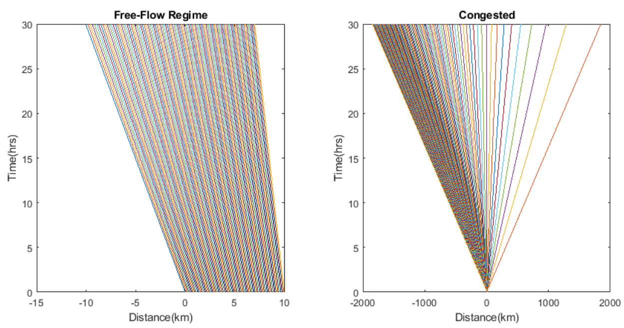

Figure 1: Characteristic Profiles for Edie's Model

On kinematic wave and stable shockwave property, the analysis is also presented in two cases. Case one for the free-flow regime and case two for the congested regime. In each case the first and second derivatives of flow with respect to density is presented.

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Case 1:

$$\frac{dq}{d\rho} = -0.335e^{-\frac{\rho}{163.9}} < 0, \quad \text{and} \quad \frac{dq^2}{d\rho^2} = 0.002e^{-\frac{\rho}{163.9}} > 0 \text{ as } \rho \to \rho_j$$

Case 2:

$$\frac{dq}{d\rho} = 26.8\left[\ln\left(\frac{162.5}{\rho}\right) - 1\right] < 0, \quad \text{and} \quad \frac{dq^2}{d\rho^2} = \frac{-26.8}{\rho} < 0$$

ρ_j is the jam density. These equations are an expression of both the first and second-order derivative property test of speed-density-flow models. The direction of the slopes explains the propagation speed of a disturbance during the cluster. The ability of the model to capture nonlinear traffic phenomena is explicated by the curvature of the flow function.

2.2 Drake's Two-Regime Model

Drake proposed two two-regime models and one three-regime model. The first two-regime model proposed by Drake makes use of Greenshields-type linear model for both the free-flow regime and the congested regime with speed defined as

$$u = \begin{cases} 60.9 - 0.525\rho, & \rho \leq 65 \\ 40 - 0.265\rho, & \rho \geq 65 \end{cases} \tag{2.6}$$

From a similar analysis from section 2.1 The free-flow function for this two-regime model is given by

$$q = 60.9\rho - 0.515\rho^2$$

with

$$\frac{dq}{d\rho} = 60.9 - 1.03\rho = \frac{dx}{dt}$$

Then, the equation that defines the characteristic in the space-time plane is given by

$$x(t) = (60.9 - 1.03\rho_0)t + x_0 \tag{2.7}$$

For the congested regime, the flow equation is expressed as

$$q = 40\rho - 0.265\rho^2$$

with the corresponding characteristic

$$x(t) = (40 - 0.53\rho_0)t + x_0 \tag{2.8}$$

The dynamic properties are as follows:
Case I:

$$\frac{dq}{d\rho} = 60.9 - 1.03\rho < 0, \quad \text{and} \quad \frac{dq^2}{d\rho^2} = -1.03 < 0$$

The sloping property is applicable when density is greater than 59.1. Hence the acceptable choice of values satisfying this property is for density to lie within $59.1 < \rho < 65$.

Case II:
For the congested regime

$$\frac{dq}{d\rho} = 40 - 0.53\rho < 0, \quad \text{and} \quad \frac{dq^2}{d\rho^2} = -0.53 < 0$$

With this regime, the first order derivative property is applicable when density exceeds $75.5veh/km$. This occurs in circumstances where traffic is more denser.

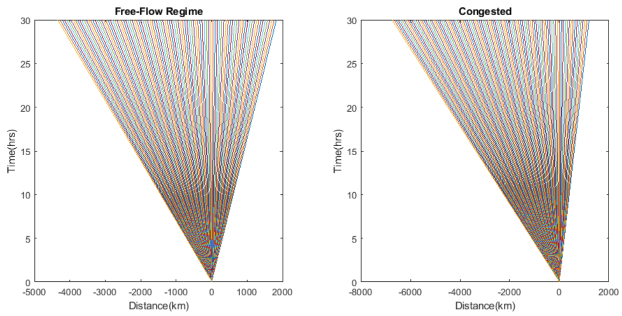

Figure 2: Characteristic Lines for Drake's Model

2.3 Modified Greenberg's Multi-Regime Model

The second two-regime model proposed by Drake suggests a constant speed for the free-flow regime and a Greenberg model for the congested-flow regime. The mathematical presentation of the equations is expressed by (2.9).

$$u = \begin{cases} 48, & \rho \leq 35 \\ 32\ln\left(\frac{145.5}{\rho}\right), & \rho \geq 35 \end{cases} \tag{2.9}$$

The derivation of the characteristic equations for the modified Greenberg fundamental equations is as follows. For the free-flow regime, flow is expressed as

$$q = 48\rho$$

This means the slope of the flow function will still be a constant value. Equation (2.2) reduces to a linear differential equation

$$\frac{\partial \rho}{dt} + 48\frac{\partial \rho}{\partial x} = 0 \tag{2.10}$$

Therefore the freeway characteristic is given by the equation

$$x(t) = 48t + x_0 \tag{2.11}$$

In the case of congestion

$$q = 32\rho\ln\left(\frac{145.5}{\rho}\right)$$

with

$$\frac{dq}{d\rho} = 32\left[\ln\left(\frac{145.5}{\rho}\right) - 1\right] = \frac{dx}{dt}$$

and

$$x(t) = 32\left[\ln\left(\frac{145.5}{\rho_0}\right) - 1\right]t + x_0 \tag{2.12}$$

For properties regarding the kinematic wave and shock wave, it is again classified into two cases. Case one for free-flowing traffic, while case two explains the properties of the congested regime. Case I: given that $q = 48\rho$, then

$$\frac{dq}{d\rho} = 48 > 0$$

with the second derivative given as

$$\frac{dq^2}{d\rho^2} = 0$$

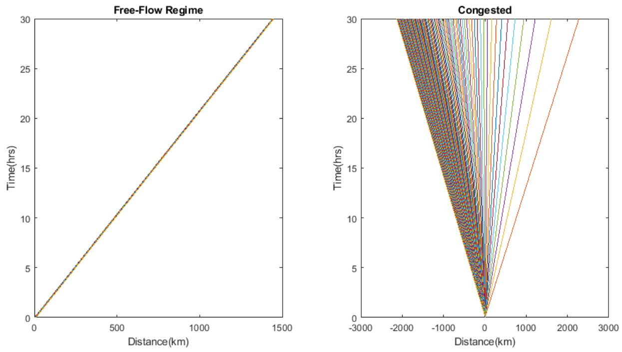

Figure 3: Characteristic Curves for Modified Greenberg's Model

For the second case

$$\frac{dq}{d\rho} = 32\left[\ln\left(\frac{145.5}{\rho}\right) - 1\right]$$

This first derivative will be less than zero when density is greater than 53.53. This presupposes that this property is not attainable. Because free-flow density should have a maximum of 35 vehicles per kilometer. The second derivative

$$\frac{dq^2}{d\rho^2} = \frac{-32}{\rho}$$

will result in a negative constant since density is always positive.

2.4 The Three-regime Model

The three-regime model developed by Drake takes a linear form for all three regimes. This model is represented by three different flow equations; the free-flow, the transitional flow, and the congested flow.

$$u = \begin{cases} 50 - 0.098\rho, & \rho \leq 40 \\ 81.4 - 0.913\rho, & 40 \leq \rho \leq 65 \\ 40 - 0.265\rho, & \rho \geq 65 \end{cases}$$

The respective equations representing flux for the various regimes are

$$q = \begin{cases} 50\rho - 0.098\rho^2, & \rho \leq 40 \\ 81.4\rho - 0.913\rho^2, & 40 \leq \rho \leq 65 \\ 40\rho - 0.265\rho^2, & \rho \geq 65 \end{cases}$$

From thence, we can derive the corresponding characteristic equations as follows

$$\frac{dx}{dt} = \begin{cases} 50 - 0.196\rho, & \rho \leq 40 \\ 81.4 - 1.826\rho, & 40 \leq \rho \leq 65 \\ 40 - 0.53\rho, & \rho \geq 65 \end{cases}$$

with the specific information paths as

$$x = \begin{cases} (50 - 0.196\rho_0)t + x_0, & \rho \leq 40 \\ (81.4 - 1.826\rho_0)t + x_0, & 40 \leq \rho \leq 65 \\ (40 - 0.53\rho_0)t + x_0 & \rho \geq 65 \end{cases} \tag{2.13}$$

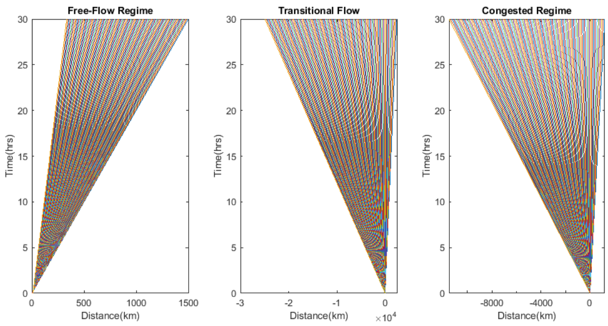

Figure 4: Characteristic Profiles for Drake's Three-Regime Model

For the kinematic wave and shockwave property, the analysis is categorized by their regimes. Free-flow regime:

$$\frac{dq}{d\rho} = 50 - 0.196\rho$$

The above derivative, will be less than zero when $\rho > 255.1$. This value of density does not make this alternative achievable. For the second derivative we obtain

$$\frac{dq^2}{d\rho^2} = -0.196 < 0$$

Transitional regime:

$$\frac{dq}{d\rho} = 81.4 - 1.826\rho$$

This also will be less than zero if $\rho > 44.6$. Therefore the applicable interval is $\rho \in [44.6, 65]$. Differentiating again

$$\frac{dq^2}{d\rho^2} = -1.826 < 0$$

Congested regime:

$$\frac{dq}{d\rho} = 40 - 0.53\rho$$

Here, the first derivative will be less than zero when density is $75.5veh/km$ or more.

$$\frac{dq^2}{d\rho^2} = -0.53 < 0$$

Since the speed-density functions are linear in nature, all the second derivatives are constants less than zero.

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3 Model Analysis and Discussions

From Figures 1-4, the characteristic curves for all regimes are presented, left is the free flow regime, the middle is the transitional regime if applicable, with the obstructed regime on the right. The curves for the congested flow functions have the same resemblance. For Drake's two-regime model, both jam traffic and free moving traffic have similar characteristic lines. An exception observed was the free-flow trajectories from the modified Greenberg's model with all others exhibiting wave-fans. For Edie's formulation, the requirement for kinematic wave speed property is satisfied for both regimes. However, a stable shockwave could only be observed during a relieved traffic. For Drake's models, the dynamic property regarding the wave speed was eligible for some range of density values. The acceptable values should range between $59veh/km$ and $65veh/km$. The second derivatives of Drake's functions were all negative. This suggests that the flow functions are concave as opposed to the usual convex shape. The situation was not different from the modified Greenberg's equation. Again, the slopes of Greenberg's flow functions were either zero or a positive real constant. These values give a delusive threshold for determining the propagating speed of traffic. The additional equation by Drake captioned synchronized regime did not produce any more efficient results. With the exception of the first derivative test for the transitional flow, the three-regime model failed all the dynamic property criterion for speed-density fundamental equations.

4 Conclusion

The main variables that characterize vehicular traffic are speed, density and flow. The relationship among these variables are evinced either through one-equation models or multiple equation models. In this paper, the characteristic profiles of vehicles are presented using a combination of some multi-regime model plus the first order traffic equation. The models considered include the two-regime Edie's models, the two-regime Greenberg's model, and two multi-regime models developed by Drake. The characteristic equations for free-flow regime, the transitional regime and the congested regime along with their corresponding curves are presented. Most of the characteristics evinced backward traveling trajectory features, more particularly the jam phase. This suggests that, these multi-regime models are not suitable for predicting trajectory paths of vehicles. These models also failed the test criterion for identifying stable shock waves. Implying that it would be more difficult using these multi-regime equations to explain nonlinear traffic phenomena such as phantom jams and traffic hysteresis.

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