Original Research Article

Numerical Solution of Two Dimensional Laplace's Equation on a Regular Domain Using Chebyshev Differentiation Matrices

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ABSTRACT

This work presents an efficient procedure based on Chebychev spectral collocation method for computing the 2D Laplace's equation on a rectangular domain. The numerical results and comparison of finite difference and finite element methods are presented. We obtained a satisfactory result when compared with other numerical solutions.

*Keywords: [*Chebychev spectral collocation method, Regular domain, pseudospectral method, Laplacian problems]

1. INTRODUCTION

A variety of problems arise throughout applied mathematics, classical and quantum mechanics require the solution of Laplace's equation in different domains. The use of high numerical methods for the computational solution of Laplacian problems is important in many fields of physics and engineering.
The general form of two dimensional steady-state Laplacian problems as given in the following equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (x, y) \subset \Omega$$

$$u(x, y) = g(x, y), \qquad (x, y) \subset \partial\Omega \qquad (1)$$

where *u* is the potential of heat, solute, etc. Here, Ω is a regular domain with $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.
Solutions by many numerical methods have been proposed. These numerical methods range from finite difference, finite element, and boundary integral methods, through to analytical techniques such as conformal mapping and series solutions. Not much work has been done on Chebyshev differentiation matrix for computing Laplacian problems. Spectral collocation methods have aroused great interest in recent decades and have given rise to a large body of literature, including the books that are practically oriented and more advanced (Berrut and Trefethen, 2004).
Taher et al, (2012) proposed efficient technique based on the Chebyshev spectral collocation method for computing the eigenvalues of fourth-order Sturm–Liouville boundary value problems. Weideman, (2006) used spectral differentiation matrices for the numerical solution of Schrodinger's equation, Hermite spectral collocation method for solving Schrodinger's equation was demonstrated through a few examples. Kong & Wu, (2008) researched on Chebyshev tau matrix method for Poisson-type equations in irregular domain, Poisson-type problems, including standard Poisson problems, Helmholtz problems, problems with variable coefficients and nonlinear problems were computed. Numerical schemes for Laplacian problems often encounter the problems of numerical dispersion and high computational effort (Li et al, 1997). The Chebyshev differentiation interpolation matrix was studied systematically by Gottlieb et. al. (1984), Solomonoff and Turkel (1986), and Peyret (1986).
In the year 2000, Trefethen (2000) gave a MATLAB code to solve fourth-order differential equations equipped with only the clamped boundary conditions. Weideman and Reddy (2000) published a book on MATLAB differentiation matrix suite based on pseudospectral method.

In this paper, we propose a new technique based on Chebyshev differentiation matrix for computing the solution of the two dimensional Laplace's equation on a regular domain. This method is able to deal with Dirichlet boundary conditions on a regular domain.

2. METHODOLOGY

2.1. Chebyshev Spectral Collocation

Spectral methods arise from the fundamental problem of approximation of a function by interpolation on an interval. Multidimensional domains of a rectilinear shape are treated as products of simple intervals and more complicated geometries are sometimes divided into rectilinear pieces (Trefethen, 1994).
Here, we restrict ourselves to the fundamental interval $[-1,1]$. Let $N \geq 1$ be an integer, even or odd, and let $x_0, \ldots, x_N$ or $x_1, \ldots, x_N$ be a set of distinct point in $[-1,1]$. For definiteness let the numbering be in reverse order:

$$1 \geq x_0 > x_1 > \ldots > x_{N-1} > x_N \geq -1$$

The Chebyshev collocation points of the first kind or Gauss_Lobatto points are defined as

$$x_j = \cos\left(\frac{\pi j}{N}\right), \qquad j = 0, \ldots, N \tag{2}$$

The Chebyshev collocation points of the second kind is defined as

$$y_j = \cos\left(\frac{\pi j}{N+1}\right), \qquad j = 0, \ldots, N \tag{3}$$

Consider the interpolation polynomial $g_j(x)$ of degree $\leq N$ satisfying $g_j(x) = \delta_{jk}$ for the Chebyshev nodes which we can express as the projections of equispaced points on the upper half of the unit circle as

$$x_k = \cos\left(\frac{\pi k}{N}\right), \qquad k = 0, \ldots, N \tag{4}$$

where the number of collocation points used is N+1. A spectral differentiation matrix for the Chebyshev collocation points is obtained by interpolating a polynomial through the collocation points, i.e. the polynomial

$$f(x) \approx \sum_{i=0}^{N} f_i g_i(x) \tag{5}$$

interpolates the points (x_i, f_i), such that $f(x_i) = f_i$, x_i is the collocation points. It can be shown that

$$g_i(x) = \frac{(-1)^{i+1}\left(1 - x^2\right)T_N'(x)}{c_i N^2\left(x - x_j\right)}, \qquad i = 0, \ldots, N \tag{6}$$

The nth -order derivative of the interpolating polynomial at the nodes is given by

$$f_i^n = \sum_{j=0}^{N} D_{ij}^{(n)} f_j, \qquad i = 0, \ldots, N \tag{7}$$

where the *i, j*th element of the differentiation matrices f_i^n is $g_j^n(x)$. For each $N \geq 1$, let the rows and columns of the $(N+1)\times(N+1)$ Chebyshev differentiation matrix D_N be indexed from 0 to N. Then the entries of the matrix are (Trefethen, 2000)

$$(D_N)_{00} = \frac{2N^2+1}{6}, \qquad (D_N)_{NN} = -\frac{2N^2+1}{6},$$

$$(D_N)_{jj} = \frac{-x_j}{2\left(1 - x_j^2\right)}, \qquad j = 1, \ldots, N-1,$$

$$(D_N)_{ij} = \frac{c_i(-1)^{i+j}}{c_j(x_i - x_j)}, \qquad i \neq j, \quad i, j = 1, \ldots, N-1$$

where $c_i = \begin{cases} 2 & i = 0 \quad or \quad N, \\ 1 & otherwise. \end{cases}$ (8)

2.2. Convergence Of Chebyshev Spectral Differentiation

Following Trefethen (2000), suppose u is analytic on and inside the ellipse with foci ± 1 on which the Chebyshev potential takes the value ϕ_f , that is, the ellipse whose semi-major and semi-minor axis lengths sum to $K = e^{\phi_f + \log 2}$. Let w be the $\upsilon\ th$ Chebyshev spectral derivative of u $(\upsilon \geq 1)$. Then

$$\left|w_j - u^{(\upsilon)}(x_j)\right| = O\left(e^{\phi_f + \log 2}\right) = O(K^{-N}) \qquad as \qquad N \to \infty. \tag{9}$$

The asymptotic convergence factor for the spectral differentiation process is at least as small as K^{-1}:

$\lim_{N\to\infty} sup \left|w_j - u^{(\upsilon)}(x_j)\right|^{\frac{1}{N}} \leq K^{-1}$.

3. RESULTS AND DISCUSSION

Laplace's Equation: Electric Potential over a Plate with Point Charge.
Consider the following Laplace's equation (Yang et al. 2005):

$$\nabla^2 u(x, y) = \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = f(x, y) \tag{10}$$

for $-1 \leq x \leq +1, -1 \leq y \leq +1$

where
$$f(x, y) = \begin{cases} -1 & for \quad (x, y) = (0.5, 0.5) \\ +1 & for \quad (x, y) = (-0.5, -0.5) \\ 0 & elsewhere \end{cases} \tag{11}$$

and the boundary condition is $u(x, y) = 0$ for all boundaries of the rectangular domain.

For this problem, we naturally set up a grid based on Chebyshev points independently in each direction, called a tensor product grid. According to Trefethen, (2000), for 1D, a Chebyshev grid is $2/\pi$ times as dense in the middle as an equally spaced grid, in *d* dimensions it becomes $(2/\pi)^d$.

We wish to approximate the Laplacian by differentiating spectrally in the x and y directions independently. The differentiation matrix 11×11 with N=12 in 1D is given by:

$$\tilde{D}_{12}^{2} = \begin{pmatrix} -934.4289 & 344.7380 & -85.5692 & 40.1610 & -24.3923 & 17.0718 & -13.1068 & 10.7672 & -9.3333 & 8.4671 & -8.0000 \\ 165.7234 & -206.6667 & 100.9898 & -24.3923 & 11.1295 & -6.6667 & 4.6603 & -3.6077 & 3.0102 & -2.6667 & 2.4869 \\ -33.7256 & 70.2929 & -99.3333 & 53.4558 & -13.1068 & 6.0000 & -3.6077 & 2.5442 & -2.0000 & 1.7071 & -1.5598 \\ 7.7821 & -13.1068 & 43.4085 & -65.3333 & 37.1472 & -9.3333 & 4.3519 & -2.6667 & 1.9249 & -1.5598 & 1.3855 \\ -3.6077 & 4.9676 & -9.3333 & 33.2328 & -52.2377 & 30.9282 & -8.0000 & 3.8390 & -2.4308 & 1.8273 & -1.5598 \\ 2.1436 & -2.6667 & 4.0000 & -8.0000 & 29.8564 & -48.6667 & 29.8564 & -8.0000 & 4.0000 & -2.6667 & 2.1436 \\ -1.5598 & 1.8273 & -2.4308 & 3.8390 & -8.0000 & 30.9282 & -52.2377 & 33.2328 & -9.3333 & 4.9676 & -3.6077 \\ 1.3855 & -1.5598 & 1.9249 & -2.6667 & 4.3519 & -9.3333 & 37.1472 & -65.3333 & 43.4085 & -13.1068 & 7.7821 \\ -1.5598 & 1.7071 & -2.0000 & 2.5442 & -3.6077 & 6.0000 & -13.1068 & 53.4558 & -99.3333 & 70.2929 & -24.3923 \\ 2.4869 & -2.6667 & 3.0102 & -3.6077 & 4.6603 & -6.6667 & 11.1295 & -24.3923 & 100.9898 & -206.6667 & 165.7234 \\ -8.0000 & 8.4671 & -9.3333 & 10.7672 & -13.1068 & 17.0718 & -24.3923 & 40.1610 & -85.5692 & 344.7380 & -934.4289 \end{pmatrix} \quad (12)$$

If I is 11×11 identity matrix, then the second derivative with respect to *x* will be computed by $kron(I, \tilde{D}_{12}^{2})$ and the second derivative with respect to y will be computed by $kron(\tilde{D}_{12}^{2}, I)$.
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$$L_N = I \otimes \tilde{D}_{12}^{2} + \tilde{D}_{12}^{2} \otimes I \quad (13)$$

The solution appears as a mesh plot in figure 1 and as a contour plot in figure 2. The first shows the locations of the 2541 nonzero entries in the 121×121 matrix L_{121}.

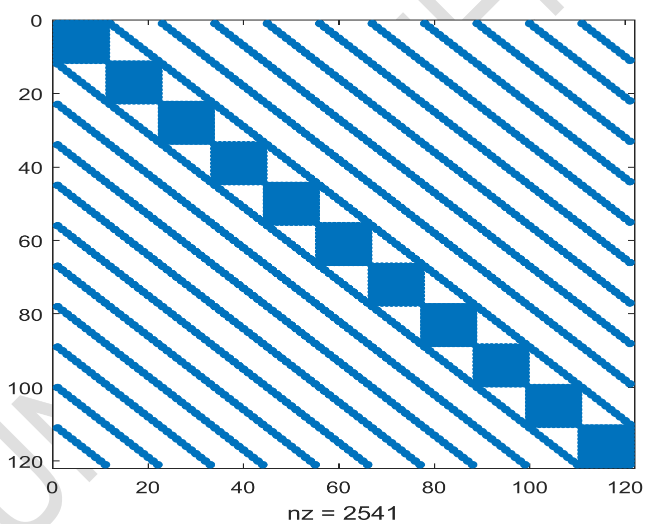

Figure 1: Sparsity plot of the $\mathbf{121} \times \mathbf{121}$ discreet Laplacian

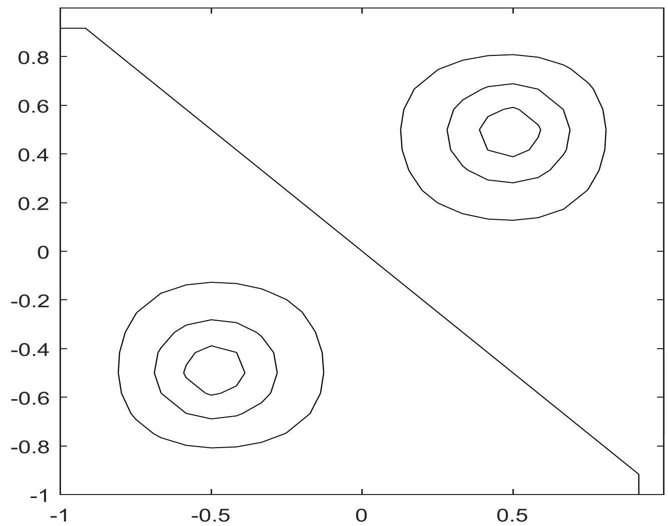

Figure 2: Some results represented as contour plot

We have made the size of the subregions small and their density high around the points (+0.5,+0.5) and [(−0.5,−0.5), since they are only two points at which the value of the right-hand side of Eq. (10) is not zero, and consequently the value of the solution u(x, y) is expected to change sensitively around them.
For comparison, the solution of the Laplace equation was carried out by three different methods, the Chebyshev differentiation matrix, finite element method and finite difference method to solve the same equation. The results obtained are depicted in Figures 3-5.

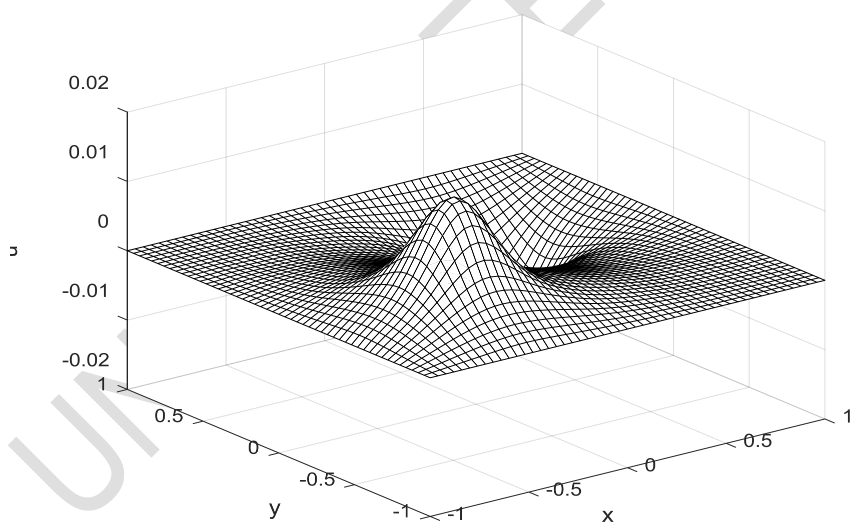

Figure 3: Solution of the Laplace equation (1.2). The result has been interpolated to a finer rectangular grid for plotting.

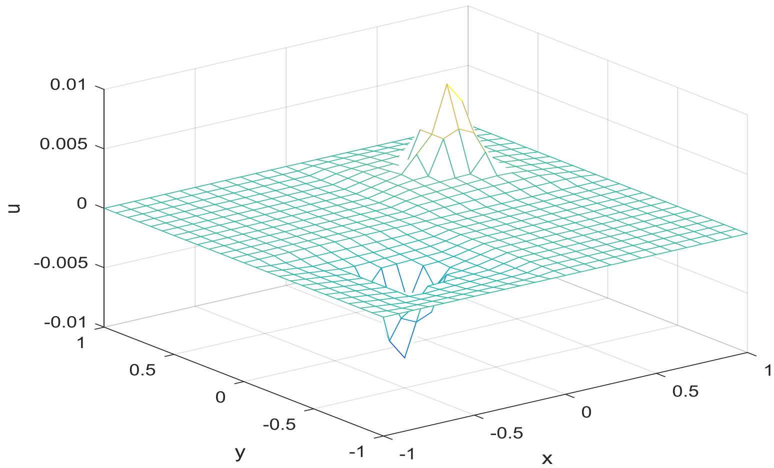

Figure 4: Solution by Finite Element Method.

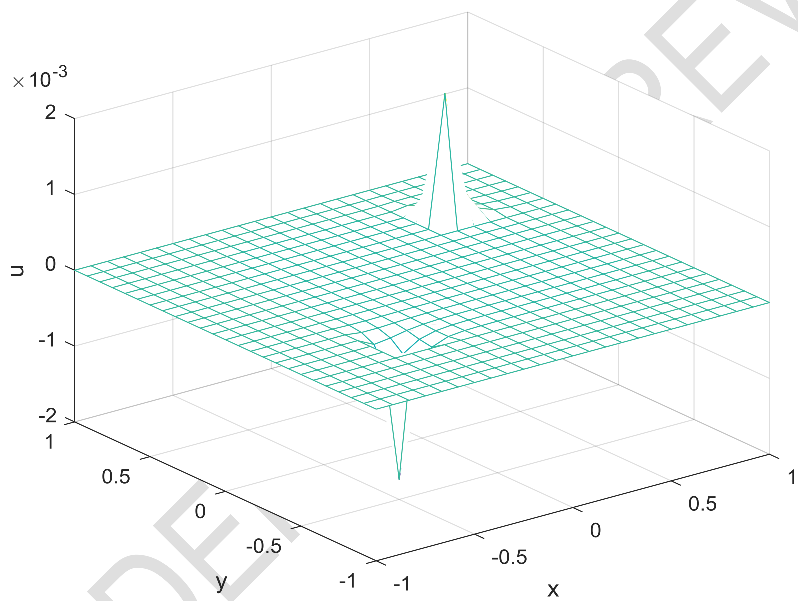

Figure 5: Solution by Finite Difference Method.

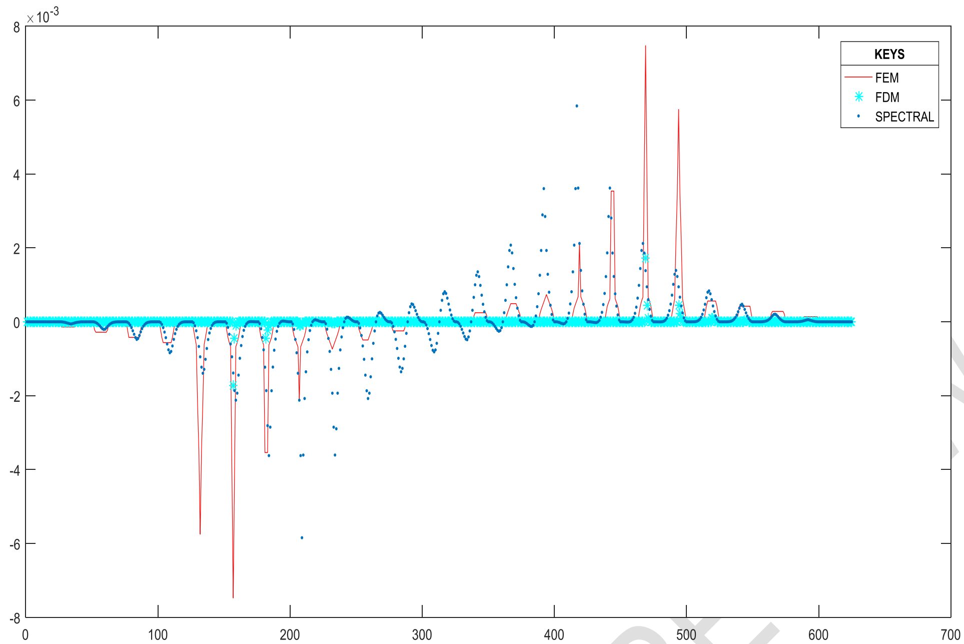

Figure 6: Graphical Comparison of Spectral method, Finite element method and Finite Difference method.

4. CONCLUSION

The two dimensional Laplace's equation on a rectangular domain was formulated in terms of Kronecker products. It was observed that the matrix, though not dense, is not as sparse as the typical matrix obtained with finite difference method or finite element method. However, we may hope to obtain satisfactory results with dimensions in the hundreds rather than the thousands or tens of thousands.

REFERENCES

1. Gottlieb, D., Hussaini, M.Y & Orszag, S.A. (1984). Introduction: theory and applications of spectral methods in Spectral
2. Methods for Partial Differential Equations, R.G. Voigt, D. Gottlieb, M.Y. Hussaini (Eds.), SIAM
3. Peyret, R. (1986). Introduction to Spectral Methods, Von Karman Institute.
4. Solomonoff, A. & Turkel, E. (1986). Global collocation methods for approximation and the solution of partial differential equations, ICASE Rep. No. 86-60, NASA Langley Research Center, Hamptom, VA.
5. Trefethen, L.N. (1996). Finite Difference and Spectral Methods for Ordinary and Partial Differential Equations. http://web.comlab.ox.ac.uk/oucl/work/nick.trefethen/pdetext.html.
6. Trefethen, L.N. (2000). Spectral Methods in Matlab, SIAM Press, Philadelphia.
7. Weideman, J.A.C. & Reddy, S.C. (2000). A MATLAB differentiation matrix suite, ACM Trans. Math. Softw. 26, 465–519.
8. Yang, W.Y, Cao, W, Tang, T.S. & Morris, J. (2005). Applied Numerical Methods using MATLAB. John Wiley & Sons, Inc., New Jersey.