Competitive Reaction-Diffusion Systems: Travelling Waves and Numerical Solutions

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Abstract

In this paper, we consider a competitive reaction-diffusion model to describe the existence of travelling wave solutions of two competing species. Moreover, the non-linear system is also studied by introducing different competitive-cooperative coefficients; constant and spatially distributed which leads to the persistence and extinction of organisms in a heterogeneous environment of population biology. If the diffusion coefficients and other parameters are positive constant, it is seen that one species is in extinction by the other and coexistence is also possible under certain conditions on carrying capacity. The results are numerically investigated by using the Finite difference method (FDM).

Keywords: Nonlinear PDEs, Travelling wave solutions, Reaction-diffusion, Crank-Nicolson scheme.
AMS Subject Classification: 92D25, 35K57 (primary), 35K61, 37N25.

1. Introduction

In nature there are two or more species compete for the same limited food source or in some way inhibit each other's growth. This type of interspecies interactions is known as mutual competitive suppression or competition for a common resource [1]. Their dynamics are considerably very rich, and also of great importance for the functioning of ecosystems. To describe the dynamics of two competing populations, the basic 2-species Lotka–Volterra competition model with diffusion can be used [2], which has the following set of equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(1 - au - \gamma v) \\[2ex] \dfrac{\partial v}{\partial t} = d_2 \Delta v + v(1 - bu - \delta v) \end{cases} \qquad (x,t) \in \mathbb{R} \times [0, \infty) \tag{1.1}$$

where u and v are the density of the two interacting species, $1/a$, $1/\delta$ are the carrying capacities, γ, b is the competition coefficients and d_1, d_2 are the diffusion coefficients, respectively. It is noted that all parametric values are non-negative. The symbol Δ is the well-known Laplacian operator which can also be written as $\Delta = \frac{\partial^2}{\partial x^2}$. Note that the competition model (1.1) is reaction-diffusion type and not a conservative system like its Lotka–Volterra predator-prey counterpart.

In modern mathematics, the theory of travelling wave solution of the partial differential equation is applied to describe different phenomena in ecology [3], farming [4], forestry [5], cell culture [6] and other natural sciences [7]. In this paper, we will study the travelling wave solution of the competitive reaction-diffusion system (1.1). We evaluate an approximate transformation of the travelling wave equations into monotone form and we reduce the existence proof to the application of well-defined theory about monotone travelling wave systems [8]. Let us now consider the system (1.1) as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{d}\dfrac{\partial^2 u}{\partial x^2} + u(1 - au - \gamma v) \\ \\ \dfrac{\partial v}{\partial t} = \mathrm{d}\dfrac{\partial^2 v}{\partial x^2} + v(1 - bu - \delta v) \end{cases} \qquad (x, t) \in \mathbb{R} \times [0, \infty) \tag{1.2}$$

For travelling wave solutions of the above systems, we will consider the following hypotheses:

$[A1]$ $\quad a < b$

$[A2]$ $\quad \gamma < \delta$

We will discuss the existence and uniqueness of the travelling wave solutions of the form $\left(u\left(\sqrt{\frac{1}{\mathrm{d}}}x + ct\right), v\left(\sqrt{\frac{1}{\mathrm{d}}}x + ct\right)\right)$ joining the equilibria $\left(0,\ \frac{1}{\delta}\right)$ and $\left(\frac{1}{a},\ 0\right)$ as $\sqrt{\frac{1}{\mathrm{d}}}x + ct$ moves from $-\infty$ to $+\infty$. It means, when the second species move from carrying the capacity to extinction, first species move from extinction to carrying capacity. If the inequality of hypothesis in $[A1]$ is interchanged, the existence of travelling wave solutions activating from $(0, 0)$ to positive coexistence equilibrium which proved in [9]. However, if $[A2]$ is interchanged, [10] and [11] assured us the existence of travelling wave solutions activating from one equilibrium on one positive axis to the equilibrium on another positive axis. Generally, we can observe that in some papers [8, 9] and [10] are used to solve the existence of travelling wave solutions using dynamical system and ordinary differential equation methods. We get help for studying about travelling wave solutions on other interacting species in related papers [13, 14, 15] and [16]. We can also be found various types of boundary value problems including the system (1.2) in [17, 18, 19, 20] and [21]. These books are not related to travelling wave solutions. The novelty of this work is that we use an alternative method of upper-lower solutions to prove the existence of travelling wave solutions. Moreover, we make the resulting system into a monotone system by changing the variable in the second equation of system (1.2) with reversing order. Recently, several researchers consider the monotone dynamical system for two species populations with competition-cooperation and mutualistic relations [24-28]. In weak competition it is shown that there is a possible coexistence of both populations and established that there is an ideal free pair; population maximize their fitness and any movement will reduce their fitness.

The main and important objectives of this paper are designed as follows:

- We established the travelling wave solutions of our governing equations analytically under some hypotheses in Section 2 and boundaries are open in $\mathbb{R}$.
- For numerical study, we consider two systems of partial differential equations (PDEs) for 2-species competition models. The first problem is defined in sub-section 3.1 with constant coefficients and homogeneous Neumann boundary conditions. The results are presented by varying the parameters in a finite domain.

- In sub-section 3.2, it is considered that the competition coefficients are spatially distributed and investigated the problems for different diffusion coefficients for various times.
- To solve PDEs, we employed the Crank-Nicolson finite difference scheme as well as *pdepe* package of MATLAB.

In the following section, we will study to find the travelling wave solutions of (1.2).

2. Existence of Travelling Wave Solution

In this section, we will show the existence of travelling wave solution and explore the system (1.2) which has of the form $\left(u\left(\sqrt{\frac{1}{\mathrm{d}}}x+ct\right),\ v\left(\sqrt{\frac{1}{\mathrm{d}}}x+ct\right)\right)$ adding the equilibria $\left(0,\ \frac{1}{\delta}\right)$ and $\left(\frac{1}{a},\ 0\right)$ as $\sqrt{\frac{1}{\mathrm{d}}}x+ct$ moves from $-\infty$ to $+\infty$.

Let us consider

$$t=\bar{t}\quad \text{and}\quad x=\sqrt{\mathrm{d}}\bar{x} \tag{2.1}$$

Equation (1.2) can be written as

$$\begin{cases}\dfrac{\partial u}{\partial \bar{t}}=\mathrm{d}\dfrac{1}{\left(\sqrt{\mathrm{d}}\right)^2}\dfrac{\partial^2 u}{\partial \bar{x}^2}+u(1-au-\gamma v)\\[2ex] \dfrac{\partial v}{\partial \bar{t}}=\mathrm{d}\dfrac{1}{\left(\sqrt{\mathrm{d}}\right)^2}\dfrac{\partial^2 v}{\partial \bar{x}^2}+v(1-bu-\delta v)\end{cases}$$

where $\bar{x}\in\mathbb{R},\ \bar{t}\in\mathbb{R}^+$. Now we can simplify the above system such that

$$\begin{cases}\dfrac{\partial u}{\partial \bar{t}}=\dfrac{\partial^2 u}{\partial \bar{x}^2}+u(1-au-\gamma v)\\[2ex] \dfrac{\partial v}{\partial \bar{t}}=\dfrac{\partial^2 v}{\partial \bar{x}^2}+v(1-bu-\delta v)\end{cases}\qquad \bar{x}\in\mathbb{R},\ \bar{t}\in\mathbb{R}^+ \tag{2.2}$$

Let

$$u=yM,\qquad v=wN, \tag{2.3}$$

where $y=\frac{1}{a}$ and w is a constant satisfying

$$\frac{1}{\delta}<w<\frac{1}{\gamma} \tag{2.4}$$

Then the system (2.2) becomes

$$\begin{cases}\dfrac{\partial (yM)}{\partial \bar{t}}=\dfrac{\partial^2 (yM)}{\partial \bar{x}^2}+yM(1-ayM-\gamma wN)\\[2ex] \dfrac{\partial (wN)}{\partial \bar{t}}=\dfrac{\partial^2 (wN)}{\partial \bar{x}^2}+wN(1-byM-\delta wN)\end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial M}{\partial \bar{t}} = \dfrac{\partial^2 M}{\partial \bar{x}^2} + M(1 - M - \gamma w N) \\ \dfrac{\partial N}{\partial \bar{t}} = \dfrac{\partial^2 N}{\partial \bar{x}^2} + N\left(1 - \dfrac{b}{a} M - \delta w N\right) \end{cases}$$

After rearranging the above system, we get

$$\begin{cases} \dfrac{\partial M}{\partial \bar{t}} = \dfrac{\partial^2 M}{\partial \bar{x}^2} + M(1 - M - zN) \\ \dfrac{\partial N}{\partial \bar{t}} = \dfrac{\partial^2 N}{\partial \bar{x}^2} + N(\rho_1 - b_1 M - \rho_1(1 + \rho_2)N) \end{cases} \tag{2.5}$$

where

$$z = \gamma w, \qquad \rho_1 = 1, \qquad b_1 = \frac{b}{a}, \qquad \rho_2 = \delta w - 1, \tag{2.6}$$

Here from $[A1]$ and (2.4), we have

$$z \in (0,1), \qquad \rho_2 > 0 \tag{2.7}$$

We can make ρ_2 arbitrarily small by taking w close to $\frac{1}{\delta}$ in (2.4).

Theorem 2.1 [3] Let us consider the system (1.2) under $[A1]$ and $[A2]$. For transforming the system (1.2) into the system (2.5), we use the change of variables (2.1) and (2.3) with w satisfying (2.4). The parameters in (2.5) are related to those in (1.2) by (2.6) and the parameters z, ρ_1, ρ_2 and b_1 satisfy the inequalities in (2.7).

If $(M(\bar{t}, \bar{x}),\ N(\bar{t}, \bar{x}))$ is a solution of (2.5) we can easily verify that

$$(u(t,x), v(t,x)) = \left(u(\bar{t}, \sqrt{\mathrm{d}}\bar{x}), v(\bar{t}, \sqrt{\mathrm{d}}\bar{x})\right) = (yM(\bar{t}, \bar{x}), wN(\bar{t}, \bar{x})) \tag{2.8}$$

is a solution of (1.2), where y and w are introduced in (2.3), (2.4). Now we have to find for solution of system (2.5). Let us consider the transformation

$$\left(M(\bar{t}, \bar{x}), N(\bar{t}, \bar{x})\right) = \left(M(s^*), N(s^*)\right) \text{ where } s^* = \bar{x} + c\bar{t}$$

and it satisfies

$$\begin{cases} \lim\limits_{s \to -\infty} \left(M(s^*), N(s^*)\right) = \left(0, \dfrac{1}{1 + \rho_2}\right) \\ \lim\limits_{s \to +\infty} \left(M(s^*), N(s^*)\right) = (1, 0) \end{cases} \tag{2.9}$$

Using this transformation, relating to (2.5), we are now finding for the solution of

$$\begin{cases} c\dfrac{\partial M}{\partial s^*} = \dfrac{\partial^2 M}{\partial {s^*}^2} + M(1 - M - zN) \\ c\dfrac{\partial N}{\partial s^*} = \dfrac{\partial^2 N}{\partial {s^*}^2} + N(\rho_1 - b_1 M - \rho_1(1 + \rho_2)N) \end{cases} \qquad s^* \in (-\infty, \infty) \tag{2.10}$$

Theorem 2.2 System (1.2) has a travelling wave solution of the form

$$(u(t,x),v(t,x)) = \left(yM\left(\sqrt{\frac{1}{\mathrm{d}}}x+ct\right), wN\left(\sqrt{\frac{1}{\mathrm{d}}}x+ct\right)\right) \tag{2.11}$$

for any $c > 2$ under the hypotheses [A1], [A2] and newly [A3] such that

[A3] $b \le 2a$.

Now, (M,N) is a function of one variable which is denoted by s^* satisfying (2.10) for $\mathrm{s}^* \in (-\infty,\infty)$ and (2.9) as $s^* \to \pm\infty$ and also $M(s^*)$ and $N(s^*)$ are positive monotonic functions for $s^* \in (-\infty,\infty)$. Remarkable thing is that

$$\begin{cases} \lim\limits_{t\to-\infty}\big(u(t,x),v(t,x)\big) = \left(0,\dfrac{1}{\delta}\right) \\ \lim\limits_{t\to+\infty}\big(u(t,x),v(t,x)\big) = \left(\dfrac{1}{a},0\right) \end{cases} \tag{2.12}$$

Proof: The change of variables

$$\begin{aligned} u_1(s^*) &= M(s^*) \\ u_2(s^*) &= \frac{1}{1+\rho_2} - N(s^*) \end{aligned} \tag{2.13}$$

For $s^* \in (-\infty,\infty)$, turns (2.10) into

$$\begin{cases} c\dfrac{\partial u_1}{\partial s^*} = \dfrac{\partial^2 u_1}{\partial {s^*}^2} + u_1\left(1-u_1-z\left(\dfrac{1}{1+\rho_2}-u_2\right)\right) \\ c\dfrac{\partial u_2}{\partial s^*} = \dfrac{\partial^2 u_2}{\partial {s^*}^2} + \left(\dfrac{1}{1+\rho_2}-u_2\right)(b_1u_1-\rho_1(1+\rho_2)u_2) \end{cases}$$

$$= \begin{cases} c\dfrac{\partial u_1}{\partial s^*} = \dfrac{\partial^2 u_1}{\partial {s^*}^2} + u_1\left(1-u_1-\dfrac{z}{1+\rho_2}+zu_2\right) \\ c\dfrac{\partial u_2}{\partial s^*} = \dfrac{\partial^2 u_2}{\partial {s^*}^2} + \left(\dfrac{1}{1+\rho_2}-u_2\right)(b_1u_1-\rho_1(1+\rho_2)u_2) \end{cases}$$

$$= \begin{cases} -\dfrac{\partial^2 u_1}{\partial {s^*}^2} + c\dfrac{\partial u_1}{\partial s^*} = u_1\left(\dfrac{1+\rho_2-z}{1+\rho_2}-u_1+zu_2\right), \\ -\dfrac{\partial^2 u_2}{\partial {s^*}^2} + c\dfrac{\partial u_2}{\partial s^*} = \left(\dfrac{1}{1+\rho_2}-u_2\right)(b_1u_1-\rho_1(1+\rho_2)u_2), \end{cases} \tag{2.14}$$

This equation is also monotone for following conditions such that

$$0 \le u_1, 0 \le u_2 \le \frac{1}{1+\rho_2}.$$

Now we have to construct a pair of coupled upper solutions for the system (2.14). Let us consider an increasing function $\bar{u}(s^*)$ satisfying the following Kolmogorov-Petrovskii-Piscunov (KPP) equation for $c > 2$ such that

$$-\frac{d^2\bar{u}}{ds^{*2}} + c\frac{d\bar{u}}{ds^*} = \bar{u}(1-\bar{u}) \tag{2.15}$$

For $s^* \in (-\infty, \infty)$ and also $\lim_{s^*\to-\infty} \bar{u}(s^*) = 0$ and $\lim_{s^*\to\infty} \bar{u}(s^*) = 1$. Let

$$\bar{u}_1(s^*) = \bar{u}(s^*), \qquad \bar{u}_2(s^*) = \frac{1}{1+\rho_2}\bar{u}(s^*), \tag{2.16}$$

For $0 \le u_2 \le \bar{u}_2(s^*)$, we can easily observe that

$$\begin{aligned}
-\frac{d^2\bar{u}_1}{ds^{*2}} + c\frac{d\bar{u}_1}{ds^*} - \bar{u}_1\left(\frac{1+\rho_2-z}{1+\rho_2} - \bar{u}_1 + zu_2\right)& \\
&= -\frac{d^2\bar{u}}{ds^{*2}} + c\frac{d\bar{u}}{ds^*} - \bar{u}\left(1 - \frac{z}{1+\rho_2} - \bar{u} + zu_2\right)\\
&= \bar{u}(1-\bar{u}) - \bar{u}\left(1 - \frac{z}{1+\rho_2} - \bar{u} + zu_2\right)\\
&= \bar{u}\left(1 - \bar{u} - 1 + \frac{z}{1+\rho_2} + \bar{u} - zu_2\right)\\
&= \bar{u}\left(\frac{z}{1+\rho_2} - zu_2\right)\\
&\ge \frac{z}{1+\rho_2}\bar{u}(1-\bar{u}) > 0
\end{aligned} \tag{2.17}$$

for all $s^* \in (-\infty, \infty)$. For $0 \le u_1 \le \bar{u}_1(s^*)$, we also can check that

$$\begin{aligned}
&-\frac{d^2\bar{u}_2}{ds^{*2}} + c\frac{d\bar{u}_2}{ds^*} - \left(\frac{1}{1+\rho_2} - \bar{u}_2\right)(b_1u_1 - \rho_1(1+\rho_2)\bar{u}_2)\\
&\quad= \frac{1}{1+\rho_2}\left(-\frac{d^2\bar{u}}{ds^{*2}}\right) + c\frac{1}{1+\rho_2}\frac{d\bar{u}}{ds^*} - \left(\frac{1}{1+\rho_2} - \frac{\bar{u}}{1+\rho_2}\right)\left(b_1u_1 - \rho_1(1+\rho_2)\frac{\bar{u}}{(1+\rho_2)}\right)\\
&\quad= -\frac{1}{1+\rho_2}\frac{d^2\bar{u}}{ds^{*2}} + \frac{c}{1+\rho_2}\frac{d\bar{u}}{ds^*} - \frac{1}{1+\rho_2}(1-\bar{u})(b_1u_1 - \rho_1\bar{u})\\
&\quad= \frac{1}{1+\rho_2}\left[-\frac{d^2\bar{u}}{ds^{*2}} + c\frac{d\bar{u}}{ds^*} - (1-\bar{u})(b_1u_1 - \rho_1\bar{u})\right]\\
&\quad= \frac{1}{1+\rho_2}[\bar{u}(1-\bar{u}) + (1-\bar{u})(\bar{u}\rho_1 - b_1u_1)]\\
&\quad\ge \frac{1}{1+\rho_2}\bar{u}(1-\bar{u})(1+\rho_1-b_1)\\
&\quad\ge 0
\end{aligned}$$

for all $s^* \in (-\infty, \infty)$. We can say that the inequalities are the true cause

$$1 + \rho_1 - b_1 = 1 + 1 - \frac{b}{a} = 2 - \frac{b}{a} \ge 0$$

by hypothesis $[A3]$. Consider a pair of functions denoted by $\bar{\eta}_1(s^*)$ and $\bar{\eta}_2(s^*)$ and defined by

$$\bar{\eta}_1(s^*) = \bar{u}_1(-s^*), \bar{\eta}_2(s^*) = \bar{u}_2(-s^*) \tag{2.18}$$

Now let us consider the monotone system

$$\begin{cases} \dfrac{\partial^2 \eta_1}{\partial s^{*2}} + c\dfrac{\partial \eta_1}{\partial s^*} + \eta_1\left(\dfrac{1+\rho_2 - z}{1+\rho_2} - \eta_1 + z\eta_2\right) = 0, \\ \dfrac{\partial^2 \eta_2}{\partial s^{*2}} + c\dfrac{\partial \eta_2}{\partial s^*} + \left(\dfrac{1}{1+\rho_2} - \eta_2\right)(b_1\eta_1 - \rho_1(1+\rho_2)\eta_2) = 0, \end{cases} \tag{2.19}$$

For $s^* \in (-\infty, \infty)$, the problem reduces to

$$\begin{cases} \dfrac{\partial^2 \bar{\eta}_1}{\partial s^{*2}} + c\dfrac{\partial \bar{\eta}_1}{\partial s^*} + \bar{\eta}_1\left(\dfrac{1+\rho_2 - z}{1+\rho_2} - \bar{\eta}_1 + z\eta_2\right) \le 0, \\ \dfrac{\partial^2 \bar{\eta}_2}{\partial s^{*2}} + c\dfrac{\partial \bar{\eta}_2}{\partial s^*} + \left(\dfrac{1}{1+\rho_2} - \bar{\eta}_2\right)(b_1\eta_1 - \rho_1(1+\rho_2)\bar{\eta}_2) \le 0, \end{cases} \tag{2.20}$$

For $s^* \in (-\infty, \infty)$, all$0 \le \eta_2 \le \bar{\eta}_2(s^*)$, $0 \le \eta_1 \le \bar{\eta}_1(s^*)$. In the region, $0 \le \eta_1 \le 1$ and $0 \le \eta_2 \le \frac{1}{1+\rho_2}$, the system (2.19) is monotone. When $\eta_1 = \bar{\eta}_1(s^*)$is the first equation and $\eta_2 = \bar{\eta}_2(s^*)$ is the second equation for all $s^* \in (-\infty, \infty)$, particularly (2.20) is true. Here let

$$f_1(\eta_1, \eta_2) = \eta_1\left(\frac{1+\rho_2 - z}{1+\rho_2} - \eta_1 + z\eta_2\right)$$

$$f_2(\eta_1, \eta_2) = \left(\frac{1}{1+\rho_2} - \eta_2\right)(b_1\eta_1 - \rho_1(1+\rho_2)\eta_2)$$

Hence $f_i\left(S, \frac{Sb_1}{2\rho_1(1+\rho_2)}\right) > 0$ for $i = 1,2$ and $S > 0$ is sufficiently small. Let V_1 be a class of vector valued functions $\vec{\eta}(s^*) \in C^2(-\infty, \infty)$ is monotonically decreasing and satisfying $\lim_{s^* \to \pm\infty} \vec{\eta}(s^*) = \overrightarrow{M_\pm}$ with $\vec{f} = (f_1, f_2), \overrightarrow{M_+} = (0,0)$ and $\overrightarrow{M_-} = \left(1, \frac{1}{1+\rho_2}\right)$.

We have $c \ge M^*$ for the existence of the function $(\bar{\eta}_1(s^*), \bar{\eta}_2(s^*))$ satisfying (2.20) where

$$M^* = max\left\{Inf_{\vec{\eta} \in V_1}\left\{Sup_{s^*, V_1}\frac{\frac{d^2\eta_{V_1}}{ds^{*2}} + f_{V_1}(\vec{\eta}(s^*))}{\frac{d\eta_{V_1}}{ds^*}}\right\}, 0\right\}.$$

Since the function $\vec{f}$ can be reduced at the top left corner of the rectangle $[\overrightarrow{M_+}, \overrightarrow{M_-}] = [0,1] \times [0, \frac{1}{1+\rho_2}]$, then the system (2.14) has a solution which is a function denoted defined by

$$(\hat{u}_1(s^*), \hat{u}_2(s^*)) \coloneqq (\hat{\eta}_1(s^*), \hat{\eta}_2(s^*))$$

After setting $M(s^*) = \hat{u}_1(s^*)$ and $N(s^*) = \frac{1}{1+\rho_2} - \hat{u}_2(s^*)$ for $s^* \in (-\infty, \infty)$ as in (2.13), then $(u(t,x), v(t,x))$ as defined in (2.11) is a travelling wave solution of system (1.2) for $x \in (-\infty, \infty)$, $t > 0$, satisfying (2.12) as described in the statement of theorem 2.2.

3. Numerical Examples and Applications

3.1 Effects of competitive constant coefficients

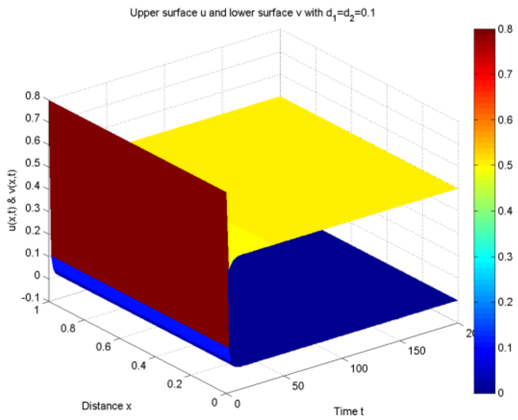
We consider the following system of partial differential equations subject to initial and boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{d}\dfrac{\partial^2 u}{\partial x^2} + u(1 - 2u - v) \\ \dfrac{\partial v}{\partial t} = \mathrm{d}\dfrac{\partial^2 v}{\partial x^2} + v(1 - 3u - \dfrac{19}{10}v) \end{cases} \tag{3.1}$$

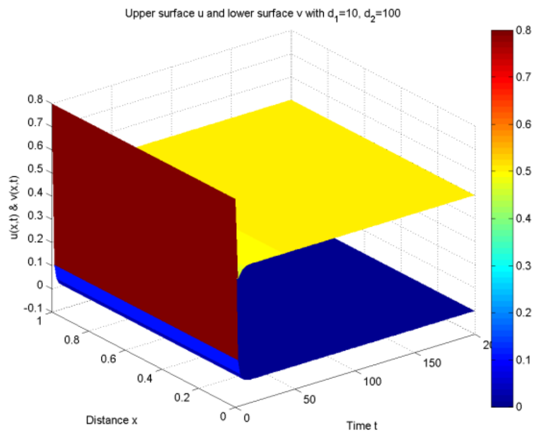
Where $a = 2, \gamma = 1, b = 3, \delta = \frac{19}{10}$, with the domain $\sigma = (0,1)$ and homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0$$

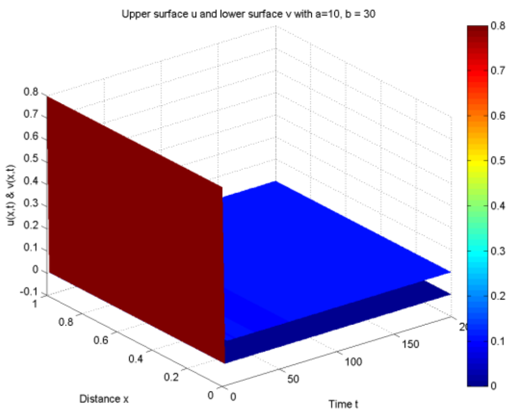
Here $[A1]$ to $[A2]$ are readily satisfied and $[A3]$ is always true for $b \leq 2a$. Now, our goal is to solve these equations numerically by using Implicit Finite Difference Method such as Crank-Nicolson method [22]. Constructing the algorithm in FORTRAN languages by code block software and *pdepe* package for partial differential equations in MATLAB; we can have the solutions which are graphically presented by the following figures.



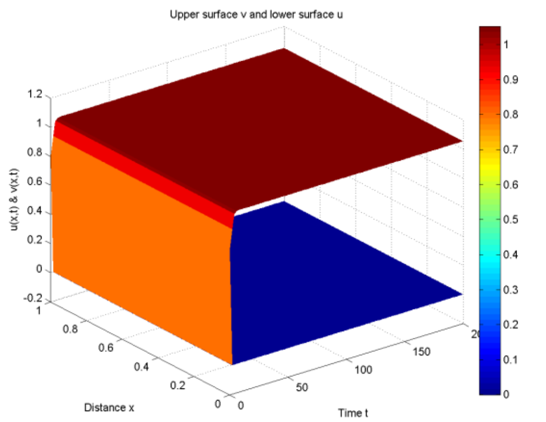
(a)



(b)



(c)



(d)

Figure 3.1: Dynamics of the system of equation (3.1) which corresponds (1.1) for various changes of the parameter.

By taking time-domain maximum 200 and spatial domain 1, we see from Figure 3.1 (a, b, c, d) that for different diffusion strategies the domination of species $u(t,x)$ over $v(t,x)$ does not change. Besides, when $b = 30 \nleq 2a = 20$, we get almost extinction of both the species. Nevertheless, using $\gamma = 10$, $u(t,x)$ shows steady eradication nature during whole time whereas $u(t,x)$ moves up to 1. More dynamics under different diffusion and time domain are given below.

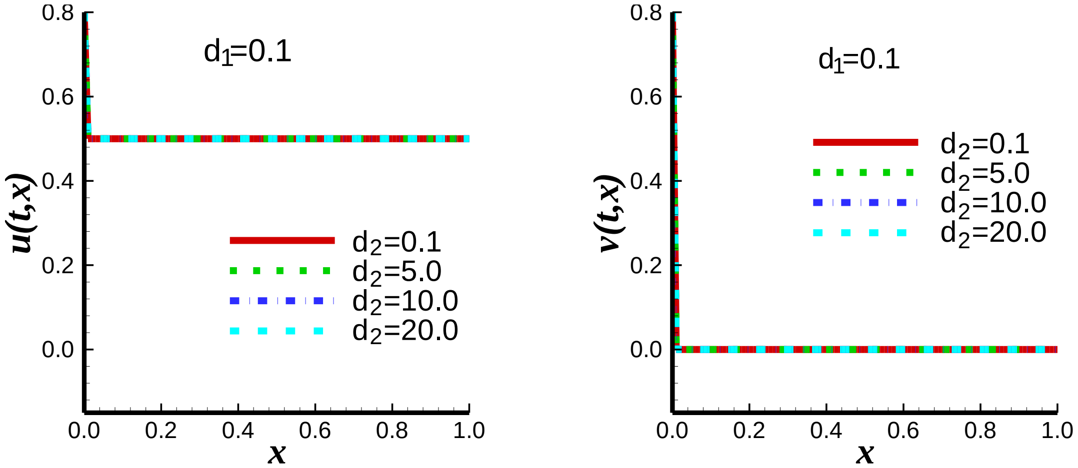


Figure 3.2: The illustration of the solutions (3.1) for different diffusion coefficients at time $t = 200$ (left) $u(t,x)$ and (right) $v(t,x)$.

The behaviour of diffusion coefficients is reported in Figure 3.2. We take different values of diffusion coefficients at time $t = 200$ over the habitat. Considering one diffusion coefficient is fixed such as $d_1 = 0.1$ and another one is replaced by various values such as $d_2 = 0.1, 5, 10, 20$ and it is observed that the solutions are coinciding separately. Biologically, it means that the solutions $u(t,x)$ and $v(t,x)$ are independent of diffusion coefficients.

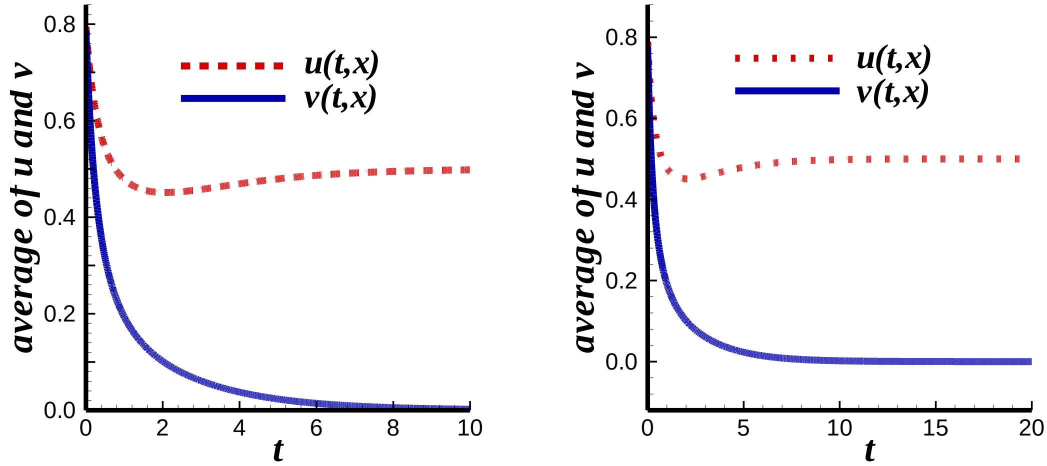


Figure 3.3: The graphical representation of average solutions at different times $t = 10,\ 20$ with diffusion coefficient $d_1 = d_2 = 0.1$.

Figure 3.3 represents the nature of average solutions versus time. By taking identical as diffusion coefficient $d_1 = d_2 = 0.1$ at different times $t = 10$ (left) and $t = 20$ (right), we see that the average solutions vary on time and the density of populations are changing. One species is survived and the other one is in extinction.

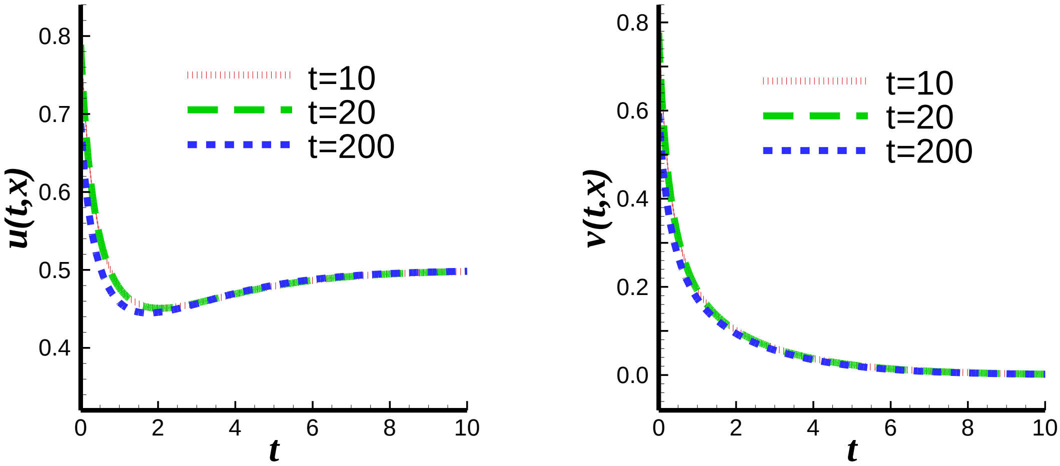


Figure 3.4: Comparison at different times $t = 10, t = 20, t = 200$ and corresponding average solutions of $u(t, x)$ (left) and $v(t, x)$ (right) for same diffusion coefficient $\mathrm{d} = 0.1$ according to (3.1).

From Figure 3.3, we have known that the average solutions are time-dependent. The descriptions of Figure 3.2 are still valid for Figure 3.4. Here we represent the multiple plots of solutions at time $t = 10$ (solid), $t = 20$ (long dashed) and $t = 200$ (dashed). So, it is generalized that solutions of the system (3.1) are independent of diffusion coefficients but obviously, the persistence and extinction depend on diffusion speed.

3.2 Effects of spatially distributed competition coefficients

Let us now consider a generalized form of (1.2), when competition coefficients are space dependent:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \dfrac{\partial^2 u}{\partial x^2} + u(k(x) - a(x)u - \gamma(x)v) \\ \dfrac{\partial v}{\partial t} = d_2 \dfrac{\partial^2 v}{\partial x^2} + v(k(x) - b(x)u - \delta(x)v) \end{cases} \tag{3.2}$$

where $k(x)$ is the carrying capacity and $a(x)$, $\gamma(x)$, $b(x)$, $\delta(x)$ are all function of x, positive and defined as the competition coefficient. It is noted that the boundary conditions of (3.2) are the same as defined in the earlier section.

Our next step is to establish some results using the following parametric functional $k(x) = 1$, $a(x) = 1.1 + sin(\pi x) < b(x) = 1.2 + sin(\pi x)$, and $\gamma(x) = 2.0 + cos(\pi x) < \delta(x) = 2.2 + cos(\pi x)$ of (3.2) over the domain $\sigma = (0, 1)$.

Using the same numerical strategy of section 3.1, we produce the following results:

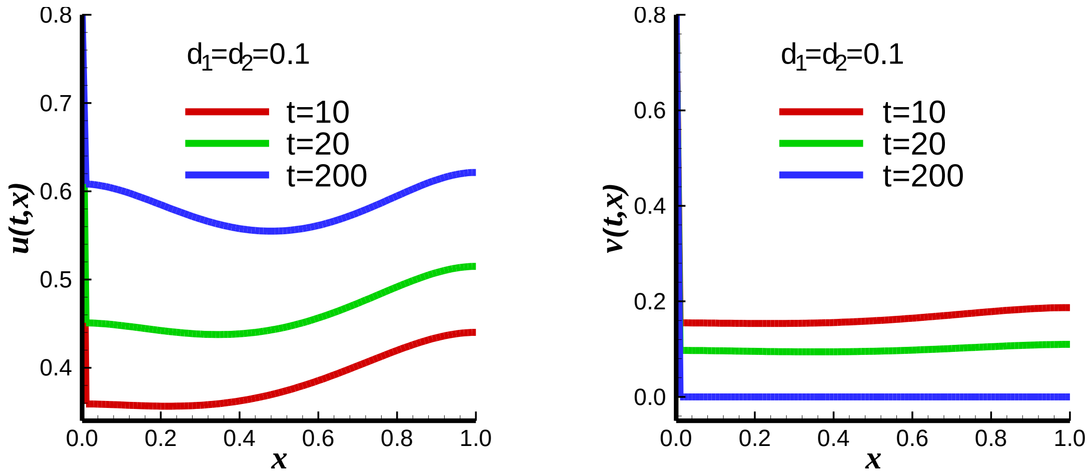


Figure 3.5: Solutions of (3.2) for same diffusion coefficient $d_1 = d_2 = 0.1$, initial value $u_0 = v_0 = 0.8$ at times $t = 10, t = 20$ and $t = 200$ over the domain.

We can see that $u(t,x)$ and $v(t,x)$, the solutions of (3.2) which show that the density of first species are increasing over the domain for same diffusion coefficient at different times while the second population density is decreasing. The result satisfies the third hypothesis $b(x) < 2a(x)$. Now we consider (3.2) with different diffusion coefficient as $d_1 = 0.1$ and $d_2 = 0.5, 1.0, 5.0, 10.0, 20.0$ at *t=200*. It is observed (see, Figure 3.6) that all the solutions for different diffusion coefficients coincide. So, the variation of diffusion coefficient does not effect on populations size.

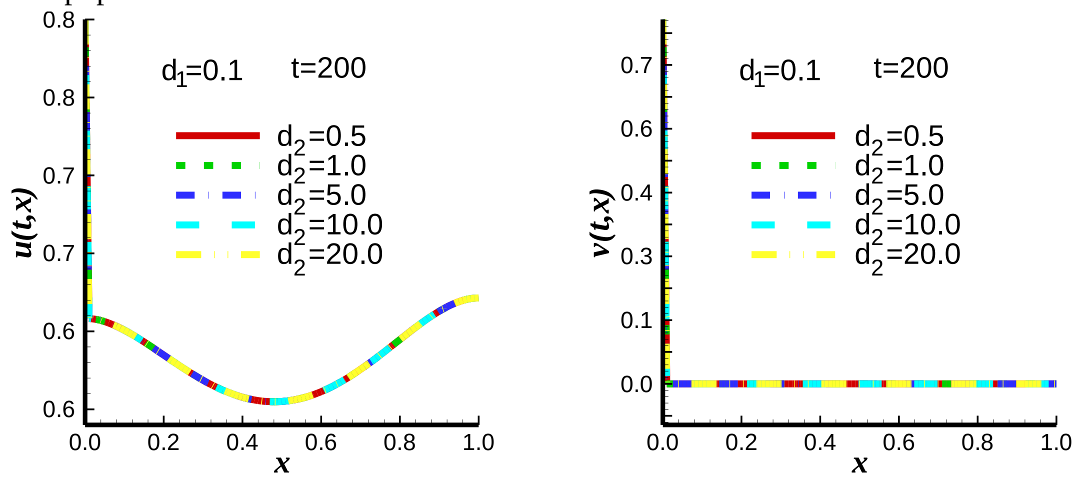


Figure 3.6: Solutions of (3.2) for $d_1 = 0.1$, $\sigma = (0,1)$, $u_0 = v_0 = 0.8$ for various $d_2 = 0.5, 1.0, 5.0, 10.0$ and 20.0 at time $t = 200$.

We can investigate the solutions for increasing of times from 10 to 200 using the same diffusion coefficient which is depicted in the following Figure 3.7. Solutions are indicating by $u(t,x)$ and $v(t,x)$. When time varies, we observe that both populations are coexisting with $u_0 = v_0 = 0.8$ over the domain.

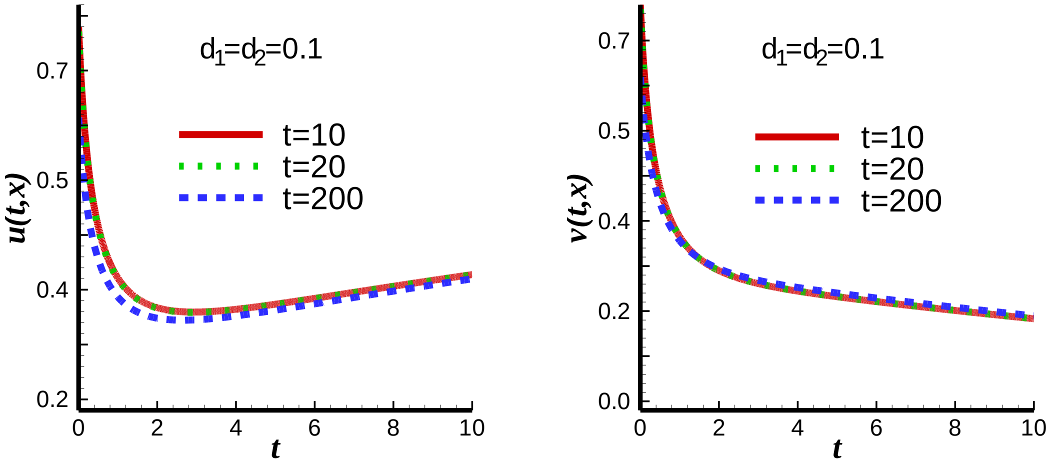


Figure 3.7: Solutions of (3.2) for same diffusion coefficient 0.1 at different times $t = 10, t = 20$ and $t = 200$, respectively.

The following Figure 3.8 establishes for carrying capacity, $k(x) = 2.5 + \cos(\pi x)$ which is bigger than all other parameters such that $a(x) = 1.1 + sin(\pi x)$, $b(x) = 1.2 + sin(\pi x)$, $\gamma(x) = 2.0 + cos(\pi x)$ and $\delta(x) = 2.2 + cos(\pi x)$ and diffusion coefficients $d_1 = d_2 = 0.5$.

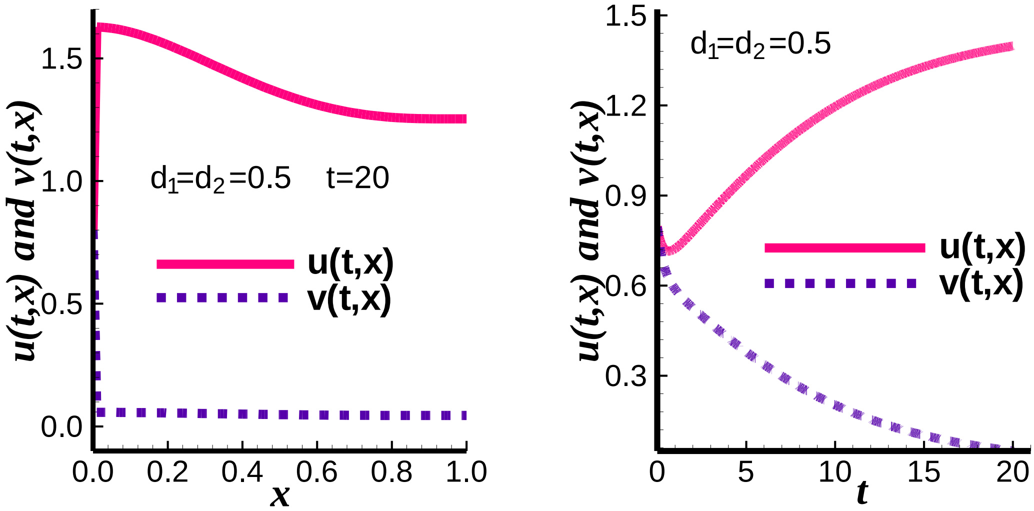


Figure 3.8: Solutions (left) and average solutions (right) of (3.2) using carrying capacity $k(x) = 2.5 + \cos(\pi x)$.

The above figures depict, if the carrying capacity is larger than all other parameters, then left illustration shows that $u(t, x)$, increasing and $v(t, x)$, decreasing. Similarly, the right illustration shows that average solutions have similar behaviour species at time $t = 20$ for same diffusion coefficient d_1=d_2=0.5 and same initial value 0.8. It's obvious that with smaller carrying capacity in the competitive species, there is a formidable chance for both the species step toward extinction.

4. Conclusion

In this paper, we introduced an appropriate transformation of the travelling wave solutions using three hypotheses and the realistic significances of these hypotheses. The models have presented the interconnection between growth, competition, diffusion coefficients etc. for two species population dynamics and it is observed that the travelling wave can exist. We investigated the characteristics of competitive reaction-diffusion equations for a couple of species. The selected equations do not depend on the changes of diffusion coefficients over the domain and the density of one species are decreasing at a certain time while the rest one is increasing. It is also constructed two different forms of governing equations for numerical simulations and observed that the persistence and extinction speed for different times are independent of diffusion coefficients. One species is increasing and the left one is decreasing over the domain if we take the larger carrying capacity for identical or different diffusion coefficients. The numerical results are obtained by an implicit finite difference method.

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Conflict of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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