Theoretical verification of formula for charge function in time q = c * v in RC circuit for charging/discharging of fractional & ideal capacitor

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Abstract
In this paper we apply the newly developed charge storage expression as a function of time i.e. via convolution operation of time varying capacity function and applied voltage function to a capacitor i.e. q = c* v. This new formula is different to usual and conventional way of writing capacitance multiplied by voltage to get charge stored in a capacitor i.e. q = cv. We apply this new formula to RC circuit as charging/discharging the capacitors (ideal and fractional ones) via constant dc voltage or current sources. This paper gives validity of usage of this new formula in RC circuits.

Keywords
Mittag-Leffler function, Time varying Capacity Function, Fractional Capacitor, Ideal loss-less Capacitor, Convolution Operation, Laplace Transform, Fractional derivative, Supercapacitor, Self-Discharging, Memory Effect

1. Introduction
This is continuation of our earlier deliberations regarding verification of the new formula $q(t) = c(t) * v(t)$; [1], [40]. This paper is from deliberations regarding usage of this formula in Project: Design & Development of Power-packs with Aerogel Supercapacitors & Fractional Order Modeling BRNS Santion No. 36(3)/14/50B/2014-BRNS/2620 dated 11.03.2015; where we wish to use this new developed formula.

The voltage change when appears at a capacitor, it reacts or relaxes via relaxation current. The time varying capacity function $c(t)$ is the one that defines the response function; and by principle of causality [1] we write $q(t) = c(t) * v(t)$ where $v(t)$ is the input impressed voltage. This is different to usual formula $q(t) = c(t)v(t)$. This new formulation is deliberated in detail with $c(t)$ as for ideal loss less capacitor case, as well as time varying capacity function (fractional capacitor case) in [1]. The capacity function $c(t)$ is the function which decays with time, and has the form $c(t) \propto t^{-\alpha}$; $0 < \alpha < 1$ and acts only at the time of application of voltage change. For ideal case of loss-less capacitor the capacity function is $c(t) \propto \delta(t)$; [1]. In this paper we will always take the power-exponent of power-law of decaying capacity function i.e. α as between zero and one, i.e. $0 < \alpha < 1$. This power-law decay function is in singular at origin and is in tune with singular power law decay relaxation current given by Curie-von Schweidler (universal law) of dielectric relaxation [2]-[5]. In this universal dielectric relaxation law, the relaxing current is a decaying power-law as $i(t) \propto t^{-\alpha}$, when uncharged system of dielectric is stressed by a constant voltage. The use of this universal dielectric relaxation law gives current voltage relation of a capacitor as given by fractional derivative [6]-[10]. The non-singular decaying function gives all together different form of current voltage relations in capacitor is discussed in [11], [38]. The use of non-singular kernel in integration for the formula for fractional derivative and application is developing topic. This concept is used and studied in pioneering works [23]-[36], for several dynamic systems.

Here we are taking singular function $c(t)$ as 'time varying capacity function', as because the same gets derived from basic universal dielectric relaxation law $i(t) \propto t^{-\alpha}$ of Curie-von Schweidler [1], [40]. In this paper we will take capacitor with time varying capacity function $c(t) = C_\alpha t^{-\alpha}$ (i.e. a fractional capacitor), and will use the formula [1], where the voltage excitation $v(t)$ is applied at time $t = 0$ to an uncharged capacitor

$$q(t) = c(t) * v(t) = \int_{-\infty}^{t} c(t-\tau)v(\tau)d\tau = \int_{-\infty}^{t} c(\tau)v(t-\tau)d\tau$$

With this new formula $q(t) = c(t) * v(t)$ applied we discuss various cases of $q(t)$ i.e. charge stored in capacitor and $i(t)$, the circuital current etc. for RC charging/discharging circuit with ideal capacitor and fractional capacitor.

We note a priori that the constant C_α is proportionality constant of the relation of time varying capacity function i.e. $c(t) \propto t^{-\alpha}$, and not Fractional Capacity. The fractional capacity of a fractional capacitor we will represent as $C_{F-\alpha}$ which has units of $\text{Farad} / \sec^{1-\alpha}$, and we will use $C_{F-\alpha} = C_\alpha \Gamma(1-\alpha)$ to relate the two [1], [40]. The voltage, $v(t)$ across a capacitor or dielectric changes at a rate in proportion to the current: $i(t) = D_t^1\left(c(t) * v(t)\right)$, with $c(t) = C_\alpha t^{-\alpha}$ we get $i(t) = \left(c(t)\right)\left(v(0)\right) + c(t) * D_t^1 v(t)$; [1]. The equation of current and voltage, and impedance for fractional capacitor is following, given by fractional derivative $D_t^\alpha \equiv d^\alpha / dt^\alpha$ [6], [7] [8], [12], [13]; comes from $q(t) = c(t) * v(t)$, [1]. The fractional derivative operator is Riemann-Liouville type (Refer Appendix) as derived in [1]; and in [6], [7].

$$i(t) = C_{F-\alpha} \frac{d^\alpha v(t)}{dt^\alpha}; \quad Z(s) = \frac{1}{s^\alpha C_{F-\alpha}}; \quad 0 < \alpha < 1$$

With limit $\alpha \to 1$ we get classical ideal loss less capacitor that is following

$$i(t) = C \frac{d\ v(t)}{dt}; \quad Z(s) = \frac{1}{s\ C}$$

The fractional capacitor appears in studies with super-capacitors and other memory based relaxation phenomena [14]-[22]. We assume that the fractional capacitor has no resistance, (like ideal capacitor has no resistance) and is excited by ideal voltage sources (that has zero output impedance), in the RC charging circuits. We will use Laplace Transform technique in all our analysis. In all the cases in subsequent sections, we will apply this new formula $q(t) = c(t) * v(t)$ and give the validity justification. Recently this formula $q(t) = c(t) * v(t)$ is getting experimentally validated [39], for super-capacitors.

Therefore charge in a capacitor is $q(t) = c(t) * v(t)$, is given via convolution operation and not with the usual way that we write as $q(t) = c(t)v(t)$. Let us have a capacitor with capacity function in time as power-law $c(t) = C_\alpha t^{-\alpha}$ $(0 < \alpha < 1)$, that is fractional capacitor, is charged via resistance R. Let a voltage $v_{in}(t)$ or current $i_{in}(t)$ be applied to an uncharged capacitor in the

RC circuit at time $t=0$. Then charge function in time is given as convolution $(*)$ operation as $q(t)=c(t)*v_0(t)$, with $v_0(t)$ is the voltage profile on the capacitor, in the RC circuit of Figure-1. This charge $q(t)$ is also $q(t)=\int_0^t i(\tau)d\tau$, where $i(t)$ is current flowing through the capacitor in the RC circuit. This comes from normal circuit theory application, and we will show that this $q(t)=c(t)*v_0(t)$ is same that we get from normal circuit theory. For each case we also study the ideal loss less capacitor given by capacity function as $c(t)=C\delta(t)$, [1], [40] and apply $q(t)=c(t)*v_0(t)$.

We will validate and verify this new formula $q(t)=c(t)*v(t)$ in circuit theory with RC circuit, in this paper. The aim of the paper is not to show profiles of circuit voltage current or charge, with variation of α; but rather validate the new formula $q(t)=c(t)*v_0(t)$, with that of solution obtained by circuit theory techniques. Thus we are not drawing MATLAB simulated figures for voltage current and charge functions. We will also validate self-discharge mechanism of fractional capacitor (super-capacitor) exhibiting memory effect, by using this new formula $q(t)=c(t)*v(t)$.

2. Charge storage $q(t)$ by step input voltage $v_{in}(t) = V_m u(t)$ excitation to RC circuit with ideal loss less capacitor

In classical circuit theory, if we charge an ideal capacitor, C (initially uncharged) through a resistor R, via a step input voltage $v_{in}(t) = V_m u(t)$ (Figure-1) we get voltage across capacitor as exponential rise as $v_0(t)=V_m(1-e^{-t/RC})$; $t\ge 0$. In Figure-1 consider $Z_1(s)=R$, and $Z_2(s)$ is ideal capacitor with capacity function as $c(t)=C\delta(t)$, [1], [40]. Therefore we have following impedance function

$$Z_2(s)=\frac{1}{s\mathcal{L}\{c(t)\}}=\frac{1}{s\mathcal{L}\{C\delta(t)\}}=\frac{1}{sC} \tag{1}$$

The above Eq. (1) is new way of writing $Z(s)$ for capacitor ideal or fractional we got from application of formula $q(t)=c(t)*v(t)$ in our earlier discussion [40]. That we got by differentiating this convolution expression to get $i(t)$ and taking Laplace transform to arrive at Eq. (1), i.e. $Z(s)=V(s)/I(s)=\left(s\mathcal{L}\{c(t)\}\right)^{-1}$.

We have from circuit theory and from Figure-1 the following expressions

$$\begin{aligned} V_0(s) &= \frac{Z_2(s)}{Z_1(s)+Z_2(s)}\mathcal{L}\{v_{in}(t)\}, \quad v_{in}(t)=V_m u(t), \quad \mathcal{L}\{v_{in}(t)\}=\frac{V_m}{s} \\ &= \frac{V_m}{RCs\left(s+\frac{1}{RC}\right)} = V_m\left(\frac{1}{s}-\frac{1}{s+\frac{1}{RC}}\right) \end{aligned} \tag{2}$$

The inverse Laplace Transform of Eq. (2) gives following voltage charging equation for capacitor

$$v_0(t)=V_m(1-e^{-t/RC}); \quad t\ge 0 \tag{3}$$

We have $\lim_{t\uparrow\infty} v_0(t)=V_m$, the current flowing in the RC circuit at $t\ge 0$ is the following

$$i(t)=\mathcal{L}^{-1}\left\{\frac{V_m/s}{R+\frac{1}{Cs}}\right\}=\mathcal{L}^{-1}\left\{\frac{V_m}{R}\left(\frac{1}{s+\frac{1}{RC}}\right)\right\}=\frac{V_m}{R}e^{-t/RC} \tag{4}$$

Therefore the charge function $q(t)$ is

$$q(t) = \int_0^t i(\tau)d\tau = \int_0^t \frac{V_m}{R} e^{-\tau/RC} \\ = V_m C(1 - e^{-t/RC}); \quad t \ge 0 \tag{5}$$

We apply the formula $q(t) = c(t) * v(t)$ to ideal capacitor given by $c(t) = C\delta(t)$ across which we are having a voltage profile as $v_0(t) = V_m(1 - e^{-t/RC})$, to write following

$$Q(s) = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right) \\ = \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\{V_m(1 - e^{-t/RC})\}\right) = C\left(\frac{V_m}{s} - \frac{V_m}{\left(s + \frac{1}{RC}\right)}\right) \tag{6}$$

The inverse Laplace transform of Eq. (6) above gives

$$q(t) = V_m C(1 - e^{-t/RC}); \quad t \ge 0 \tag{7}$$

Eq. (7) is same as Eq. (5) that we got via circuit theory applying $q(t) = \int_0^t i(\tau)d\tau$. This gives validation of formula $q(t) = c(t) * v(t)$ for classical ideal loss less capacitor case.

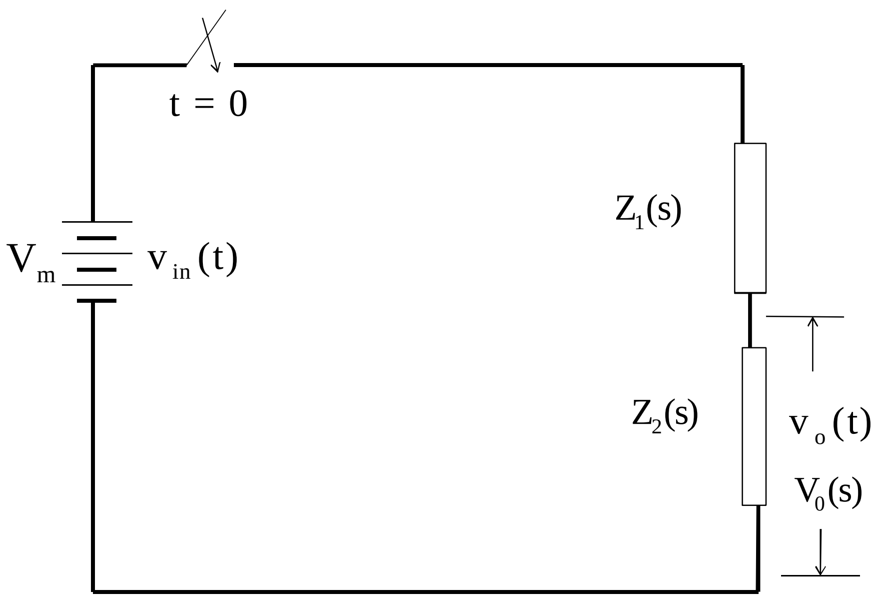

Figure- 1: The constant voltage charging RC circuit

3. Charge storage $q(t)$ by step input voltage $v_{in}(t) = V_m u(t)$ excitation to RC circuit with fractional capacitor

In Figure-1 consider $Z_1(s) = R$, and $Z_2(s)$ is fractional capacitor with capacity function as $c(t) = C_\alpha t^{-\alpha}$; with $0 < \alpha < 1$. Therefore we have following impedance function [40]

$$Z_2(s)=\frac{1}{s\mathcal{L}\{c(t)\}}=\frac{1}{s\mathcal{L}\{C_\alpha t^{-\alpha}\}}=\frac{1}{s\left(C_\alpha\Gamma(1-\alpha)s^{\alpha-1}\right)}$$
$$=\frac{1}{s^\alpha C_\alpha\Gamma(1-\alpha)}=\frac{1}{s^\alpha C_{F-\alpha}};\quad C_{F-\alpha}=C_\alpha\Gamma(1-\alpha) \tag{8}$$

Here we will use a constant voltage excitation of V_m from time $t=0$, to time $t=T_c$ (as charging phase, through a known resistor R) and thereafter we will switch to discharging phase i.e. voltage source will be made zero (Figure-3). By this we record the charging and discharging profile $v_0(t)$, and then apply $q(t)=c(t)*v_0(t)$ to get charge, and then current. From the circuit diagram of Figure-1, we write the following [37]

$$V_0(s)=\frac{Z_2(s)}{Z_1(s)+Z_2(s)}\mathcal{L}\{v_{in}(t)\},\quad v_{in}(t)=V_m u(t),\qquad \mathcal{L}\{v_{in}(t)\}=\frac{V_m}{s}$$
$$=\frac{V_m}{RC_{F-\alpha}s\left(s^\alpha+\frac{1}{RC_{F-\alpha}}\right)}=\frac{V_m ks^{-1}}{(s^\alpha+k)};\quad k=\frac{1}{RC_{F-\alpha}} \tag{9}$$

Now use $\mathcal{L}\left\{t^{\alpha p+\beta-1}E^{(p)}_{\alpha,\beta}(at^\alpha)\right\}=\frac{p!s^{\alpha-\beta}}{s^\alpha-a}$ [10], [12], [13] to get $\mathcal{L}^{-1}\left\{\frac{s^{-1}}{s^\alpha-a}\right\}=t^\alpha E_{\alpha,\alpha+1}(at^\alpha)$, by putting $p=0$, $\alpha=\alpha$, $\beta=\alpha+1$, where the $E_{\alpha,\beta}(at^\alpha)$ is two parameter Mittag-Leffler function (Refer Appendix); as defined in infinite series in following expression

$$E_{\alpha,\beta}(x)=\sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)},\qquad E_{\alpha,(\alpha+1)}(-kt^\alpha)=\sum_{m=0}^{\infty}\frac{(-kt^\alpha)^m}{\Gamma(m\alpha+\alpha+1)}$$
$$E_{\alpha,1}(x)=E_\alpha(x)=\sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+1)} \tag{10}$$

With this we obtain the following from Laplace inverse of Eq. (9)

$$v_0(t)=\mathcal{L}^{-1}\left\{\frac{V_m k}{s(s^\alpha+k)}\right\}=V_m kt^\alpha E_{\alpha,\alpha+1}(-kt^\alpha)$$
$$=\frac{V_m}{RC_{F-\alpha}}t^\alpha E_{\alpha,\alpha+1}\left(-\frac{t^\alpha}{RC_{F-\alpha}}\right) \tag{11}$$

We have alternate derivation via series expansion [13], [37] as follows

$$V_0(s)=\frac{V_m k}{s(s^\alpha+k)}=\frac{V_m k}{s^{\alpha+1}}\left(1+\frac{k}{s^\alpha}\right)^{-1};\quad (1+x)^{-1}=1-x+x^2-x^3+\ldots$$
$$=\frac{V_m k}{s^{\alpha+1}}\left(1-\frac{k}{s^\alpha}+\frac{k^2}{s^{2\alpha}}-\frac{k^3}{s^{3\alpha}}+\ldots\right) \tag{12}$$
$$=V_m\left(\frac{k}{s^{\alpha+1}}-\frac{k^2}{s^{2\alpha+1}}+\frac{k^3}{s^{3\alpha+1}}-..\right)$$

Use Laplace pair $\frac{\Gamma(n+1)}{s^{n+1}}=\mathcal{L}\left\{t^n\right\}$ to invert term by term the above Eq. (12) to get following

$$v_0(t) = V_m\left(\frac{kt^{\alpha}}{\Gamma(\alpha+1)} - \frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)} - \dots\right)$$

$$= V_m\left(1 - \left[1 - \frac{kt^{\alpha}}{\Gamma(\alpha+1)} + \frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots\right]\right) \tag{13}$$

$$= V_m\left(1 - \sum_{n=0}^{\infty}\frac{(-kt^{\alpha})^n}{\Gamma(n\alpha+1)}\right) = V_m\left[1 - E_{\alpha}(-kt^{\alpha})\right] = V_m\left[1 - E_{\alpha}\left(-\tfrac{t^{\alpha}}{RC_{F\text{-}\alpha}}\right)\right]$$

Where, $E_{\alpha}(x)$ is one parameter Mittag-Leffler function (Refer Appendix) used in Eq. (13), with $E_1(x) = e^x$. Therefore for classical ideal capacitor with limit $\alpha \to 1$, we have normal exponential charging $v_0(t) = V_m(1 - e^{-t/RC})$; writing $C_{F-\alpha}\big|_{\alpha\to1} \equiv C$.

For voltage charging expression for fractional order impedance $Z_2(s) = s^{-\alpha}\, C_{F-\alpha}^{-1}$, Eq. (8) we have from Eq. (11) and Eq. (13) the following

$$v_0(t) = V_m\left(1 - E_{\alpha}\left(-\tfrac{t^{\alpha}}{RC_{F\text{-}\alpha}}\right)\right) = \frac{V_m}{RC_{F\text{-}\alpha}}\, t^{\alpha}E_{\alpha,\alpha+1}\left(-\tfrac{t^{\alpha}}{RC_{F\text{-}\alpha}}\right) \tag{14}$$

We have $\lim_{t\uparrow\infty} v_0(t) = V_m$. For charging current of circuit of Figure-1 with $Z_1 = R$ and $Z_2(s) = \frac{1}{s^{\alpha}C_{F\text{-}\alpha}}$, we have $Z(s) = Z_1(s) + Z_2(s)$ and write the following

$$I(s) = \frac{1}{Z(s)}\left(\frac{V_m}{s}\right) = \frac{V_m}{s\left(R + \frac{1}{s^{\alpha}C_{F\text{-}\alpha}}\right)} = \frac{V_m}{R}\left(\frac{s^{\alpha-1}}{s^{\alpha} + \frac{1}{RC_{F\text{-}\alpha}}}\right) \tag{15}$$

Using $\mathcal{L}\left\{E_n(at^n)\right\} = \frac{s^{n-1}}{s^n - a}$, [10], [12], [13] we get inverse Laplace transform of above Eq. (15) as

$$i(t) = \frac{V_m}{R}E_{\alpha}\left(-\tfrac{t^{\alpha}}{RC_{F\text{-}\alpha}}\right) \tag{16}$$

Clearly for ideal case i.e. in limit $\alpha \to 1$ case we get $i(t) = \frac{V_m}{R}e^{-t/RC}$. Therefore the charge $q(t)$ is from Eq. (16) the following

$$q(t) = \int_0^t i(\tau)d\tau = \int_0^t \frac{V_m}{R}E_{\alpha}\left(-\tfrac{\tau^{\alpha}}{RC_{F\text{-}\alpha}}\right)d\tau \tag{17}$$

We apply the formula $q(t) = c(t) * v(t)$ to fractional capacitor given by $c(t) = C_{\alpha}\, t^{-\alpha}$ across which we are having a voltage profile as $v_0(t) = V_m\left(1 - E_{\alpha}\left(-\tfrac{t^{\alpha}}{RC_{F\text{-}\alpha}}\right)\right)$, to write following steps

$$Q(s) = \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right)$$

$$= \left(\mathcal{L}\{C_\alpha t^{-\alpha}\}\right)\left(\mathcal{L}\left\{V_m\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right)\right\}\right)$$

$$= \left(C_\alpha \Gamma(1-\alpha) s^{\alpha-1}\right)\left(\tfrac{V_m k}{s(s^\alpha + k)}\right) = \frac{V_m C_{F\text{-}\alpha}\left(\frac{1}{RC_{F\text{-}\alpha}}\right)}{s^{2\text{-}\alpha}\left(s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}\right)}; \quad k = \tfrac{1}{RC_{F\text{-}\alpha}}, \quad \tfrac{C_{F\text{-}\alpha}}{\Gamma(1-\alpha)} = C_\alpha \tag{18}$$

$$= \left(\frac{V_m}{R}\right)\frac{s^{\alpha-2}}{\left(s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}\right)} \qquad \mathcal{L}\{E_\alpha(-kt^\alpha)\} = \frac{s^{\alpha\text{-}1}}{s^\alpha + k}$$

$$= \left(\frac{V_m}{R}\right)\left(s^{-1}\left(\frac{s^{\alpha-1}}{\left(s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}\right)}\right)\right) = \left(\frac{V_m}{R}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right\}\right)$$

Taking inverse Laplace transform of Eq. (18) by recognizing $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = s^{-1}F(s)$ we write

$$q(t) = \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau \tag{19}$$

The same result as in Eq. (17) we got by using $q(t) = \int_0^t i(\tau)d\tau$ validates the verification of formula $q(t) = c(t) * v(t)$. Put $\alpha = 1$ in Eq. (19) and we get ideal loss-less capacitor with $C_{F\text{-}\alpha} \equiv C$, and $E_1(x) = e^x$ to write the following case

$$\begin{aligned} q(t) &= \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau\Big|_{\alpha=1} = \int_0^t \frac{V_m}{R} e^{\text{-}\tau/RC} d\tau \\ &= V_m C\left(1 - e^{\text{-}t/RC}\right) \end{aligned} \tag{20}$$

The above Eq. (20) is charge build up relation for ideal-loss less capacitor, same as Eq. (5) and Eq. (7).

We take the integration of Mittag-Leffler function as $\int_0^t E_\alpha\left(\text{-}k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ with $E_{\alpha,\beta}(x) = \sum_{m=0}^\infty \frac{(x)^m}{\Gamma(\alpha m+\beta)}$ (Refer Appendix for proof). So we have charge build up function on a fractional capacitor in RC charging circuit as follows

$$\begin{aligned} q(t) &= \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau \\ &= \frac{V_m t}{R}\left(E_{\alpha,2}(-t^\alpha / RC_{F\text{-}\alpha})\right); \quad t \ge 0 \end{aligned} \tag{21}$$

Let us verify this for $\alpha = 1$ from Eq. (21) where we get $q(t) = \frac{V_m t}{R}\left(E_{\alpha,2}(-t^\alpha / RC_{F\text{-}\alpha})\right)\Big|_{\alpha=1; C_{F\text{-}\alpha}=C}$, We use $E_{\alpha,2}(-a\,x^\alpha) = \sum_{m=0}^\infty \frac{(-1)^m a^m x^{\alpha m}}{\Gamma(\alpha m+2)}$ and get $q(t) = V_m C\left(1 - e^{-t/RC}\right)$, by simple algebraic manipulations and tricks. Thus we have verified the validity of formula $q(t) = c(t) * v(t)$ in RC charging circuit with fractional capacitor.

4. Charge holding at large times for fractional capacitor

We have from Eq. (21) at $t = T_c$ the charge stored is $q(T_c) = \left(\frac{V_m}{R}\right) T_c \left(E_{\alpha,2}(-T_c^{\alpha} / RC_{F\text{-}\alpha})\right)$. Now we see if we keep the unit step voltage $v_{in}(t) = V_m u(t)$ for large time say $T_c \uparrow \infty$ for a fractional capacitor, what is $\lim_{T_c \uparrow \infty} q(T_c) = \lim_{T_c \uparrow \infty} \left(\frac{V_m}{R}\right) T_c \left(E_{\alpha,2}(-T_c^{\alpha} / RC_{F\text{-}\alpha})\right)$, that we analyze. Whereas for classical ideal capacitor $\lim_{T_c \uparrow \infty} q(T_c) = \lim_{T_c \uparrow \infty} CV_m(1 - e^{-T_c/RC}) = V_m C$, is a constant independent of $t = T_c$.

This we study from recurring property of $E_{\alpha,\beta}(x)$ which is $E_{\alpha,\beta}(x) = \frac{1}{x} E_{\alpha,\beta-\alpha}(x) - \frac{1}{x\Gamma(\beta-\alpha)}$ from which Poincare asymptotic expansion is $E_{\alpha,\beta}(x) \sim -\sum_{n=1}^{\infty} \frac{1}{x^n \Gamma(\beta - n\alpha)}$ valid for $x \to -\infty$ (Refer Appendix). In the expression asymptotic expansion of $E_{\alpha,2}(-T_c^{\alpha} / RC_{F\text{-}\alpha})$ taking $x = -kT_c^{\alpha}$, where $k = \frac{1}{RC_{F\text{-}\alpha}}$ we write for $T_c \uparrow \infty$ as following

$$\begin{aligned} \lim_{T_c \uparrow \infty} E_{\alpha,2}(-k T_c^{\alpha}) &= \frac{T_c^{-\alpha}}{k\Gamma(2-\alpha)} - \frac{T_c^{-2\alpha}}{k^2\Gamma(2-2\alpha)} - \frac{T_c^{-3\alpha}}{k^3\Gamma(2-3\alpha)} - \dots \\ &\sim \frac{T_c^{-\alpha}}{k\Gamma(2-\alpha)} \end{aligned} \tag{22}$$

We approximate above infinite series Eq. (22) by neglecting higher powers exponents of power law, as the higher terms will be decaying much faster than the first term. Therefore we write the following

$$\begin{aligned} \lim_{T_c \uparrow \infty} q(T_c) &= \lim_{T_c \uparrow \infty} \left(\frac{V_m}{R}\right) T_c \left(E_{\alpha,2}(-T_c^{\alpha} / RC_{F\text{-}\alpha})\right); \quad 0 < \alpha < 1 \\ &\sim \frac{V_m}{R} T_c \left(\frac{T_c^{-\alpha}}{k\Gamma(2-\alpha)}\right) = \frac{V_m C_{F\text{-}\alpha}}{\Gamma(2-\alpha)} T_c^{1-\alpha}; \quad \Gamma(m+1) = m\Gamma(m) \\ &= \frac{V_m C_{F\text{-}\alpha}}{(1-\alpha)\Gamma(1-\alpha)} T_c^{1-\alpha} = \infty \end{aligned} \tag{23}$$

In [1] we got $q(t) = \frac{C_\alpha V_m t^{1-\alpha}}{1-\alpha}$ for a fractional capacitor with capacity function $c(t) = C_\alpha t^{-\alpha}$ as a charge build up formula for a fractional capacitor. In [1] we showed $\lim_{t \uparrow \infty} q(t) = \infty$ by use of formula $q(t) = c(t) * v(t)$ for an uncharged fractional capacitor, charged directly from ideal voltage source (i.e. in RC of circuit Figure-1 with $R = 0\Omega$). Here in RC circuit case we see that steady state of charge holding will be never obtained (as we get for an ideal loss less capacitor). For the fractional capacitor case, the charge will keep growing to infinity, leading to electro-static break down of capacitors [1], [6], [7]. Using $C_{F\text{-}\alpha} = C_\alpha \Gamma(1-\alpha)$ in the derived formula for large times in RC charging in asymptotic approximation is $q(t) \sim \frac{V_m C_{F\text{-}\alpha}}{(1-\alpha)\Gamma(1-\alpha)} t^{1-\alpha} = \frac{V_m C_\alpha}{(1-\alpha)} t^{1-\alpha}$ that is same that we got in [1]. Here if we put $\alpha \to 1$, we have classical ideal capacitor $C_{F\text{-}\alpha} = C_\alpha \Gamma(1-\alpha) \equiv C$ and thus $q(t) = V_m C$ for any $t \geq 0$; that is true for classical ideal capacitor case.

In case of classical capacitors, we have $q(t) = V_m C\left(1 - e^{-t/RC}\right)$ and here we get steady-state at $\lim_{t\uparrow\infty} q(t) = V_m C$. This is fundamental to memory effect as observed in a fractional capacitor case [40]. There is no memory effect in the classical capacitor cases the charge store is steady constant $q(t) = CV_m$ for any holding time for $v_{in}(t) = V_m u(t)$. While the charge storage in a fractional capacitor depends on holding time for step voltage, more the holding time more the charge stored in fractional capacitor [1], [40].

5. Self-Discharging a fractional capacitor after holding a step input voltage for a long time--The memory effect, explained by the formula q = c * v

A fractional capacitor is charged from time $t = -T_c$ to time t with a constant step input $v_{in}(t) = V_m u(t - (-T_c))$. That is step voltage us applied at time $t = -T_c$. The charging current is from general charge equation by following convolution expression $q_{CH}(t) = \left(c(t) * v(t)\right) = \int_{-\infty}^{t} c(t - x)v(x)dx$; [1]. For a fractional capacitor with capacity function $c(t) = C_\alpha t^{-\alpha}$ we write the convolution expression with lower limit of integration as $-T_c$ that is the time where the voltage change is applied, [1].

$$q_{CH}(t) = \left(c(t) * v(t)\right)\Big|_{-T_c}^{t} = \int_{-T_c}^{t} C_\alpha (t - x)^{-\alpha} v(x)dx \qquad (24)$$

Where $v(t)$ we say voltage across the capacitor assumed to be at V_m in $t = -T_c$, and $v(t) = 0$, for $t < -T_c$. This assumption is valid when we say $t >> -T_c$, that is neglecting the rise part of the charging equation $v(t + T_c) = V_m\left(1 - E_\alpha\left(-\frac{(t+T_c)^\alpha}{RC_{F-\alpha}}\right)\right)$ is $v(t + T_c) \cong V_m$ for $t \gg T_c$. The charging current is following

$$\begin{aligned} i_{CH}(t) &= \frac{dq_{CH}(t)}{dt} = \frac{d}{dt}\left(c(t) * v(t)\right)\Big|_{-T_c}^{t}, \quad c(t) = C_\alpha t^{-\alpha} \\ &= \frac{d}{dt}\int_{x=-T_c}^{x=t} C_\alpha (t - x)^{-\alpha} v(x)dx = C_\alpha \frac{d}{dt}\int_{-T_c}^{t} \frac{v(x)dx}{(t - x)^\alpha} \end{aligned} \qquad (25)$$

The integration by parts for term $\int_{-T_c}^{t} (t - x)^{-\alpha} v(x)dx$ in Eq. (25) gives following result

$$\begin{aligned} \int_{-T_c}^{t} \frac{v(x)dx}{(t - x)^\alpha} &= \left[v(x)\int \frac{dx}{(t - x)^\alpha}\right]_{x=-T_c}^{x=t} - \int_{-T_c}^{t}\left(v^{(1)}(x)\int \frac{dx}{(t - x)^\alpha}\right)dx \\ &= v(x)\left(-\frac{(t - x)^{1-\alpha}}{1 - \alpha}\right)\Bigg|_{x=-T_c}^{x=t} - \int_{-T_c}^{t} v^{(1)}(x)\left(\frac{(-1)(t - x)^{1-\alpha}}{1 - \alpha}\right)dx \\ &= \frac{v(-T_c)}{1 - \alpha}(t + T_c)^{1-\alpha} + \int_{-T_c}^{t} \frac{v^{(1)}(x)}{1 - \alpha}(t - x)^{1-\alpha} dx \end{aligned} \qquad (26)$$

Using the derivation of Eq. (26) and using the definition of fractional derivative for $0 < \alpha < 1$ is Riemann –Liouville (RL) ${}_aD_t^\alpha$ and Caputo ${}_a^CD_t^\alpha$ (Refer Appendix) we write the following steps

$$
\begin{aligned}
i_{CH}(t) &= C_\alpha \frac{d}{dt}\int_{-T_c}^{t}\frac{v(x)dx}{(t-x)^\alpha} = C_\alpha \frac{d}{dt}\left(\frac{v(-T_c)}{1-\alpha}(t+T_c)^{1-\alpha} + \int_{-T_c}^{t}\frac{v^{(1)}(x)}{1-\alpha}(t-x)^{1-\alpha}dx\right) \\
&= C_\alpha v(-T_c)\frac{d}{dt}\left(\frac{(t+T_c)^{1-\alpha}}{1-\alpha}\right) + C_\alpha\int_{-T_c}^{t}\left(\frac{v^{(1)}(x)}{1-\alpha}\right)\frac{d}{dt}(t-x)^{1-\alpha}dx \\
&= C_\alpha\frac{v(-T_c)}{(t+T_c)^\alpha} + C_\alpha\int_{-T_c}^{t}\frac{v^{(1)}(x)}{(t-x)^\alpha}dx \\
&= C_\alpha\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\left(\frac{v(-T_c)}{(t+T_c)^\alpha} + \int_{-T_c}^{t}\frac{v^{(1)}(x)dx}{(t-x)^\alpha}\right)\right), \quad C_\alpha\left(\Gamma(1-\alpha)\right) = C_{F\text{-}\alpha} \\
&= \frac{C_{F\text{-}\alpha}}{\Gamma(1-\alpha)}\left(\frac{v(-T_c)}{(t+T_c)^\alpha}\right) + C_{F\text{-}\alpha}\left({}_{-T_c}^{\ C}D_t^\alpha\left[v(t)\right]\right) \\
&= C_{F\text{-}\alpha}\left({}_{-T_c}D_t^\alpha\left[v(t)\right]\right), \qquad 0<\alpha<1
\end{aligned}
\tag{27}
$$

We set $v(-T_c)\approx V_m$ for $t \gg T_c$ and $v^{(1)}(t)=0$ for a constant voltage $v(t)=V_m$ for $t \gg T_c$ and write

$$
\begin{aligned}
i_{CH}(t) &= C_\alpha\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\left(\frac{v(-T_c)}{(t+T_c)^\alpha} + \int_{-T_c}^{t}\frac{v^{(1)}(x)dx}{(t-x)^\alpha}\right)\right) \\
&= \frac{C_\alpha V_m}{(t+T_c)^\alpha}
\end{aligned}
\tag{28}
$$

The above $i_{CH}(t)$ in Eq. (28) is Curie-Von Schewdler relaxation current power law for dielectric relaxation when the dielectric is stressed by a constant voltage at time (in this case) $t \approx -T_c$. This we get by other method too as depicted below

$$
\begin{aligned}
i_{CH}(t) &= C_{F-\alpha}\left({}_{-T_c}D_t^\alpha v(t)\right); \quad 0<\alpha<1 \\
&= C_{F\text{-}\alpha}\left.\frac{d^\alpha V_m}{dt^\alpha}\right|_{t=-T_c}^{t=t}, \quad C_{F\text{-}\alpha} = C_\alpha\Gamma(1-\alpha) \\
&= C_\alpha\Gamma(1-\alpha)\left.\frac{d^\alpha V_m}{dt^\alpha}\right|_{-T_c}^{t} = C_\alpha\Gamma(1-\alpha)\left(\frac{V_m}{\Gamma(1-\alpha)}(t-(-T_c))^{-\alpha}\right) \\
&= C_\alpha\frac{V_m}{(t+T_c)^\alpha} \qquad 0<\alpha<1 \qquad (t+T_c)>0
\end{aligned}
\tag{29}
$$

In above steps of Eq. (29) we used formula for RL fractional derivative of a constant K as ${}_aD_x^\alpha K = K\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$, with $a=-T_c$ that is start point of fractional differentiation process, and $x=t$, and $K=V_m$ (Refer Appendix). We note that ${}_a^CD_x^\alpha K = 0$, that appears in Eq. (28).

At $t=0$ the voltage source $v_{in}(t)=V_m u(t)$ is disconnected, or we keep the fractional capacitor at open-circuited condition, after keeping this for a long-long time from $t=-T_c$ .Thus there will be a self discharging of the charged fractional capacitor, and the self discharge current will be proportional to decaying open circuited voltage $v_{oc}(t)$, given as follows from time $t=0$ the time the fractional capacitor was kept open circuited, to time $t \ge 0$. The self-discharging current we write as follows $i_{DIS}(t) = C_{F-\alpha}\left({}_0D_t^{\alpha} v_{oc}(t)\right)$, that is

$$i_{DIS}(t) = C_{F-\alpha} \left. \frac{d^{\alpha} v_{oc}(t)}{dt^{\alpha}} \right|_{t=0}^{t=t} = C_{\alpha}\Gamma(1-\alpha) \left. \frac{d^{\alpha} v_{oc}(t)}{dt^{\alpha}} \right|_{t=0}^{t=t} \tag{30}$$

We will see in subsequent section that $i_{DIS}(t)$ of Eq. (30) is not the conventional current of discharge that flows out to a shunt resistance put for discharging the stored charge, but gives a notion due to special re-distribution of charges inside a spatially distributed system infinite RC circuit-we call it notional discharge current (we will discuss later).

The coulomb of charge $q_{CH}(t)$ pumped into the capacitor plus self-discharged coulombs of charge say $q_{DIS}(t)$ is zero that is $q_{CH}(t) = -q_{DIS}(t)$. Differentiating this we get $i_{CH}(t) + i_{DIS}(t) = 0$ which gives the following. We write the following

$$\begin{aligned} & i_{CH}(t) + i_{DIS}(t) = 0 \\ & C_{F-\alpha}\left({}_{-T_c}D_t^{\alpha} v(t)\right) + C_{F-\alpha}\left({}_0D_t^{\alpha} v_{oc}(t)\right) = 0 \end{aligned} \tag{31}$$

That is the following we get using Eq. (28) or Eq. (29)

$$C_{\alpha}\frac{V_m}{(t+T_c)^{\alpha}} + C_{\alpha}\Gamma(1-\alpha)\frac{d^{\alpha} v_{oc}(t)}{dt^{\alpha}} = 0 \tag{32}$$

Our interest is finding $v_{oc}(t)$, from $t \ge 0$ that is in self-discharge phase. We do the fractional integration ${}_0I_t^{\alpha}$ (from time 0 to time t) of the above Eq. (32) and write the following

$${}_0I_t^{\alpha}\left[C_{\alpha}\frac{V_m}{(t+T_c)^{\alpha}}\right] + C_{\alpha}\Gamma(1-\alpha)\left({}_0I_t^{\alpha}\left[\frac{d^{\alpha} v_{oc}(t)}{dt^{\alpha}}\right]\right) = 0 \tag{33}$$

We write $\left({}_0I_t^{\alpha}\left[{}_0D_t^{\alpha} v_{oc}(t)\right]\right) = v_{oc}(t)\big|_0^t = v_{oc}(t) - v_{oc}(0)$ with $v_{oc}(0) = V_m$ for the second term and write the following

$${}_0I_t^{\alpha}\left[C_{\alpha}\frac{V_m}{(t+T_c)^{\alpha}}\right] + C_{\alpha}\Gamma(1-\alpha)\left[v_{oc}(t) - V_m\right] = 0 \tag{34}$$

To the first term we apply Riemann formula of Fractional Integration (Refer Appendix) that is ${}_0I_t^{\alpha}\left[f(t)\right] = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(x)dx}{(t-x)^{1-\alpha}}$ and get

$$C_{\alpha}V_m\frac{1}{\Gamma(\alpha)}\int_0^t \frac{dx}{(T_c+x)^{\alpha}(t-x)^{1-\alpha}} + \left[C_{\alpha}\Gamma(1-\alpha)v_{oc}(t) - C_{\alpha}\Gamma(1-\alpha)V_m\right] = 0 \tag{35}$$

Rearranging the Eq. (35) we write

$$v_{oc}(t) = V_m - \frac{V_m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t \frac{dx}{(T_c+x)^{\alpha}(t-x)^{1-\alpha}} \tag{36}$$

In Eq. (36) put $T_c + x = \tau$, $dx = d\tau$, therefore for $x=0$, $\tau = T_c$ and $x = t$, $\tau = T_c + t$ we have

$$v_{oc}(t)=V_m-\frac{V_m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{T_c}^{T_c+t}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}=V_m-\frac{V_m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{T_c}^{T_c+t}F(\tau)d\tau$$
$$F(\tau)=\frac{1}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}} \qquad (37)$$

Now we break $\int_{T_c}^{T_c+t}F(\tau)d\tau$ as $\int_{T_c}^{T_c+t}F(\tau)d\tau=\int_{T_c}^{0}F(\tau)d\tau+\int_{0}^{T_c+t}F(\tau)d\tau$ and call the second term as $I_N(t)$. We write $I_N(t)$ in terms of convolution of two functions with substitution $T_c+t=\overline{t}$

$$I_N(t)=\int_0^{T_c+t}F(\tau)d\tau=\int_0^{T_c+t}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}=\int_0^{\overline{t}}\frac{d\tau}{\tau^{\alpha}(\overline{t}-\tau)^{1-\alpha}}=\left(\frac{1}{t^{\alpha}}\right)*\left(\frac{1}{t^{1-\alpha}}\right) \qquad (38)$$

Now we use Laplace pair $\mathcal{L}\left\{t^m\right\}=\frac{\Gamma(m+1)}{s^{m+1}}$ to write $\mathcal{L}\left\{I_N(t)\right\}=I_N(s)=\left(\mathcal{L}\left\{t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{t^{-(1-\alpha)}\right\}\right)$

$$I_N(s)=\left(\frac{\Gamma(-\alpha+1)}{s^{-\alpha+1}}\right)\left(\frac{\Gamma\left(-(1-\alpha)+1\right)}{s^{(-(1-\alpha)+1)}}\right)=\frac{\Gamma(1-\alpha)\Gamma(\alpha)}{s} \qquad (39)$$

Recognizing $\mathcal{L}\left\{u(t)\right\}=s^{-1}$, we write $\mathcal{L}\left\{I_N(s)\right\}=I_N(t)$, and write

$$I_N(t)=\begin{cases}\Gamma(1-\alpha)\Gamma(\alpha) & ;t\geq 0\\ 0 & ;t<0\end{cases} \qquad (40)$$

Therefore we have $\int_0^{T_c+t}F(\tau)d\tau=\int_0^{T_c+t}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}=\Gamma(1-\alpha)\Gamma(\alpha)$. Thus we write the expression for open circuit voltage $v_{oc}(t)$ for a charged fractional capacitor that is charged for a long time T_c to voltage V_m and at $t=0$ kept at self-discharge mode, we get

$$\begin{aligned}v_{oc}(t)&=V_m-\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\left[\int_0^{T_c+t}F(\tau)d\tau+\int_{T_c}^{0}F(\tau)d\tau\right]\\ &=V_m-\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(\Gamma(1-\alpha)\Gamma(\alpha)\right)-\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_{T_c}^{0}F(\tau)d\tau\\ &=\frac{-V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_{T_c}^{0}F(\tau)d\tau=\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}F(\tau)d\tau\\ &=\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}\end{aligned} \qquad (41)$$

In Eq. (41) $v_{oc}(t)$ is the voltage over open capacitor at self discharge mode (oc). This $v_{oc}(t)$ function of time depends on the total time T_c the capacitor has been on the voltage source of constant voltage V_m. More the T_c more $q_{CH}(T_c)$ and more time $v_{oc}(t)$ will take to self-discharge, from charged voltage V_m. This formula for self discharge voltage i.e. $v_{oc}(t)=\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$ is noted in [6]; here we derived the same by using the concept $q(t)=\left(c(t)*v(t)\right)$.

We mention here the formula for self discharge as described above is only valid for a constant voltage excitation or a step input case. For a triangular voltage impressed at $t=-T_c$ reaching voltage V_m at time T_{cm} described as $(V_m / T_{cm})(t + T_c)$ will be having different $v_{oc}(t)$ self-discharge profile.

The Figure-2 shows self discharge of a super-capacitor when charged with different times, showing memory effect. Here T_c is 4hr, 8hr and 16hr, charged to $V_m = 2.2$ (Courtesy: BRNS Funded joint Project CMET Thrissur-BARC Development of CAG Super-capacitors and application in electronics circuits); [41], [42]. The Figure-2 shows that self discharging curves $v_{oc}(t)$ for each T_c is different, indicating memory effect.

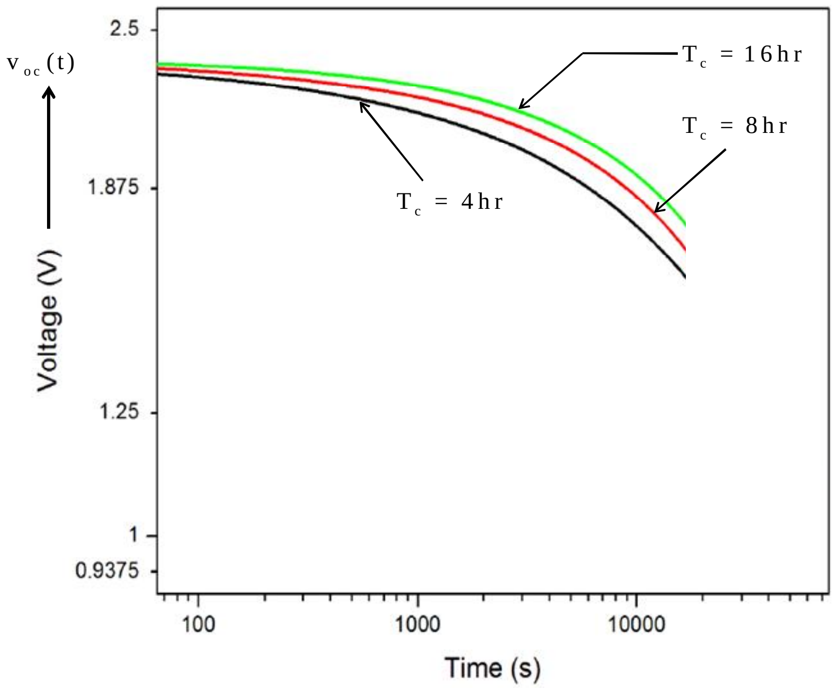

Figure-2: Self-discharge of fractional capacitor, more time we place fractional capacitor on a constant voltage more time it takes decay: Memorizing the charging history.

6. Self discharging of a classical ideal capacitor

We have a constant voltage source applied at $t = -T_c$ for a constant capacitor case with capacity function as $c(t) = C\delta(t)$, [1]. For this case we have the relation Eq. (42) i.e. $i_{CH}(t) = C\delta(t + T_c)\left(v(-T_c)\right) + C\left(v^{(1)}(t)\right)$; that we derive from formula $q_{CH}(t) = c(t) * v(t)$. Compare what we got for a fractional capacitor with $c(t) = C_\alpha t^{-\alpha}$ i.e. $i_{CH}(t) = C_\alpha \frac{v(-T_c)}{(t+T_c)^\alpha} + C_\alpha \int_{-T_c}^{t} \frac{v^{(1)}(x)}{(t-x)^\alpha} dx$, Eq. (28). We follow following steps

$$
\begin{aligned}
i_{CH}(t) &= \frac{dq_{CH}(t)}{dt} = \frac{d}{dt}\left(c(t)*v(t)\right)\Big|_{-T_c}^{t}, \quad c(t) = C\,\delta(t) \\
&= \frac{d}{dt}\int_{x=-T_c}^{x=t} C\delta(t-x)v(x)dx = \frac{d}{dt}\left(C\left(v(t)\right)\right), \qquad t \geq -T_c \\
&= v(t)\frac{dC}{dt}\Big|_{t\geq -T_c} + C\frac{dv(t)}{dt}\Big|_{t\geq -T_c} \\
&= \left(v(t)\right)\left(C\left(\delta(t+T_c)\right)\right) + C\frac{dv(t)}{dt} = C\left(v(-T_c)\delta(t+T_c)\right) + C\frac{dv(t)}{dt} \\
&= i(-T_c) + i(t), \qquad t \geq -T_c
\end{aligned}
\tag{42}
$$

The first term at RHS of above Eq. (42) i.e. $i(-T_c)$ indicate the value of current at $t = -T_c$. The constant function starting at $t = -T_c$ i.e. C when differentiated gives $C\delta(t+T_c)$. This unit delta functions at $t = -T_c$, i.e. $\delta(t+T_c)$ when multiplied by $v(t)$ gives $v(-T_c)\delta(t+T_c)$. This comes from property $\int\left(\delta(x_0-x)\right)\left(f(x)\right)dx = f(x_0)$, differentiation of this gives $\left(\delta(x_0-x)f(x)\right) = \frac{d}{dx}f(x_0) = f(x_0)\delta(x)$. Thus at $t = -T_c$ we have $i(-T_c) = Cv(-T_c)$ and $i(-T_c) = 0$ for $t > -T_c$. Compositely we write $i(-T_c) = C_1 v(-T_c)\left(\delta(t+T_c)\right)$, i.e. specifying its value at only $t = -T_c$. The second term is $i(t)$ for $t \neq -T_c$, that is $i(t) = C\left(v^{(1)}(t)\right)$.

The obtained expression $i_{CH}(t) = C\delta(t+T_c)\left(v(-T_c)\right) + C\left(v^{(1)}(t)\right)$ is by the formulation $q(t) = c(t)*v(t)$. As an example, we take $v(t) = V_m u(t+T_c)$ a step input at time $t = -T_c$, to an uncharged capacitor. We have $v^{(1)}(t) = 0$ for $t > -T_c$; and at $t = -T_c$ we have $v(-T_c) = V_m$. Using this we get $i(-T_c) = CV_m\left(\delta(t+T_c)\right)$; this makes $i_{CH}(t) = CV_m\left(\delta(t+T_c)\right), \quad t \geq -T_c$. At any time t the coulomb $q_{CH}(t)$ pumped charge into the capacitor plus self-discharged coulombs of charge say $q_{DIS}(t)$ is zero that is $q_{CH}(t) = -q_{DIS}(t)$. Differentiating this we get $i_{CH}(t) + i_{DIS}(t) = 0$ which gives $i_{CH}(t) + i_{DIS}(t) = 0$. That is the following

$$
CV_m\left(\delta(t+T_c)\right) + C\frac{dv_{oc}(t)}{dt} = 0 \tag{43}
$$

Our interest is finding $v_{oc}(t)$, from $t \geq 0$ self-discharge phase. We do the integration ${}_0I_t^1$ (from time 0 to time t) of the above Eq. (43) and write the following

$$
\int_0^t d\tau\left(CV_m\left(\delta(\tau+T_c)\right)\right) + \int_0^t d\tau C\frac{dv_{oc}(\tau)}{d\tau} = 0; \quad t \geq 0 \tag{44}
$$

The first integration term is zero since the delta function is outside of the region of integration, thus $C\int_0^t d\tau\left(V_m\left(\delta(\tau+T_c)\right)\right) = 0$.

For the second term in Eq. (44) we have $C\int_0^t v_{oc}^{(1)}(\tau)d\tau = C\left[v_{oc}(t)\right]_{t=0}^{t} = C\left(v_{oc}(t) - v(0)\right)$. The value $v(0) = V_m$ that is ideal capacitor is charged to full value of voltage. Using these results we have for ideal classical capacitor $v_{oc}(t) = V_m$, from Eq. (44).

This is very true observation. That an ideal loss less classical capacitor, once charged to V_m Volts would retain its charge that is finite and equilibrium value CV_m coulombs; and the terminal voltage $v_{oc}(t)$ will be held constant indefinitely. Now if a resistance is shunted across the charged capacitor, say R, this voltage $v_{oc}(t) = V_m$ will decay as $v_{DIS}(t) = \left(v_{oc}(t)\right)e^{-t/RC}$ or $v_{DIS}(t) = V_m e^{-t/RC}$, for $t \geq 0$ from the time the resistance was shunted.

Similarly for a case of fractional capacitor the self discharge voltage say $v_{oc}(t) = \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$, Eq. (41) will additionally discharge if the fractional capacitor is shunted by R, and we will record for a fractional capacitor $v_{DIS}(t)$ as following expression

$$v_{DIS}(t) = \left(v_{oc}(t)\right)E_{\alpha}(-t^{\alpha}/RC_{F-\alpha})$$
$$= \left(\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}\right)E_{\alpha}(-t^{\alpha}/RC_{F-\alpha}) \tag{45}$$

The term $E_{\alpha}(-t^{\alpha}/RC_{F-\alpha})$ is discharge decay function of Mittag-Leffler, for a fractional capacitor (that we will derive in subsequent section), is similar to decay function $e^{-t/RC}$ as for the case for a classical loss less capacitor.

7. Self-discharge is a misnomer

While we keep the charged fractional capacitor in ideal open circuit condition, (assume ideal infinite open circuit resistance or the ideal case this fractional capacitor having no leakage resistance), then we question why shall the terminal voltage $v_{oc}(t)$ once charged to V_m Volts, decay. We say in ideal case while shunt resistances are infinite there is no discharge current flowing out of fractional capacitor. Yet we observe decay as $v_{oc}(t) = \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$ for different T_c pumping various amounts of charge $q(T_c)$.

A fractional capacitor is like lossy semi-infinite transmission line-that is electrode structure being porous [8], [9], [41]. This infinite transmission line is composed of per unit series resistance r_u and shunt capacitance c_u, giving terminal relation of current and voltage as, [10]

$$i(t) = \sqrt{\frac{c_u}{r_u}}\frac{d^{\alpha}v(t)}{dt^{\alpha}}; \quad \alpha = \tfrac{1}{2}, \quad C_{F-\alpha} = \sqrt{\frac{c_u}{r_u}} \tag{46}$$

Therefore the fractional capacitor we say is spatially distributed system too, having infinite elements. When we connect a voltage source V_m to this semi infinite transmission line, though the first capacitor (say c_{u-1} gets charged to V_m, yet, the charging current keeps flowing to charge infinite number of $c_{u\text{-}2}$, $c_{u\text{-}3}\ldots$, $c_{u-\infty}$,(charges diffuse spatially). Therefore at time $T_c = \infty$, we have $q_{CH}(\infty) = \infty$, with all the voltages at each distributed capacitors of infinite numbers at V_m.

This system when kept in open ideal circuit condition will maintain $v_{oc}(t) = V_m$. But see the actual case, we have a limited $T_c < \infty$, but large enough that gives the terminal voltage, say to capacitor $c_{u\text{-}1}$ almost $\sim V_m$ with other capacitors $c_{u\text{-}2}$, $c_{u\text{-}3}$....., which are spatially farther away, with lesser terminal voltage as compared to the first capacitor $c_{u\text{-}1}$. While in ideal open circuited condition-this unequally charged semi-infinite transmission line, will have internal spatial charge distribution, to have voltage balancing to equal voltage to all the unit capacitors that are spatially distributed. This gives the notion as if $v_{oc}(t)$ is self-discharging or decaying, though there is no real discharge current flowing out of the fractional capacitor. Since this semi-infinite lossy transmission line has infinite elements, thus this process goes on infinitely for a long time, to have infinite capacitors have infinitesimal small charges and adding up to zero-and while the charge balancing is at play, at open circuited condition the current that flows in all the section will dissipate the stored electrostatic energy. Therefore, a fractional capacitor is a truly lossy capacitor, unlike an ideal loss-less capacitor which holds the stored charge and thus the open circuit voltage) indefinitely. This analysis is assuming that ideal capacitor or fractional capacitor doesn't to have any leakage resistance. Therefore, self-discharging term is misnomer; actually it is voltage redistribution taking place spatially-via diffusion process.

8. Charging/discharging a super-capacitor in RC circuit

8-a) Charging Phase

The differential equation corresponding to Figure-1 for $\alpha = 1$, is ordinary differential equation (ODE), with $Z_1(s) = R$ and $Z_2(s) = \frac{1}{sC}$ is following

$$RC\frac{dv_0(t)}{dt} + v_0(t) = v_{in}(t) \tag{47}$$

For $\alpha \neq 1$we get fractional differential equation (FDE), with $Z_1(s) = R$ and $Z_2(s) = \frac{1}{s^\alpha C_{F\text{-}\alpha}}$ is following

$$RC_{F\text{-}\alpha}\frac{d^\alpha v_0(t)}{dt^\alpha} + v_0(t) = v_{in}(t) \tag{48}$$

A super-capacitor is modeled as Equivalent Series Resistance (ESR) series with impedance of a Fractional Capacitor of order α [15]-[22]. We now consider a lumped ESR (R_s) for super-capacitor, thus for Figure-1 we have $Z_2(s) = R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}} = \frac{s^\alpha R_s C_{F\text{-}\alpha} + 1}{s^\alpha C_{F\text{-}\alpha}}$ while charging impedance remains at $Z_1(s) = R$. Therefore for any input voltage $V_{in}(s) = \mathcal{L}\{v_{in}(t)\}$, we write the charging current (in Laplace domain) as

$$I_{CH}(s) = \frac{V_{in}(s)}{R + R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}}} = \frac{s^\alpha C_{F\text{-}\alpha} V_{in}(s)}{s^\alpha C_{F\text{-}\alpha}(R + R_s) + 1} \tag{49}$$

Output voltage across $Z_2(s)$ in Laplace domain is therefore is

$$V_0(s)=\left(I_{CH}(s)\right)\left(Z_2(s)\right)=\left(\frac{V_{in}(s)s^{\alpha}C_{F\text{-}\alpha}}{s^{\alpha}C_{F\text{-}\alpha}(R+R_s)+1}\right)\left(\frac{s^{\alpha}R_sC_{F\text{-}\alpha}+1}{s^{\alpha}C_{F\text{-}\alpha}}\right)$$

$$=\frac{V_{in}(s)+V_{in}s^{\alpha}R_sC_{F\text{-}\alpha}}{s^{\alpha}C_{F\text{-}\alpha}(R+R_s)+1}=\frac{\frac{V_{in}(s)}{C_{F\text{-}\alpha}(R+R_s)}+\frac{V_{in}(s)s^{\alpha}R_s}{(R+R_s)}}{s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}}\qquad \text{put}\qquad V_{in}(s)=\frac{V_m}{s} \tag{50}$$

$$=\left(\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}\right)\left(\frac{1}{s\left(s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right)+\left(\frac{V_mR_s}{R+R_s}\right)\left(\frac{s^{\alpha\text{-}1}}{s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}}\right)$$

To get $v_0(t)$ we do inverse Laplace transform of Eq. (50) as following

$$v_0(t)=\mathcal{L}^{-1}\left\{V_0(s)\right\}=\mathcal{L}^{-1}\left\{\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)s\left(s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right\}+\mathcal{L}^{-1}\left\{\frac{V_mR_ss^{\alpha\text{-}1}}{(R+R_s)\left(s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right\} \tag{51}$$

Use formula $\mathcal{L}\left\{t^{\alpha p+\beta\text{-}1}E^{(p)}_{\alpha,\beta}(at^{\alpha})\right\}=p!\frac{s^{\alpha\text{-}\beta}}{s^{\alpha}\text{-}a}$. [10], [12], [13] with $p=1$, $\alpha=\alpha$, $\beta=\alpha+1$ and $p=0$, $\alpha=\alpha$, $\beta=1$, to write from Eq. (51) the inverse Laplace as

$$v_0(t)=\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}t^{\alpha}E_{\alpha,\alpha+1}\left(-\frac{t^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right)+\frac{V_RR_s}{R+R_s}E_{\alpha,1}\left(-\frac{t^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right) \tag{52}$$

Let us keep the step input from time $t=0$ to $t=T_c$, and then at time $t=T_c$, the output voltage is

$$v_0(T_c)=\frac{V_RT_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}E_{\alpha,\alpha+1}\left(-\frac{T_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right)+\frac{V_RR_s}{R+R_s}E_{\alpha,1}\left(-\frac{T_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right) \tag{53}$$

The charge $q(t)$ will be held only in the element $C_{F\text{-}\alpha}$. We calculate now the voltage profile $v_c(t)$ and then voltage at $t=T_c$, i.e. $v_c(T_c)$ for only fractional impedance part i.e. $\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}$ of the impedance $Z_2(s)$ comprising of R_s plus this fractional impedance $\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}$, the voltage is thus

$$V_c(s)=I_{CH}\left(\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}\right)=\left(\frac{s^{\alpha}C_{F\text{-}\alpha}V_{in}(s)}{s^{\alpha}C_{F\text{-}\alpha}(R+R_s)+1}\right)\left(\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}\right)\qquad \text{put}\qquad V_{in}(s)=\frac{V_m}{s}$$
$$=\left(\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}\right)\left(\frac{1}{s\left(s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right) \tag{54}$$

Using the Laplace identity of Mittag-Leffler function $\mathcal{L}\left\{E_n(at^n)\right\}=\frac{s^{n\text{-}1}}{s^n\text{-}a}$, [10], [12], [13] we write

$$v_c(t)=\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}t^{\alpha}E_{\alpha,\alpha+1}\left(-\frac{t^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right)$$
$$v_c(t)=V_m\left(1-E_{\alpha}\left(-\frac{t^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right),\quad 0\le t\le T_c \tag{55}$$

At $t=T_c$ we thus have the voltage at the fractional impedance

$$v_c(T_c)=\frac{V_mT_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}E_{\alpha,\alpha+1}\left(-\frac{T_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right)=V_m\left(1-E_{\alpha}\left(-\frac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \tag{56}$$

The charge $q(t)$ is $q(t)=c(t)*v_c(t)$ with fractional capacitor with capacity function as $c(t)=C_{\alpha}t^{-\alpha}$ having voltage profile and that is $v_c(t)=V_m\left(1-E_{\alpha}\left(-\frac{t^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$ as following

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right) \\
&= \left(\mathcal{L}\left\{C_\alpha\, t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{V_m\left(1-E_\alpha\left(-\tfrac{t^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)\right\}\right) \\
&= \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{V_m\left(\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}{s\left(s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}\right) = \frac{V_m C_{F\text{-}\alpha}\left(\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}{s^{2\text{-}\alpha}\left(s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}; \quad k=\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\ , \quad \frac{C_{F\text{-}\alpha}}{\Gamma(1-\alpha)}=C_\alpha \\
&= \left(\frac{V_m}{R+R_s}\right)\frac{s^{\alpha-2}}{\left(s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)} \qquad \mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\}=\frac{s^{\alpha\text{-}1}}{s^\alpha+k} \qquad (57) \\
&= \left(\frac{V_m}{R+R_s}\right)\left(s^{-1}\left(\frac{s^{\alpha-1}}{\left(s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}\right)\right) \\
&= \left(\frac{V_m}{R+R_s}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\tfrac{t^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right\}\right)
\end{aligned}
$$

Taking inverse Laplace transform of Eq. (57) by recognizing $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=s^{-1}F(s)$ we write

$$
q(t)=\int_0^t \frac{V_m}{R+R_s}E_\alpha\left(-\tfrac{\tau^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau=\frac{V_m t}{R+R_s}\left(E_{\alpha,2}(-t^\alpha/(R+R_s)C_{F\text{-}\alpha})\right) \qquad (58)
$$

At $t=T_c$ we have charge as

$$
q(T_c)=\frac{V_m T_c}{R+R_s}E_{\alpha,2}\left(-\tfrac{T_c{}^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right) \qquad (59)
$$

For $Z_2(s)=R_s+\frac{1}{sC}$ i.e. with an ideal capacitor with ESR, we have the following

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right) \\
&= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\left\{V_m\left(1-e^{-\frac{t}{(R+R_s)C}}\right)\right\}\right) \\
&= C\left(\frac{V_m\left(\frac{1}{(R+R_s)C}\right)}{s\left(s+\frac{1}{(R+R_s)C}\right)}\right)=\frac{V_m C\left(\frac{1}{(R+R_s)C}\right)}{s\ \left(s\ +\frac{1}{(R+R_s)C}\right)}=V_m C\left(\frac{1}{s}-\frac{1}{s+\frac{1}{(R+R_s)C}}\right) \qquad (60) \\
q(t) &= V_m C\left(1-e^{-\frac{t}{(R+R_s)C}}\right)
\end{aligned}
$$

Charge at the end of $t=T_c$ is

$$
q(T_c)=V_m C\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right) \qquad (61)
$$

The charging current is following from Eq. (60)

$$
i_{CH}(t)=\frac{dq(t)}{dt}=\frac{V_m e^{-\frac{t}{(R+R_s)C}}}{(R+R_s)}, \quad 0\le t\le T_c \qquad (62)
$$

The voltage at the end of $t=T_c$ is $v_c(T_c)=V_m(1-e^{-\frac{T_c}{(R+R_s)C}})$.

8-b) Discharging Phase

After $t = T_c$ we make the voltage $v_{in}(t) = 0$ i.e. we are draining out the stored charge i.e. $q(T_c) = V_m C(1 - e^{-T_c/((R+R_s)C)})$ during the discharge phase ($t \geq T_c$); Figure-3. In the discharge phase the voltage $v_c(T_c)$ will decay as $v_c(t') = \left(v_c(T_c)\right) e^{-t'/(R+R_s)C}$, for $t \geq T_c$, writing $t' = t - T_c$. At this point the capacity function $c(t') = C\delta(t')$ will again appear, as there is sudden change (differentiability is lost) in voltage from V_m to 0 at $t' = 0$ (i.e. $t = T_c$). Therefore the discharging charge profile $q(t')$ we write as $q(t') = c(t') * v_c(t')$ as follows

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\{c(t')\}\right)\left(\mathcal{L}\{v_c(t')\}\right), \quad t \geq T_c \\ &= \left(\mathcal{L}\{C\delta(t')\}\right)\left(\mathcal{L}\left\{\left(v_c(T_c)\right) e^{-t'/(R+R_s)C}\right\}\right) \\ &= (C)\left(\frac{\left(v_c(T_c)\right)}{s + \frac{1}{(R+R_s)C}}\right) \end{aligned} \tag{63}$$

$$\begin{aligned} q(t') &= C v_c(T_c) e^{-\frac{t'}{(R+R_s)C}}; \quad v_c(T_c) = V_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right) \\ &= V_m C\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right) e^{-\frac{t'}{(R+R_s)C}}; \quad t' \geq 0; \quad t \geq T_c \end{aligned}$$

The discharging current $t \geq T_c$ is as follows

$$\begin{aligned} i_{DIS}(t') &= \frac{dq(t')}{dt'} = C v_c(T_c) \frac{de^{-\frac{t'}{(R+R_s)C}}}{dt'}, \quad t \geq T_c \\ &= -\frac{v_c(T_c)}{(R+R_s)} e^{-\frac{t'}{(R+R_s)C}} = -\frac{V_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right)}{(R+R_s)} e^{-\frac{t'}{(R+R_s)C}} \end{aligned} \tag{64}$$

The negative sign in Eq. (64) indicates that discharge current is opposite to that of charging current. Now we carry on with the above logic for a fractional capacitor with $Z_2(s) = R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}}$.

This value $v_c(T_c) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$; Eq. (56) becomes the initial voltage while we discharge the super-capacitor with time defined as $t' = t - T_c$, for discharge phase where $v_{in}(t') = 0$.

Now we see the discharge profile, as the charged fractional capacitor $C_{F\text{-}\alpha}$ with above value $v_c(T_c)$ Eq. (56) discharges through R . The discharge current is now for $t' \geq 0$, negative to the charging current is following

$$I_{DIS}(s) = -\frac{v_c(T_c)/s}{R + R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}}} = -\frac{v_c(T_c) s^{\alpha-1}}{(R+R_s)\left(s^\alpha + \frac{1}{s^\alpha C_{F\text{-}\alpha}(R+R_s)}\right)} \tag{65}$$

The inverse Laplace transform of Eq. (65) gives discharge current for $t \geq T_c$ as following

$$i_{DIS}(t') = \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)/s}{R+R_s+\frac{1}{s^\alpha C_{F\text{-}\alpha}}}\right\}$$
$$= -\frac{v_c(T_c)}{R+R_s}E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right); \quad t\ge T_c, \quad v_c(0)=V_m\left(1-E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \tag{66}$$

This $i_{DIS}(t')$ is real discharge current flowing out of the capacitor, unlike notional discharge current that we used in explaining the self discharge phenomena. For $\alpha=1$ we have for ideal loss less capacitor $C_{F\text{-}\alpha}=C$ from Eq. (66)

$$i_{DIS}(t') = \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)/s}{R+R_s+\frac{1}{sC}}\right\} = -\frac{v_c(T_c)}{R+R_s}e^{-\frac{t'}{(R+R_s)C}}; \quad t\ge T_c, \quad v_c(T_c)=V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right) \tag{67}$$

The discharging profile of $q(t')$ with initial charge $q(0)=q(T_c)$ is

$$q(t') = q(0) + \int_0^{t'} -\frac{v_c(T_c)}{R+R_s}e^{-\frac{\tau}{(R+R_s)C}}d\tau = \left[Cv_c(T_c)e^{-\frac{\tau}{(R+R_s)C}}\right]_{\tau=0}^{\tau=t'}; \quad t>T_c$$
$$= q(0) + Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}} - Cv_c(T_c) \tag{68}$$
$$q(T_c) = q(0) = V_mC\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right) = Cv_c(T_c)$$

Thus we get $q(t')$ for $t\ge T_c$ with $t'=t-T_c$ as following

$$q(t') = Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}}; \quad v_c(T_c)=V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right); \quad t\ge T_c \tag{69}$$

The voltage profile across the fractional capacitor while discharging process is

$$v_c(t') = v_c(T_c)E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right), \quad t\ge T_c, \quad v_c(T_c)=V_m\left(1-E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \tag{70}$$

We mention here that Eq. (70) is only having discharge though shunt resistor R . If we consider the self-discharge phenomena of the fractional capacitors, then we have from earlier derivation

$$v_c(t') = \left(\frac{V_m\left(1-E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)}{\Gamma(1\text{-}\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^\alpha(T_c+t'-\tau)^{1\text{-}\alpha}}\right)E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right); \quad t'\ge 0 \tag{71}$$

The self discharge part due to spatial charge diffusion into distributed structure, is a very-very slow process, thus we generally avoid that while calculating the discharge profiles through external shunt resistance.

The charge $q(t')$ profile during the discharge phase is $q(t')=c(t')*v_c(t')$ for $t\ge T_c$ is following

$$Q(s) = \left(\mathcal{L}\{c(t')\}\right)\left(\mathcal{L}\{v_c(t')\}\right)$$
$$= \left(\mathcal{L}\left\{C_\alpha(t')^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{v_c(T_c)E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right\}\right); \quad C_{F\text{-}\alpha}=C_\alpha\Gamma(1-\alpha)$$
$$= \left(C_\alpha\Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{v_c(T_c)s^{\alpha\text{-}1}}{s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}}\right) = C_{F\text{-}\alpha}v_c(T_c)s^{\alpha-1}\frac{s^{\alpha-1}}{s^\alpha+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}} \tag{72}$$
$$= C_{F\text{-}\alpha}v_c(T_c)\left(s^{-1}\mathcal{L}\left\{D_{t'}^\alpha E_\alpha(-kt'^\alpha)\right\}\right); \quad k=\frac{1}{(R+R_s)C_{F\text{-}\alpha}}$$

In above steps Eq. (72), we have used $s^{\alpha}F(s)\equiv D_t^{\alpha}f(t)$, for $F(s)=\frac{s^{\alpha-1}}{s^{\alpha}+k}$, $f(t)=E_{\alpha}(-kt^{\alpha})$. Consider the fractional derivative operator D_t^{α} as Caputo fractional derivative. We have the Caputo fractional derivative of Mittag-Leffler function $E_{\alpha}(\lambda x^{\alpha})$ as $D_x^{\alpha}E_{\alpha}(\lambda x^{\alpha})=\lambda E_{\alpha}(\lambda x^{\alpha})$; [13] (Refer Appendix). Using this we write the following

$$Q(s)=C_{F\text{-}\alpha}v_c(T_c)\left(s^{-1}\mathcal{L}\left\{-kE_{\alpha}(-kt'^{\alpha})\right\}\right);\quad k=\tfrac{1}{(R+R_s)C_{F\text{-}\alpha}} \tag{73}$$

Using inverse Laplace Transform we have

$$\begin{aligned} q(t') &= q(0)+\left(-C_{F\text{-}\alpha}v_c(T_c)\int_0^{t'}kE_{\alpha}(-k\tau^{\alpha})d\tau\right);\quad k=\tfrac{1}{(R+R_s)C_{F\text{-}\alpha}} \\ &= q(0)+\left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'}E_{\alpha}(-k\tau^{\alpha})d\tau\right);\quad t\geq T_c \end{aligned} \tag{74}$$

Where we have $q(0)=q(T_c)=\frac{V_mT_c}{R+R_s}E_{\alpha,2}\left(-\frac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)$ and $v_c(T_c)=V_m\left(1-E_{\alpha}\left(-\frac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$

We use $\int_0^t E_{\alpha}\left(-k\tau^{\alpha}\right)d\tau=t\left(E_{\alpha,2}(-kt^{\alpha})\right)$ (Refer Appendix) and write the following

$$\begin{aligned} q(t') &= q(0)+\left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'}E_{\alpha}\left(-\tfrac{\tau^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau\right);\quad t\geq T_c \\ &= \frac{V_mT_c}{R+R_s}E_{\alpha,2}\left(-\tfrac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)-\frac{V_m\left(1-E_{\alpha}\left(-\tfrac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)}{(R+R_s)}\left[t'\left(E_{\alpha,2}\left(-\tfrac{t'^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)\right] \end{aligned} \tag{75}$$

We put $\alpha=1$ in $q(t')=q(0)+\left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'}E_{\alpha}\left(-\frac{\tau^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau\right)$; Eq. (75) and we get what we got for classical ideal capacitor $C_{F\text{-}\alpha}=C$, i.e. $q(t')=q(0)+\left(-\frac{v_c(T_c)}{R+R_s}\int_0^{t'}e^{-\frac{\tau}{(R+R_s)C}}d\tau\right)$, Eq. (69).

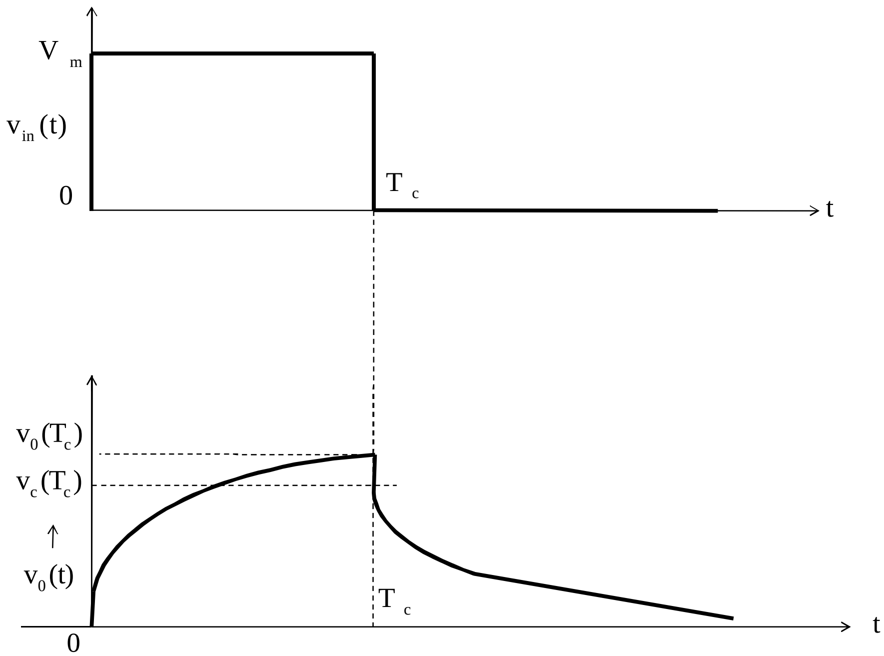

Figure-3: Constant voltage charging and discharging voltage profile at super-capacitor

The Figure-3 displays the curve of voltage profile for a constant voltage charge and discharge case. Here we point out that the charging curve though similar to exponential charging of a text book capacitor $v_0(t) \propto (1-e^{-t/RC})$, but it is not so, for fractional capacitor that is described via Mittag-Leffler function. Similarly the discharge profile though similar to exponential decay $v_0(t) \propto e^{-t/RC}$, but is not so for fractional capacitor; here too described by Mittag-Leffler function. All the relations we obtained and also verified our formula $q(t) = c(t) * v(t)$.

9. Charge storage $q(t)$ by step input constant current $i_{in}(t) = I_m u(t)$ excitation to RC circuit with fractional capacitor and ideal capacitor

In the Figure-1 we take $Z_1(s) = R$, $Z_2(s) = \frac{1}{s^\alpha C_{F-\alpha}}$ and instead of $v_{in}(t) = V_m u(t)$, that is voltage source, we take, that as an ideal constant current source i.e. $i_{in}(t) = I_m u(t)$. This constant current charging we apply to initially uncharged fractional capacitor, with capacity function $c(t) = C_\alpha t^{-\alpha}$. The fractional capacitor will develop a voltage across it by law governed by fractional derivative and fractional integral as follows

$$i(t) = C_{F-\alpha} \frac{d^\alpha v(t)}{dt^\alpha}; \quad v(t) = \frac{1}{C_{F-\alpha}} \int_0^t i(\tau)\left(d\tau\right)^\alpha = \frac{1}{C_{F-\alpha}} D_t^{-\alpha} i(t); \quad 0 < \alpha < 1 \tag{73}$$

Therefore, for constant current $i(t) = I_m$ the voltage is fractional integral of a constant I_m

$$v(t) = \frac{1}{C_{F-\alpha}} D_t^{-\alpha} i(t) = \frac{1}{C_{F-\alpha}} D_t^{-\alpha} I_m = \frac{I_m}{C_{F-\alpha}\Gamma(1+\alpha)} t^\alpha; \quad t \ge 0 \tag{74}$$

for $t \ge 0$ [12], [13], [37]. We used formula $D_t^{-n} t^m = \frac{\Gamma(m+1)}{\Gamma(m+1+n)} t^{m+n}$ in Eq. (74), (Refer Appendix) Therefore the charge function $q(t)$ is $q(t) = c(t) * v(t)$ as follows

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right) \\ &= \left(\mathcal{L}\left\{C_\alpha\, t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{I_m \tfrac{1}{C_{F\text{-}\alpha}\Gamma(1+\alpha)} t^{\alpha}\right\}\right); \quad \tfrac{\Gamma(n+1)}{s^{n+1}} = \mathcal{L}\left\{t^n\right\} \\ &= \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\left(I_m \tfrac{1}{C_{F\text{-}\alpha}\Gamma(1+\alpha)} \tfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right); \quad C_\alpha\Gamma(1-\alpha) = C_{F\text{-}\alpha} \\ &= \frac{I_m}{s^2} \end{aligned} \tag{75}$$

Thus we have charge function by taking Laplace inverse of above Eq. (75) as

$$q(t) = I_m t; \quad t \geq 0 \tag{76}$$

The Eq. (76) can be expressed as $q(t) = I_m r(t)$, where $r(t)$ is unit ramp function at $t = 0$. That is $r(t) = t$ for $t \geq 0$ and $r(t) = 0$ for $t < 0$. This Eq. (76) is matter of fact is the current flowing through R and $C_{F\text{-}\alpha}$ is $i(t) = I_m$ for $t \geq 0$, and thus the charge will be

$$q(t) = \int_0^t i(\tau)d\tau = \int_0^t I_m d\tau = I_m t = I_m r(t); \quad t \geq 0 \tag{77}$$

For an ideal capacitor with $c(t) = C\delta(t)$ the voltage is $v(t) = \frac{1}{C}\int_0^t I_m d\tau = \frac{I_m}{C} t$ so the charge is $q(t) = c(t) * v(t)$ as follows

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right) \\ &= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\left\{I_m \tfrac{1}{C} t\right\}\right); \quad \tfrac{1}{s^2} = \mathcal{L}\{t\} = \mathcal{L}\{r(t)\} \\ &= (C)\left(I_m \tfrac{1}{Cs^2}\right) = \frac{I_m}{s^2} \end{aligned} \tag{78}$$

$$q(t) = I_m t = I_m r(t); \quad t \geq 0$$

Thus in the case of constant current charging, we verified the validity of $q(t) = c(t) * v(t)$ as for any capacitor fractional or ideal loss less capacitor, the $q(t) = I_m t$; that is always integration of current function, i.e. $q(t) = \int_0^t i(\tau)d\tau$, for $t \geq 0$.

10. Charge storage $q(t)$ by step input current of a square pulse $i_{in}(t)$ to RC circuit with fractional capacitor and ideal capacitor

Let the square pulse of current be described as follows

$$i(t) = I_m u(t) - 2I_m u(t - T_c) + I_m u(t - T_d) \tag{79}$$

Where $u(t - T) = 1$ for $t \geq T$ and $u(t - T) = 0$ for $t < T$, i.e. unit step function at time $t = T$ .Then with identity $\mathcal{L}\{f(t - T)\} = e^{-sT}F(s)$ with $f(t - T) = 0$ for $t < T$; we write

$$I(s) = \mathcal{L}\{i(t)\} = \frac{I_m}{s} - \frac{2I_m}{s} e^{-sT_c} + \frac{I_m}{s} e^{-sT_d} \tag{80}$$

We have voltage across $Z_2(s) = \frac{1}{s^\alpha C_{F\text{-}\alpha}}$ as follows

$$\begin{aligned} V(s) &= Z_2(s)I(s) \\ &= \left(\frac{1}{C_{F\text{-}\alpha}s^\alpha}\right)\left(\frac{I_m}{s} - \frac{2I_m}{s}e^{-sT_c} + \frac{I_m}{s}e^{-sT_d}\right) = \frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}} - \frac{2I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_c} + \frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_d} \end{aligned} \tag{81}$$

Then taking inverse Laplace of Eq. (81) we get voltage profile across $C_{F\text{-}\alpha}$ as

$$\begin{aligned} v(t) &= \frac{I_m t^{\alpha}}{C_{F\text{-}\alpha}\Gamma(\alpha+1)}u(t) - \frac{2I_m(t-T_c)^{\alpha}}{C_{F\text{-}\alpha}\Gamma(\alpha+1)}u(t-T_c) + \frac{I_m(t-T_d)^{\alpha}}{C_{F\text{-}\alpha}\Gamma(\alpha+1)}u(t-T_d) \\ &= \frac{I_m r_{\alpha}(t)}{C_{F\text{-}\alpha}\Gamma(\alpha+1)} - \frac{2I_m r_{\alpha}(t-T_c)}{C_{F\text{-}\alpha}\Gamma(\alpha+1)} + \frac{I_m r_{\alpha}(t-T_d)}{C_{F\text{-}\alpha}\Gamma(\alpha+1)} \end{aligned} \tag{82}$$

We note that $\mathcal{L}^{-1}\left\{e^{-sT}F(s)\right\} = f(t-T)$, where $f(t-T)=0$ for $t<T$. We can write explicitly $\mathcal{L}^{-1}\left\{e^{-sT}F(s)\right\} = f(t-T)u(t-T)$, where $u(t-T)$ is unit step function at $t=T$. This we used in Eq. (82). Also in Eq. (682) we define function r_{α} as $r_{\alpha}(t-\tau)=(t-\tau)^{\alpha}$ for $t \ge \tau$ and $r_{\alpha}(t-\tau)=0$ for $t<\tau$. The Laplace transform of r_{α} is, $\mathcal{L}\left\{r_{\alpha}(t)\right\} = \Gamma(\alpha+1)s^{-(\alpha+1)}$ therefore we have the identity $\mathcal{L}\left\{r_{\alpha}(t-\tau)\right\} = e^{-s\tau}\Gamma(\alpha+1)s^{-(\alpha+1)}$, which is used in Eq. (81) to get Eq. (82).

The charge function is $q(t)=c(t)*v(t)$ as follows, when the voltage profile $v(t)$; Eq. (81) is across a fractional capacitor $c(t)=C_{\alpha}t^{-\alpha}$. This $c(t)=C_{\alpha}t^{-\alpha}$ gets applied at $t=0$, $t=T_c$ and $t=T_d$; that is where there is sudden change of state of $v(t)$; (that is at points where the differentiability of $v(t)$ is lost). We write

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\left\{C_{\alpha}\, t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{v(t)\right\}\right); \quad \tfrac{\Gamma(n+1)}{s^{n+1}} = \mathcal{L}\left\{t^{n}\right\}; \quad C_{F\text{-}\alpha} = C_{\alpha}\Gamma(1-\alpha) \\ &= \left(C_{\alpha}\Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}} - \frac{2I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_c} + \frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_d}\right) \\ &= C_{F\text{-}\alpha}s^{\alpha-1}\frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}} - C_{F\text{-}\alpha}s^{\alpha-1}\frac{2I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_c} + \frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_d} \\ &= \frac{I_m}{s^2} - \frac{2I_m}{s^2}e^{-sT_c} + \frac{I_m}{s^2}e^{-sT_d} \\ q(t) &= I_m t - 2I_m(t-T_c)u(t-T_c) + I_m(t-T_d)u(t-T_d) \\ &= I_m r(t) - 2I_m r(t-T_c) + I_m r(t-T_d) \end{aligned} \tag{83}$$

In Eq. (83) we define unit ramp function r as $r(t-\tau)=(t-\tau)$ for $t \ge \tau$ and $r(t-\tau)=0$ for $t<\tau$. The Laplace transform of r is, $\mathcal{L}\left\{r(t)\right\} = s^{-2}$ therefore we have the identity $\mathcal{L}\left\{r(t-\tau)\right\} = e^{-s\tau}s^{-2}$, which is used in Eq. (683). This shows verification of our formula $q(t)=c(t)*v(t)$. In similar way we can analyze the ideal loss less capacitor $c(t)=C\delta(t)$, for this wave form of current pulse.

11. Charging/discharging when R is zero ohms in RC circuit with voltage pulses

In this case Figure-1 has $Z_1(s)=0$. Therefore the voltage source directly gets connected to the fractional or ideal capacitor represented by impedance $Z_2(s)$. This case we have studied for step, ramp and sinusoidal voltage excitation in [40]. Here we take square wave case and triangular wave case, as extension of [40].

11-a) Charge storage $q(t)$ in a square wave voltage-on for time T_c and thereafter zero

The following excitation of a square wave pulse is applied to uncharged capacitor

$$v(t)=\begin{cases}0 & , \quad t<0\\ V_m & , \quad 0\le t\le T_c\\ 0 & , \quad t>T_c\end{cases} \tag{84}$$

We construct the above Eq. (84) excitation with $u(t-\tau)=1$ for $t\ge\tau$ and $u(t-\tau)=0$ for $t<\tau$; that is unit step function at $t=\tau$ as $v(t)=V_m u(t)-V_m u(t-T_c)$. The Laplace transform is

$$V(s)=\mathcal{L}\{V_m u(t)\}-\mathcal{L}\{V_m u(t-T_c)\}=\frac{V_m}{s}-\frac{V_m}{s}e^{-sT_c} \tag{85}$$

We used $\mathcal{L}\{f(t-t_d)\}=e^{-st_d}\mathcal{L}\{f(t)\}=e^{-st_d}F(s)$ with $f(t-t_d)=0$ for $t<t_d$ in above Eq. (85). When this voltage is applied to a time varying capacity function $c(t)=C_1\delta(t)$ i.e. ideal loss less capacitor we write from $q(t)=c(t)*v(t)$ the following

$$\begin{aligned}Q(s)=\mathcal{L}\{q(t)\}&=\left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right)=\left(C_1\right)\left(\frac{V_m}{s}-\frac{V_m}{s}e^{-sT_c}\right)\\&=\frac{V_m C_1}{s}-e^{-sT_c}\frac{V_m C_1}{s}\end{aligned} \tag{86}$$

Taking inverse Laplace transform of Eq. (86) we get

$$q(t)=V_m C_1 u(t)-V_m C_1 u(t-T_c)=\begin{cases}0 & , \quad t<0\\ V_m C_1 & , \quad 0\le t\le T_c\\ 0 & , \quad t>T_c\end{cases} \tag{87}$$

Now when this square-wave is applied for a time varying capacity function as $c(t)=C_\alpha t^{-\alpha}$ i.e. for fractional capacitor we write from $q(t)=c(t)*v(t)$ the following

$$\begin{aligned}Q(s)=\mathcal{L}\{q(t)\}&=\left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right)=\left(\frac{C_\alpha\Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(\frac{V_m}{s}-\frac{V_m}{s}e^{-sT_c}\right)\\&=\frac{V_m C_\alpha\Gamma(1-\alpha)}{s^{2-\alpha}}-e^{-sT_c}\frac{V_m C_\alpha\Gamma(1-\alpha)}{s^{2-\alpha}}\end{aligned} \tag{88}$$

Taking inverse Laplace Transform of above Eq. (89) we obtain

$$q(t)=\frac{V_m C_\alpha t^{1-\alpha}u(t)}{1-\alpha}-\frac{V_m C_\alpha (t-T_c)^{1-\alpha}u(t-T_c)}{1-\alpha}=\begin{cases}0 & , \quad t<0\\ \dfrac{V_m C_\alpha}{1-\alpha}t^{1-\alpha} & , \quad 0\le t\le T_c\\ \dfrac{V_m C_\alpha}{1-\alpha}t^{1-\alpha}-\dfrac{V_m C_\alpha}{1-\alpha}(t-T)^{1-\alpha} & , \quad t>T_c\end{cases} \tag{90}$$

The charge at $t=T_c$ is $q(T_c)=\frac{V_m C_\alpha T_c^{1-\alpha}}{1-\alpha}$, charge at $t=2T_c>T_c$ $q(2T_c)=\frac{V_m C_\alpha T_c^{1-\alpha}}{1-\alpha}(2^{1-\alpha}-1)$, charge at $t=3T_c$ is $q(3T_c)=\frac{V_m C_\alpha T_c^{1-\alpha}}{(1-\alpha)}(3^{1-\alpha}-2^{1-\alpha})$. We observe that for a fractional capacitor while the voltage is zero, after $t=T_c$, there still is charge holding, as compared with ideal capacitor Eq. (87). The current wave form is

$$i(t) = \frac{dq(t)}{dt} = V_m C_\alpha (t^{-\alpha} - (t - T_c)^{-\alpha}) = \begin{cases} 0 & , \quad t < 0 \\ V_m C_\alpha t^{-\alpha} & , \quad 0 \le t \le T_c \\ V_m C_\alpha (t^{-\alpha} - (t - T_c)^{-\alpha}) & , \quad t > T_c \end{cases} \tag{91}$$

11-b) Charge storage by voltage as triangular input of voltage

The following excitation of a square wave pulse is applied to uncharged capacitor

$$v(t) = \begin{cases} 0 & , \quad t < 0 \\ \frac{V_m}{T} t & , \quad 0 \le t \le T \\ \frac{V_m}{T} t - \frac{2V_m}{T}(t - T) & , \quad T \le t \le 2T \\ 0 & , \quad t \ge 2T \end{cases} \tag{92}$$

We can write the above excitation as $v(t) = (V_m / T)r(t) - (2V_m / T)r(t - T)$ for $0 \le t \le 2T$. With $r(t)$ unit ramp at $t = 0$ and is zero for $t < 0$ and $r(t - T)$ as unit ramp at $t = T$ and zero at $t < T$. With this applied to a ideal capacitor, with $c(t) = C_1\delta(t)$, we get the following by application of $q(t) = c(t) * v(t)$

$$\begin{aligned} Q(s) &= \mathcal{L}\{q(t)\} = (\mathcal{L}\{c(t)\})(\mathcal{L}\{v(t)\}) = (C_1)\left(\frac{V_m}{Ts^2} - \frac{2V_m}{Ts^2} e^{-sT}\right) \\ &= \frac{V_m C_1}{Ts^2} - e^{-sT} \frac{2V_m C_1}{Ts^2} \end{aligned} \tag{93}$$

Doing inverse Laplace transform of Eq. (93) we get

$$q(t) = \frac{V_m C_1}{T} r(t) - \frac{2V_m C_1}{T} r(t - T) = \begin{cases} 0 & , \quad t < 0 \\ \frac{V_m C_1}{T} t & , \quad 0 \le t \le T \\ \frac{V_m C_1}{T} t - \frac{2V_m C_1}{T}(t - T) & , \quad T \le t \le 2T \\ 0 & , \quad t \ge 2T \end{cases} \tag{94}$$

Current is got by differentiation of above Eq. (94)

$$i(t)=\frac{dq(t)}{dt}=\frac{V_mC_1}{T}u(t)-\frac{2V_mC_1}{T}u(t-T)=\begin{cases}0 & , \quad t<0\\ \frac{V_mC_1}{T} & , \quad 0\le t\le T\\ -\frac{V_mC_1}{T} & , \quad T\le t\le 2T\\ 0 & , \quad t\ge 2T\end{cases} \tag{95}$$

We take a fractional capacitor and do the following as done above as in Eq. (95) by applying the formula $q(t)=c(t)*v(t)$

$$\begin{aligned}Q(s)=\mathcal{L}\{q(t)\}=\left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v(t)\}\right)&=\left(\frac{C_\alpha\Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(\frac{V_m}{Ts^2}-\frac{2V_m}{Ts^2}e^{-sT}\right)\\ &=\frac{V_mC_\alpha\Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}}-e^{-sT}\frac{2V_mC_\alpha\Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}}\end{aligned} \tag{96}$$

We take inverse Laplace transform of above Eq. (96) with following definition of a function $r_m(t-\tau)$ defined as

$$r_m(t-\tau)=\begin{cases}(t-\tau)^m, & t\ge\tau\\ 0, & t<\tau\end{cases};\qquad \mathcal{L}\{r_m(t-\tau)\}=\frac{e^{-s\tau}\Gamma(1+m)}{s^{1+m}} \tag{97}$$

Thus the charge function $q(t)$ is following from Eq. (96) and Eq. (97)

$$\begin{aligned}q(t)&=\mathcal{L}^{-1}\left\{\frac{V_mC_\alpha\Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}}\right\}-\mathcal{L}^{-1}\left\{e^{-sT}\frac{2V_mC_\alpha\Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}}\right\}\\ &=\frac{V_mC_\alpha\Gamma(1-\alpha)}{T\Gamma(3-\alpha)}r_{2-\alpha}(t)-\frac{2V_mC_\alpha\Gamma(1-\alpha)}{T\Gamma(3-\alpha)}r_{2-\alpha}(t-T)\\ &=\frac{V_mC_\alpha}{T(1-\alpha)(2-\alpha)}r_{2-\alpha}(t)-\frac{2V_mC_\alpha}{T(1-\alpha)(2-\alpha)}r_{2-\alpha}(t-T)\end{aligned} \tag{98}$$

We used $\Gamma(1+m)=m\Gamma(m)$ in above Eq. (98). We re-write above Eq. (98) as using Eq. (97)

$$q(t)=\frac{V_mC_\alpha r_{2-\alpha}(t)}{T(1-\alpha)(2-\alpha)}-\frac{2V_mC_\alpha r_{2-\alpha}(t-T)}{T(1-\alpha)(2-\alpha)}=\begin{cases}0 & , \quad t<0\\ \frac{V_mC_\alpha t^{2-\alpha}}{T(1-\alpha)(2-\alpha)} & , \quad 0\le t\le T\\ \frac{V_mC_\alpha t^{2-\alpha}}{T(1-\alpha)(2-\alpha)}-\frac{2V_mC_\alpha (t-T)^{2-\alpha}}{T(1-\alpha)(2-\alpha)} & , \quad T\le t\le 2T\end{cases} \tag{99}$$

We have at $t = T$, $q(T) = \frac{V_m C_\alpha T^{1-\alpha}}{(1-\alpha)(2-\alpha)}$ at $t = 2T$, $q(2T) = \frac{V_m C_\alpha T^{1-\alpha}(2^{2-\alpha}-2)}{(1-\alpha)(2-\alpha)}$. We observe that at $t = 2T$, the voltage is zero, but we have charge as non-zero. With $\alpha \approx 1$, we get $q(2T) \approx 0$, Eq. (99) that we have analyzed for an ideal loss less capacitor. Differentiating the above we write current as

$$i(t) = \frac{dq(t)}{dt} = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{V_m C_\alpha t^{1-\alpha}}{T(1-\alpha)} & , \quad 0 \le t \le T \\ \dfrac{V_m C_\alpha t^{1-\alpha}}{T(1-\alpha)} - \dfrac{2V_m C_\alpha (t-T)^{1-\alpha}}{T(1-\alpha)} & , \quad T \le t \le 2T \end{cases} \qquad (100)$$

Thus we verified $q(t) = c(t) * v(t)$ the formula in RC circuits with charging resistance as zero, for triangular and square pulse of voltage excitation.

12. Conclusions

This formula $q(t) = c(t) * v(t)$ is a new development. We have not yet applied this to practical cases in our project as this theoretical development very new, but plan to have further experimental and theoretical studies on this new formula, like application in estimation state of charge (SOC) in supercapacitors charge discharge applications, parameter extraction by Hysteresis plot where use this formula for supercapacitors, the insight into new way of defining loss-tangent as we obtained from this formula, and applications to several dielectric relaxation experiments where memory is observed. In this paper however we have applied this new formula of charge storage i.e. via convolution operation $q(t) = c(t) * v(t)$, of time varying capacity function and voltage stress for a fractional capacitor and ideal loss-less capacitor; for verification in RC charging/discharging circuit; with dc voltage and current sources. We have also shown the effect of memory in self-discharging cases for a fractional capacitor, by this formula. This new formulation is different to the earlier used formula of multiplication of capacity and voltage function. The circuit analysis that we described for each cases verifies this formula. Thus this new formulation of stored charge via convolution operation is applicable, and can be taken as general formula applicable to fractional capacitor as well as ideal capacitor.

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APPENDIX

A Preliminaries of fractional calculus

For a function $f(t)$ for $t \ge 0$, the Riemann-Liouville fractional integration of order $\upsilon \in \mathbb{R}^{+}$ is defined as

$$ {}_0I_t^{\upsilon}\left[f(t)\right] = \frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} f(\tau)d\tau \qquad \text{A1}$$

Where $\Gamma(\upsilon)$ is Euler's Gamma function, is generalization of factorial function we have $\Gamma(\upsilon)=(\upsilon-1)!$. The formula Eq. (A1) is ${}_0I_t^{\upsilon}\left[f(t)\right]=\left(\frac{t^{\upsilon-1}}{\Gamma(\upsilon)}\right)*f(t)$ is convolution operation, with power-law memory kernel. This is $k_{\upsilon}(t)=\frac{t^{\upsilon-1}}{\Gamma(\upsilon)}$ and is singular function for case $0<\upsilon<1$. We have $\lim_{\upsilon\to 0}k_{\upsilon}(t)=\lim_{\upsilon\to 0}\frac{t^{\upsilon-1}}{\Gamma(\upsilon)}=\delta(t)$, which gives ${}_0I_t^{0}\left[f(t)\right]=f(t)$ .The formula Eq. (A1) is appearing as generalization of Cauchy's multiple integration formula of m fold integration where $m\in\mathbb{N}$ given as follows

$$ {}_0I_t^{m}\left[f(t)\right] = \frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1} f(\tau)d\tau; \quad m=1,2,3,.... \qquad \text{A2}$$

The fractional derivative of order β for $0<\beta<1$ by Riemann-Liouville (RL) formula is

$$ {}_0D_t^{\beta}\left[f(t)\right] = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\beta} f(\tau)d\tau; \quad 0<\beta<1 \qquad \text{A3}$$

The Eq. (A3) is fractionally integrating the function by order $(1-\beta)$ by formula Eq. (A1) and then followed by one-whole differentiation. We note that Eq. (A7) is also having convolution operation and with singular kernel as $k_{\beta}(t)=\frac{t^{-\beta}}{\Gamma(1-\beta)}$ .We have thus $\lim_{\beta\to 1}k_{\beta}(t)=\lim_{\beta\to 1}\frac{t^{-\beta}}{\Gamma(1-\beta)}=\delta(t)$ and $\lim_{\beta\to 1}\left({}_0D_t^{\beta}\left[f(t)\right]\right)=\frac{d\ f(t)}{dt}$.

There is reverse operation called Caputo's fractional derivative, where we have a function $f(t)$ defined for $t\ge 0$ and is differentiable i.e. $f^{(1)}(t)$ exists for $t\ge 0$. The Caputo fractional derivative for fractional order $0<\beta<1$ is given as

$$ {}_0^{C}D_t^{\beta}\left[f(t)\right] = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta} f^{(1)}(\tau)d\tau; \quad 0<\beta<1 \qquad \text{A4}$$

Thus for Eq. (A4) we need to get first the one-whole order derivative that is $f^{(1)}(t)$, and then carry out fractional integration for order $1-\beta$, by formula Eq. (A1). The formula Eq. (A4) also employs singular kernel as $k_{\beta}(t)=\frac{t^{-\beta}}{\Gamma(1-\beta)}$, and we have $\lim_{\beta\to 1}\left({}_0^C D_t^{\beta}[f(t)]\right)=f^{(1)}(t)$. The Caputo and Riemann-Liouville (RL) fractional derivative are related by

$$ {}_0D_t^{\beta}[f(t)] = {}_0^C D_t^{\beta}[f(t)] + \frac{f(0)}{\Gamma(1-\beta)} t^{-\beta}; \quad 0<\beta<1 \qquad \text{A5}$$

We write (A5) as following, with non-zero as start point of fractional differentiation process

$$\begin{aligned} {}_aD_t^{\beta}[f(t)] &= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_a^t \frac{f(x)}{(t-x)^{\beta}}dx, \quad 0<\beta<1 \\ &= \frac{1}{\Gamma(1-\beta)}\left(\frac{f(a)}{(t-a)^{\beta}} + \int_a^t \frac{f^{(1)}(x)}{(t-x)^{\beta}}dx\right); \quad t>a \\ &= \frac{f(a)}{(t-a)^{\beta}\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{f^{(1)}(x)}{(t-x)^{\beta}}dx \\ &= \frac{f(a)}{\Gamma(1-\beta)}(t-a)^{-\beta} + {}_a^C D_t^{\beta}[f(t)] \end{aligned} \qquad \text{A6}$$

We mention that both the fractional derivatives are equal when initial value is zero i.e. $f(0)=0$. We note that fractional derivative of constant is not zero in RL sense, but is a power function (and that is singular at start point) i.e. ${}_0D_t^{\beta}[K]=\frac{K}{\Gamma(1-\beta)}t^{-\beta}$. Whereas the Caputo's fractional derivative of a constant is zero, i.e. ${}_0^C D_t^{\beta}[K]=0$.

The fractional integration and fractional differentiation of delta function is as follows

$$ {}_0I_t^{\upsilon}\delta(t)=\frac{1}{\Gamma(\upsilon)}t^{\upsilon-1}; \quad {}_0D_t^{\upsilon}\delta(t)=\frac{1}{\Gamma(-\upsilon)}t^{-\upsilon-1}, \quad 0<\upsilon<1 \qquad \text{A7}$$

Fractional derivative and fractional integration of power function $f(t)=Kt^p$ is

$$ {}_0I_t^{\upsilon}Kt^p = K\frac{\Gamma(p+1)}{\Gamma(p+\upsilon+1)}t^{p+\upsilon}, \qquad {}_0D_t^{\upsilon}Kt^p = K\frac{\Gamma(p+1)}{\Gamma(p-\upsilon+1)}t^{p-\upsilon}, \quad p>-1 \qquad \text{A8}$$

The Laplace transform of fractional integral operation is

$$\mathcal{L}\left\{{}_0I_t^{\upsilon}f(t)\right\} = s^{-\upsilon}F(s) \qquad \text{A9}$$

Laplace transform of Caputo fractional derivative for fractional order $0<\upsilon<1$ is

$$\mathcal{L}\left\{{}_0^C D_t^{\upsilon}f(t)\right\} = s^{\upsilon}F(s) - s^{\upsilon-1}f(0) \qquad \text{A10}$$

Laplace transform of Riemann-Liouville fractional derivative of order $0<\upsilon<1$ is

$$\mathcal{L}\left\{{}_0D_t^{\upsilon}f(t)\right\} = s^{\upsilon}F(s) - f^{(\upsilon-1)}(0) \qquad \text{A11}$$

In (A11) $f^{(\upsilon-1)}(0)=\lim_{t\to 0}\left({}_0I_t^{1-\upsilon}f(t)\right)$; that initial states required in (A11) for RL fractional derivative is of fractional order, types $f^{(\upsilon-1)}(0)$ whereas initial states required (A10) for Caputo type fractional derivative is integer order (classical) type $f(0)$.

B. Mittag-Leffler Function

Like in classical calculus, we have exponential function e^z; similarly, in fractional calculus we have Mittag-Leffler function. The series definition Mittag Leffler function is

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(z)^n}{\Gamma(\alpha n+\beta)}, \quad z\in\mathbb{C}; \quad \mathrm{Re}[\alpha,\beta]>0 \qquad \text{B1}$$

For $\beta = 1$ we have $E_{\alpha,1}(z) = E_\alpha(z)$; is called One-Parameter Mittag-Leffler function. The Laplace transformation of Mittag-Leffler function is following

$$\mathcal{L}\left\{E_\alpha(\lambda t^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha - \lambda} \qquad \text{B2}$$

We observe that for $E_\alpha(-bt^\alpha)\big|_{\alpha=1} = e^{-bt}$, and $E_\alpha(-at^\alpha)\big|_{\alpha=2} = \cos\sqrt{a}t$.

We point here that $f(t) = E_\alpha(\lambda t^\alpha)$ is eigen-function for fractional differential equation with Caputo derivative i.e. ${}_0^C D_t^\alpha f(t) = \lambda f(t)$; and $f(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ is eigen-function for fractional differential equation with RL fractional derivative i.e. ${}_0 D_t^\alpha f(t) = \lambda f(t)$.

Recurring property of $E_{\alpha,\beta}(x)$ is

$$E_{\alpha,\beta}(x) = \frac{1}{x}E_{\alpha,\beta-\alpha}(x) - \frac{1}{x\Gamma(\beta-\alpha)} \qquad \text{B3}$$

For one parameter Mittag-Leffler function

$$E_\alpha(x) = E_{\alpha,1}(x) = \frac{1}{x}E_{\alpha,1-\alpha}(x) - \frac{1}{x\Gamma(1-\alpha)} \qquad \text{B4}$$

We use (B3) and write following steps

$$\begin{aligned} E_{\alpha,\beta}(x) &= -\frac{1}{x\Gamma(\beta-\alpha)} + \frac{1}{x}E_{\alpha,\beta-\alpha}(x) = -\frac{1}{x\Gamma(\beta-\alpha)} + \frac{1}{x}\left(-\frac{1}{x\Gamma(\beta-2\alpha)} + \frac{1}{x}E_{\alpha,\beta-2\alpha}(x)\right) \\ &= -\frac{1}{x\Gamma(\beta-\alpha)} - \frac{1}{x^2\Gamma(\beta-2\alpha)} + \frac{1}{x^2}E_{\alpha,\beta-2\alpha}(x) \qquad \text{B5} \\ &= -\frac{1}{x\Gamma(\beta-\alpha)} - \frac{1}{x^2\Gamma(\beta-2\alpha)} - \frac{1}{x^3\Gamma(\beta-3\alpha)} + \frac{1}{x^3}E_{\alpha,\beta-3\alpha}(x) \end{aligned}$$

From (B5) we get Poincare asymptotic expansion of $E_{\alpha,\beta}(x)$ as

$$E_{\alpha,\beta}(x) \sim -\sum_{n=1}^{\infty}\frac{1}{x^n\Gamma(\beta-n\alpha)} \qquad \text{B6}$$

valid for $x \to -\infty$.

C. Proof of formula $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ used

We verify the formula used $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ as in following steps

$$\begin{aligned} \int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau &= \int_0^t\left(1 - \frac{k\tau^\alpha}{\Gamma(\alpha+1)} + \frac{k^2\tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{k^3\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right)d\tau \\ &= t - \frac{kt^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)} + \frac{k^2t^{2\alpha+1}}{(2\alpha+1)\Gamma(2\alpha+1)} - \frac{k^3t^{3\alpha+1}}{(3\alpha+1)\Gamma(3\alpha+1)} + \ldots \qquad \text{C1} \\ &= t\left(1 - \frac{kt^\alpha}{\Gamma(\alpha+2)} + \frac{k^2t^{2\alpha}}{\Gamma(2\alpha+2)} - \frac{k^3t^{3\alpha}}{\Gamma(3\alpha+2)} + \ldots\right), \quad \Gamma(m+1) = m\Gamma(m) \\ &= t\left(E_{\alpha,2}(-kt^\alpha)\right) \quad ; \quad E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)} \end{aligned}$$